# The maximum size of a partial spread in $H(4n+1, q^2)$ is $q^{2n+1}+1$

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**Abstract**

We prove that in every finite Hermitian polar space of odd dimension and even maximal dimension $\rho$ of the totally isotropic subspaces, a partial spread has size at most $q^{\rho+1}+1$, where $GF(q^2)$ is the defining field. This bound is tight and is a generalisation of the result of De Beule and Metsch for the case $\rho = 2$.

# 1 Introduction

A *partial spread* of a polar space is a set of pairwise disjoint generators. If these generators form a partition of the points of the polar space, it is said to be a *spread*. Thas [10] proved that in the Hermitian polar space $H(2n+1, q^2)$ spreads cannot exist, which has made the question on the size of a partial spread in such a space, an intriguing question. A partial spread is said to be maximal if it cannot be extended to a larger partial spread.

In [1], partial spreads of size $q^{n+1}+1$ in $H(2n+1, q^2)$ are constructed by use of a symplectic polarity of the projective space $\mathsf{PG}(2n+1, q^2)$, commuting with the associated Hermitian polarity. In the Baer subgeometry of points on which the two polarities coincide, a (regular) spread of the induced symplectic polar space $W(2n+1, q)$ can always be found, and these $q^{n+1}+1$ generators extend to pairwise disjoint generators of $H(2n+1, q^2)$. They also prove maximality of this construction for $H(5, q^2)$. Luyckx [9] generalises this result by showing that this construction does in fact yield a maximal partial spread of size $q^{2n+1}+1$ in all spaces $H(4n+1, q^2)$, and she also improves the upper bound on the size of partial spreads in $H(5, q^2)$. De Beule and Metsch [5] prove that the size of a partial

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spread in $H(5,q^2)$ can never exceed $q^3+1$. Their proof relies on counting methods, and they also obtain additional information on the structure of partial spreads which meet this bound $q^3+1$.

In this note, we will generalise the result of [5] by proving that in all $H(4n+1,q^2)$, the number $q^{2n+1}+1$ is an upper bound on the cardinality of partial spreads, hence establishing tightness of the bound. Our technique will be somewhat different from what was used in previous work. We will consider partial spreads in polar spaces as cliques with respect to the oppositeness relation on generators, and then use inequalities involving eigenvalues to obtain an upper bound. In general, the calculation of eigenvalues for this relation on $m$-spaces in a polar space is much more complex, but for our purposes these calculations are considerably shorter as the oppositeness relation can be directly associated with the dual polar graph. The dual polar graph is distance-regular and hence we readily have the required information about its eigenvalues and intersection numbers.

# 2 Background theory and notation

## 2.1 Polar spaces

We refer to [8] for definitions and properties of polar spaces. If $\mathcal{P}$ is a polar space, we will denote by $d$ the dimension of its generators. The parameter $\epsilon$ is defined as: 0 if $\mathcal{P}$ is a symplectic or parabolic space, $-1$ if $\mathcal{P}$ is a hyperbolic space, 1 if $\mathcal{P}$ is an elliptic space, $1/2$ if $\mathcal{P}$ is a Hermitian variety in even dimension, and $-1/2$ if $\mathcal{P}$ is a Hermitian variety in odd dimension. The number of points in the polar space $\mathcal{P}$ is $(q^{d+1+\epsilon}+1)(q^{d+1}-1)/(q-1)$. The size of a spread in $\mathcal{P}$ is equal to $q^{d+1+\epsilon}+1$, which is of course, also an upper bound on the size of a partial spread.

The Hermitian variety, embedded in the projective space $\mathsf{PG}(n,q^2)$, consists of those subspaces of $\mathsf{PG}(n,q^2)$, the points $(X_0,\ldots,X_n)$ of which all satisfy the homogeneous equation: $X_0^{q+1}+\ldots+X_n^{q+1}=0$.

The number of $m$-spaces in a projective space $\mathsf{PG}(n,q)$ is $\begin{bmatrix} n+1 \\ m+1 \end{bmatrix}_q$, where $\begin{bmatrix} a \\ b \end{bmatrix}_q$ is the Gaussian coefficient, which is defined as follows if $a \geq b$:

$$\begin{bmatrix} a \\ b \end{bmatrix}_q = \prod_{i=1}^{b} \frac{q^{a+1-i}-1}{q^i-1},$$

and defined to be zero if $a<b$.

## 2.2 Association schemes

Bose and Shimamoto [3] introduced the notion of a $D$-class *association scheme* on a finite set $\Omega$ as a set of symmetric relations $\mathcal{R}=\{R_0,R_1,\ldots,R_D\}$ on $\Omega$ such that the following axioms hold:

(i) $R_0$ is the identity relation,

(ii) $\mathcal{R}$ is a partition of $\Omega^2$,

(iii) there are *intersection numbers* $p_{ij}^k$ such that for $(x, y) \in R_k$, the number of elements $z$ in $\Omega$ for which $(x, z) \in R_i$ and $(z, y) \in R_j$ equals $p_{ij}^k$.

The relations $R_i$ are all symmetric regular relations with valency $p_{ii}^0$, and hence define regular graphs on $\Omega$.

It can be shown (see for instance [2]) that the real algebra $\mathbb{R}\Omega$ has an orthogonal decomposition into $D+1$ subspaces $V_i$, all of them eigenspaces of the relations $R_j$ of the association scheme. The $(D+1) \times (D+1)$-matrix $P$, where $P_{ij}$ is the eigenvalue of the relation $R_j$ for the eigenspace $V_i$, is the *matrix of eigenvalues* of the association scheme. If $\Delta_m$ is the diagonal matrix with $(\Delta_m)_{ii}$ the dimension of the eigenspace $V_i$, and if $\Delta_n$ is the diagonal matrix with $(\Delta_n)_{jj}$ the valency of the relation $R_j$, then the *dual matrix of eigenvalues* $Q = |\Omega| P^{-1}$ can be obtained by calculating $\Delta_n^{-1} P^T \Delta_m$.

In [6], the *inner distribution vector* $\mathbf{a} := (\mathbf{a}_0, \ldots, \mathbf{a}_D)$ of a non-empty subset $X$ of $\Omega$ is defined as follows:

$$\mathbf{a}_i = \frac{1}{|X|} |\{(X \times X) \cap R_i\}|, \quad \text{for all } i \in \{0, \ldots, D\}.$$

The $i$-th entry of $\mathbf{a}$ thus equals the average number of elements $x' \in X$, such that $(x, x') \in R_i$ for some $x \in X$. It follows immediately from the definitions that $\mathbf{a}_0 = 1$, and that the sum of all of its entries must equal $|X|$. Delsarte proved (see for instance [6]) that every entry of the row matrix $\mathbf{a}Q$ is non-negative, which implies that the same holds for all entries of $\mathbf{a}\Delta_n^{-1} P^T$.

## 2.3 Distance-regular graphs

Let $\Gamma$ be a connected graph with diameter $D$ on a set of vertices $V$. For every $i$ in $\{0, \ldots, D\}$, we let $\Gamma_i$ denote the graph on the same set $V$, with two vertices adjacent if and only if they are at distance $i$ in $\Gamma$, and we write $R_i$ for the corresponding symmetric relation on $V$. The graph $\Gamma$ is said to be *distance-regular* if the set of relations $\{R_0, R_1, \ldots, R_D\}$ defines an association scheme on $V$. It can be shown (see [4]) that this is equivalent with the existence of parameters $b_i$ and $c_i$, such that for every $(v, v_i) \in R_i$, there are $c_i$ neighbours $v_{i-1}$ of $v_i$ with $(v, v_{i-1}) \in R_{i-1}$, for every $i \in \{1, \ldots, D\}$, and $b_i$ neighbours $v_{i+1}$ with $(v, v_{i+1}) \in R_{i+1}$, for every $i \in \{0, \ldots, D-1\}$. These parameters $b_i$ and $c_i$ are known as the *intersection numbers* of the distance-regular graph $\Gamma$.

If $\theta$ is any eigenvalue of a distance-regular graph $\Gamma$, then there is a series of eigenvalues $\{v_i\}$ of the associated $i$-distance graphs $\Gamma_i$, recursively defined (see page 128 in [4]) by $v_0 = 1, v_1 = \theta$, and:

$$\theta v_i = c_{i+1} v_{i+1} + (k - b_i - c_i) v_i + b_{i-1} v_{i-1}, \text{ for all } i \in \{1, \ldots, D-1\}.$$

### 2.4 The dual polar graph

We will consider the *dual polar graph* $\Gamma$, associated with a polar space $\mathcal{P}$. The vertices of this graph are the generators (i.e., $d$-spaces in $\mathcal{P}$), and two vertices are adjacent if and only if they intersect in a $(d-1)$-space. This graph is distance-regular with diameter $d+1$, and two generators are at distance $i$ of each other if and only if they meet in a $(d-i)$-space. We refer to [4] for the valency $k$ and the intersection numbers $b_i$ and $c_i$ of the dual polar graph:

$$k = q^{\epsilon+1}\begin{bmatrix} d+1 \\ 1 \end{bmatrix}_q, \quad b_i = q^{i+\epsilon+1}\begin{bmatrix} d+1-i \\ 1 \end{bmatrix}_q, \quad c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}_q.$$

By Theorem 9.4.3 [4], the eigenvalues of the dual polar graph are given by:

$$q^{\epsilon+1}\begin{bmatrix} d-r \\ 1 \end{bmatrix}_q - \begin{bmatrix} r+1 \\ 1 \end{bmatrix}_q, \text{ for all } r \text{ with } -1 \leq r \leq d,$$

and especially $-\left[\begin{smallmatrix} d+1 \\ 1 \end{smallmatrix}\right]_q$ is an eigenvalue (take $r=d$).

If $\Gamma$ is the dual polar graph of a polar space $\mathcal{P}$, then $\Gamma_i$ consists of those edges connecting generators meeting in a $(d-i)$-space. Hence in $\Gamma_0$, every vertex is adjacent only to itself, while $\Gamma_1$ is just the (distance-regular) dual polar graph $\Gamma$. Finally, $\Gamma_{d+1}$ is the oppositeness graph in which we are interested. The valencies of these regular graphs $\Gamma_i$ can also be found in [4] (Lemma 9.4.2): $q^{i(i+1+2\epsilon)/2}\left[\begin{smallmatrix} d+1 \\ i \end{smallmatrix}\right]_q$. In particular, the valency of the oppositeness graph $\Gamma_{d+1}$ is $q^{(d+1)(d+2+2\epsilon)/2}$.

## 3 Calculation of a specific subset of eigenvalues of the association scheme

The eigenvalues of the dual polar graph were already given in Subsection 2.4. We will now calculate the recursively defined series of associated eigenvalues of the other graphs $\Gamma_i$ in order to obtain an eigenvalue of the oppositeness graph $\Gamma_{d+1}$.

**Lemma 3.1.** *The eigenvalue $\theta = -\left[\begin{smallmatrix} d+1 \\ 1 \end{smallmatrix}\right]_q$ of the dual polar graph yields the series of eigenvalues $v_i$ of the graphs $\Gamma_i$, with:*

$$v_i = (-1)^i q^{\binom{i}{2}}\begin{bmatrix} d+1 \\ i \end{bmatrix}_q, \text{ for all } i \in \{0,\dots,d+1\}.$$

*and hence the oppositeness graph $\Gamma_{d+1}$ has eigenvalue $(-1)^{d+1}q^{d(d+1)/2}$.*

*Proof.* We will first prove that $v_i = -q^{i-1}v_{i-1}\left[\begin{smallmatrix} d+2-i \\ 1 \end{smallmatrix}\right]_q / \left[\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right]_q$, for all $i \in \{1,\dots,d+1\}$. This is obvious if $i=1$. Now suppose that it holds for $i \in \{1,\dots,d\}$. By substituting the values for $b_i$ and $c_i$ in the recurrence relation, one obtains:

$$\begin{bmatrix} i+1 \\ 1 \end{bmatrix}_q v_{i+1} + \left( q^{\epsilon+1} \begin{bmatrix} d+1 \\ 1 \end{bmatrix}_q - q^{i+\epsilon+1} \begin{bmatrix} d+1-i \\ 1 \end{bmatrix}_q - \begin{bmatrix} i \\ 1 \end{bmatrix}_q + \begin{bmatrix} d+1 \\ 1 \end{bmatrix}_q \right) v_i + q^{i+\epsilon} \begin{bmatrix} d+2-i \\ 1 \end{bmatrix}_q v_{i-1} = 0.$$

Using the induction hypothesis to substitute for $v_{i-1}$, this can be rewritten as:

$$\begin{bmatrix} i+1 \\ 1 \end{bmatrix}_q v_{i+1} = -(q^{\epsilon+1}+1) \left( \begin{bmatrix} d+1 \\ 1 \end{bmatrix}_q - \begin{bmatrix} i \\ 1 \end{bmatrix}_q \right) v_i + q^{\epsilon+1+i} \begin{bmatrix} d+1-i \\ 1 \end{bmatrix}_q v_i.$$

As $\left[\begin{smallmatrix} d+1 \\ 1 \end{smallmatrix}\right]_q = q^i \left[\begin{smallmatrix} d+1-i \\ 1 \end{smallmatrix}\right]_q + \left[\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right]_q$, this proves the induction hypothesis for $i+1$. Using this relation, as well as the identity $\left[\begin{smallmatrix} d+2-i \\ 1 \end{smallmatrix}\right]_q \left[\begin{smallmatrix} d+1 \\ i-1 \end{smallmatrix}\right]_q = \left[\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right]_q \left[\begin{smallmatrix} d+1 \\ i \end{smallmatrix}\right]_q$ for all $i \in \{1, \ldots, d+1\}$, one can now prove by induction that $v_i = (-1)^i q^{\binom{i}{2}} \left[\begin{smallmatrix} d+1 \\ i \end{smallmatrix}\right]_q$ for every $i \in \{0, \ldots, d+1\}$. $\square$

# 4 Upper bounds on the sizes of cliques

We first state a general result on the cliques of association schemes which can be found in [7], but for which we give an alternative proof.

**Lemma 4.1.** *Let $\Gamma$ be a graph corresponding with one of the relations in an association scheme, with valency $k$. If $S$ is a clique in this graph, then for every eigenvalue $\lambda < 0$ of the graph $\Gamma$ the following inequality holds: $|S| \leq 1 - \frac{k}{\lambda}$.*

*Proof.* The inner distribution vector $\mathbf{a}$ simply has a 1 on the position corresponding with the identity relation, and $|S|-1$ on the position corresponding with the relation defining the graph $\Gamma$. We now consider the vector $\mathbf{a}\Delta_n^{-1}P^T$. Its $i$-th entry is given by $1 + \frac{|S|-1}{k}\lambda_i$, where $\lambda_i$ is the eigenvalue of $\Gamma$ corresponding with the eigenspace $V_i$. As this value must be non-negative (see Subsection 2.2), we obtain the desired inequality for every negative eigenvalue. $\square$

As the cliques of the oppositeness graph on generators of a polar space are precisely the partial spreads, we can now prove the main result.

**Theorem 4.2.** *A partial spread in $H(4n+1, q^2)$ has at most $q^{2n+1}+1$ elements.*

*Proof.* For this polar space, $d = 2n$ and $\epsilon = -1/2$. The valency $k$ of the oppositeness graph is in this case equal to $q^{(2n+1)^2}$. On the other hand, we know from Lemma 3.1 that $\lambda = -q^{2n(2n+1)}$ is an eigenvalue. Applying the bound from Lemma 4.1, we obtain the following upper bound on the size of a partial spread in $H(4n+1, q^2)$:

$$1 - \frac{k}{\lambda} = 1 + q^{2n+1}.$$

$\square$

# 5 Concluding remarks

It is in fact possible to calculate in general all eigenvalues of the oppositeness relation between generators in polar spaces. However, in most cases, the smallest eigenvalue $\lambda$ is such that the bound $1-k/\lambda$ from Lemma 4.1 is just the upper bound $q^{d+1+\epsilon}+1$; the size of a spread, if it exists. Only for partial spreads in $Q^+(4n+1,q)$ and in $H(4n+1,q^2)$ does one actually obtain a sharper bound, where the former is the trivial bound of 2.

As additional information on the structure of partial spreads meeting the bound of $q^3+1$ in $H(5,q^2)$ is also obtained in [5], the question arises whether this is possible in the general case as well.


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