Set Systems with Restricted t-wise Intersections modulo Prime Powers

Rudy X. J. Liu

Department of Mathematics, Pearl River College
Tianjin University of Finance & Economics, Tianjin, 301811, P. R. China

xiaojiethink@yahoo.com.cn

Submitted: Jan 4, 2009; Accepted: May 24, 2009; Published: Jun 5, 2009
Mathematics Subject Classifications: 05D05

Abstract

We give a polynomial upper bound on the size of set systems with restricted t-wise intersections modulo prime powers. Let $t \geq 2$. Let p be a prime and $q = p^\alpha$ be a prime power. Let $\mathcal{L} = \{l_1, l_2, \ldots, l_s\}$ be a subset of $\{0, 1, 2, \ldots, q-1\}$. If $\mathcal{F}$ is a family of subsets of an n element set X such that $|F_1 \cap \cdots \cap F_t| \pmod q \in \mathcal{L}$ for any collection of t distinct sets from $\mathcal{F}$ and $|F| \pmod q \notin \mathcal{L}$ for every $F \in \mathcal{F}$, then

$$|\mathcal{F}| \leq \frac{t(t-1)}{2} \sum_{i=0}^{2^{s-1}} \binom{n}{i}.$$

Our result extends a theorem of Babai, Frankl, Kutin, and Štefankovič, who studied the 2-wise case for prime power moduli, and also complements a result of Grolmusz that no polynomial upper bound holds for non-prime-power composite moduli.

1 Introduction

We are interested in set systems with restricted t-wise intersections modulo prime powers. Let X denote a set of n elements and $\mathcal{F}$ be a family of subsets of X. Let p be a prime and $q = p^\alpha$ be a prime power. Let $\mathcal{L}$ be a subset of $\{0, 1, 2, \ldots, q-1\}$ of size s. For an integer $t \geq 2$, a family $\mathcal{F}$ is called *t-wise q-modular $\mathcal{L}$-intersecting* if $|F_1 \cap \cdots \cap F_t| \pmod q \in \mathcal{L}$ for any collection of t distinct sets from $\mathcal{F}$ and $|F| \pmod q \notin \mathcal{L}$ for every $F \in \mathcal{F}$. It is called *q-modular $\mathcal{L}$-intersecting* for simplicity when $t = 2$. Note that, the same definition is also used when q is not a prime power.

In 2001, Babai, Frankl, Kutin, and Štefankovič proved the size of a p^α-modular $\mathcal{L}$-intersecting family is polynomial bounded as a function of n.

Theorem 1 **(Babai *et al.* [1])** *If $\mathcal{F}$ is a p^α-modular $\mathcal{L}$-intersecting family of subsets of X, then*

$$|\mathcal{F}| \leq \binom{n}{2^{s-1}} + \binom{n}{2^{s-1}-1} + \cdots + \binom{n}{0}.$$

When $q = p$, Grolmusz [5] proved the following result in 2002.

Theorem 2 **(Grolmusz [5])** *If $\mathcal{F}$ is a t-wise p-modular $\mathcal{L}$-intersecting family of subsets of X, then*

$$|\mathcal{F}| \leq (t-1) \sum_{i=0}^{s} \binom{n}{i}.$$

When $t = 2$, it is a modular version of the celebrated Frankl-Wilson Theorem. Grolmusz and Sudakov [6] gave another proof of this bound using multilinear polynomials. Recently, Cao, Hwang and West [2] improved the above bound by replacing $\binom{n}{i}$ with $\binom{n-1}{i}$ in the sum.

In the same paper [5], Grolmusz also showed that Theorem 2 does not generalize to non-prime-power composite moduli. In particular for any $t \geq 2$, $q = 6$ and $\mathcal{L} = \{1, \ldots, 5\}$, there exists a t-wise 6-modular $\mathcal{L}$-intersecting family of X of superpolynomial size in n, see Theorem 11 in [5] for detail.

In this paper, we will fill the gap between Theorem 2 (prime moduli) and Grolmusz's result (non-prime-power composite moduli, Theorem 11 in [5]) by proving a polynomial upper bound on the size of the t-wise p^α-modular $\mathcal{L}$-intersecting families for any $t \geq 2$.

Theorem 3 *If $\mathcal{F}$ is a t-wise p^α-modular $\mathcal{L}$-intersecting family of subsets of X, then*

$$|\mathcal{F}| \leq \frac{t(t-1)}{2} \sum_{i=0}^{2^{s-1}} \binom{n}{i}.$$

Clearly, the special case $t = 2$ of Theorem 3 corresponds to Theorem 1.

2 The Proof

In this section, let $q = p^\alpha$ be a prime power and we will give a proof of Theorem 3, which is motivated by the methods used in [1] and [3].

First we need the following Frankl-Wilson-type result for pairs of families of sets with restricted intersection modulo prime power, which is a slight generalization of Theorem 1.

Lemma 1 *Let $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ be two families of subsets of X such that $|A_i \cap B_i| \pmod{q} \notin \mathcal{L}$ for all $1 \leq i \leq m$ and $|A_i \cap B_j| \pmod{q} \in \mathcal{L}$ for $i \neq j$. Then*

$$m \leq \binom{n}{2^{s-1}} + \binom{n}{2^{s-1}-1} + \cdots + \binom{n}{0}.$$

Note that Theorem 1 is a special case of Lemma 1 when $A_i = B_i$ for $1 \le i \le m$. The proof of this lemma follows from the proof of Lemma 3.1 in [1], and we refer the reader there for details.

Proof of Theorem 3 Let us apply induction on t. When $t = 2$, it has been proved by Theorem 1. Now assume that $t > 2$ and the assertion is true for $t = k$, we will prove that it also holds for $t = k + 1$.

Let $\mathcal{F} = \{F_1, \dots, F_m\}$ be a $(k+1)$-wise q-modular $\mathcal{L}$-intersecting family of subsets of X. To prove the statement, we partition $\mathcal{F}$ into three families of sets $\mathcal{A}$, $\mathcal{F}_1$ and $\mathcal{F}_2$ with the following properties: there exists a family of sets $\mathcal{B}$ such that the pair $(\mathcal{A}, \mathcal{B})$ satisfies the condition of Lemma 1, $|\mathcal{F}_1| = (k-1)|\mathcal{A}|$ and the family $\mathcal{F}_2$ is k-wise q-modular $\mathcal{L}$-intersecting. To do this we repeat the following procedure. For every $0 \le r \le |\mathcal{F}| - 1$, suppose that after step r we have already constructed families of sets $\mathcal{A} = \{A_1, \dots, A_i\}$, $\mathcal{B} = \{B_1, \dots, B_i\}$, $\mathcal{F}_1$ and $\mathcal{F}_2 = \{D_1, \dots, D_j\}$ such that $|\mathcal{F}_1| = (k-1)i$. Consider three possible cases.

Case 1: If $F_{r+1} \in \mathcal{F}_1$, then proceed to the next step.

Case 2: If there are indices $r + 1 < r_1 < \cdots < r_{k-1}$ such that $F_{r_i} \notin \mathcal{F}_1$ for all $1 \le i \le k-1$ and $|F_{r+1} \cap F_{r_1} \cap \cdots \cap F_{r_{k-1}}| \pmod q \notin \mathcal{L}$, then define $A_{i+1} = F_{r+1}$, $B_{i+1} = F_{r+1} \cap F_{r_1} \cap \cdots \cap F_{r_{k-1}}$. Let $\mathcal{F}_1 = \mathcal{F}_1 \cup \{F_{r_1}, \cdots, F_{r_{k-1}}\}$ and proceed to the next step.

Case 3: Suppose that $|F_{r+1} \cap F_{r_1} \cap \cdots \cap F_{r_{k-1}}| \pmod q \in \mathcal{L}$ for every set of indices $r+1 < r_1 < \cdots < r_{k-1}$ with $F_{r_i} \notin \mathcal{F}_1$ for all $1 \le i \le k-1$. In this case define $D_{j+1} = F_{r+1}$ and continue. Clearly, by construction, $\mathcal{F}_2$ is a k-wise q-modular $\mathcal{L}$-intersecting family.

Let $\mathcal{A} = \{A_1, \dots, A_h\}$, $\mathcal{B} = \{B_1, \dots, B_h\}$, $\mathcal{F}_1$ and $\mathcal{F}_2$ be the set systems obtained in the end of our procedure. Note that, by definition, $|A_i \cap B_i| \pmod q \notin \mathcal{L}$ for $1 \le i \le h$ but $|A_i \cap B_j| \pmod q \in \mathcal{L}$ for $i \ne j$, since this is a size of intersection of $k+1$ distinct members of $\mathcal{F}$.

Now we can apply Lemma 1 to bound the size of $\mathcal{A}$. Since $\mathcal{F} = \mathcal{A} \cup \mathcal{F}_1 \cup \mathcal{F}_2$ and $|\mathcal{F}_1| = (k-1)|\mathcal{A}|$, by the induction hypothesis we obtain that

$$\begin{aligned}
|\mathcal{F}| &\le |\mathcal{A}| + |\mathcal{F}_1| + |\mathcal{F}_2| = k|\mathcal{A}| + |\mathcal{F}_2| \\
&\le k \sum_{i=0}^{2^{s-1}} \binom{n}{i} + \frac{k(k-1)}{2} \sum_{i=0}^{2^{s-1}} \binom{n}{i} \\
&= \frac{k(k+1)}{2} \sum_{i=0}^{2^{s-1}} \binom{n}{i}.
\end{aligned}$$

This completes the proof of the theorem. □

3 Concluding Remarks

The main point we make is that our bound in Theorem 3 implies a polynomial upper bound in n for the t-wise p^α-modular $\mathcal{L}$-intersecting families with $t \ge 3$. For the special case of prime power moduli q and $s = q-1$, the bound in Theorem 3 can be improved.

Theorem 4 (Grolmusz and Sudakov [6]) *Let $t \geq 2$ and r be integers. If $\mathcal{F}$ is a family of subsets of X such that $|F| \pmod q = r$ for each $F \in \mathcal{F}$ and $|F_1 \cap \cdots \cap F_t| \pmod q \neq r$ for any collection of t distinct sets from $\mathcal{F}$, then*

$$|\mathcal{F}| \leq (t-1) \sum_{i=0}^{q-1} \binom{n}{i}.$$

Still it would be interesting to obtain improved upper bound for our results.

Acknowledgments. I would like to thank my research supervisor, Professor Jiuqiang Liu for his support, especially during the last 2 years. This work was partially done when I was a student in Center for Combinatorics in Nankai University. I would also like to thank an anonymous referee for some helpful suggestions.

References

[1] L. Babai, P. Frankl, S. Kutin, and D. Štefankovič, Set systems with restricted intersections modulo prime powers, *J. Combinatorial Theory, Ser. A*, 95 (2001), 39-73.

[2] Weiting Cao, Kyung-Won Hwang, and Douglas B. West, Improved bounds on families under k-wise set-intersection constraints, *Graphs and Combinatorics*, 23 (2007), 381-386.

[3] Z. Füredi and B. Sudakov, Extremal set systems with restricted k-wise intersections, *J. Combinatorial Theory, Ser. A*, 105 (2004), 143-159.

[4] Vince Grolmusz, Superpolynomial size set-systems with restricted intersections mod 6 and explicit Ramsey graphs, *Combinatorica*, 20, No.1, (2000), 71-86.

[5] Vince Grolmusz, Set-systems with restricted multiple intersections, *Electronic J. Combinatorics*, 9 (2002), R8.

[6] Vince Grolmusz and Benny Sudakov, On k-wise set-intersections and k-wise hamming distances, *J. Combinatorial Theory, Ser. A*, 99 (2002), 180-190.