Landau's and Rado's Theorems and Partial Tournaments

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Abstract

Using Rado's theorem for the existence of an independent transversal of family of subsets of a set on which a matroid is defined, we give a proof of Landau's theorem for the existence of a tournament with a prescribed degree sequence. A similar approach is used to determine when a partial tournament can be extended to a tournament with a prescribed degree sequence.
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1 Introduction

A *tournament of order* n is a digraph obtained from the complete graph K_n of order n by giving a direction to each of its edges. Thus, a tournament T of order n has $\binom{n}{2}$ (directed) edges. The sequence $(r_1, r_2, \cdots, r_n)$ of outdegrees of the vertices $\{1, 2, \ldots, n\}$ of T, ordered so that $r_1 \leq r_2 \leq \cdots \leq r_n$, is called the *score sequence* of T. The sequence of indegrees of the vertices of T is given by $(s_1 = n-1-r_1, s_2 = n-1-r_2, \ldots, s_n = n-1-r_n)$ and satisfies $s_1 \geq s_2 \geq \cdots \geq s_n$. In the tournament T' obtained from T by reversing the direction of each edge, the indegree sequence and outdegree sequence are interchanged; the score vector of T' equals $(s_1, s_2, \ldots, s_n)$ with the s_i in nonincreasing order.

2 Landau's theorem from Rado's theorem

Landau's theorem characterizes score vectors of tournaments.

Theorem 2.1 (Landau's theorem) *The sequence $r_1 \leq r_2 \leq \cdots \leq r_n$ of integers is the score sequence of a tournament of order n if and only if*

$$\sum_{i=1}^{k} r_i \geq \binom{k}{2} \quad (k = 1, 2, \ldots, n) \tag{1}$$

with equality for $k = n$.

Note that (1) is equivalent to

$$\sum_{i \in K} r_i \geq \binom{|K|}{2} \quad (K \subseteq \{1, 2, \ldots, n\}). \tag{2}$$

There are several known short proofs of Landau's theorem (see [2, 3, 4, 7, 8]). In this section we give a short proof of Landau's theorem using Rado's theorem (see [5, 6]) for the existence of an independent transversal of a finite family of subsets of a set X on which a matroid is defined.

Let $\mathbf{M}$ be a matroid on X with rank function denoted by $\rho(\cdot)$. (We assume that the reader is familiar with the very basics of matroid theory, which can be found e.g. in [6].) Let $\mathcal{A} = (A_1, A_2, \ldots, A_n)$ be a family of n subsets of X. A *transversal* of $\mathcal{A}$ is a set S of n elements of X which can be ordered as $x_1, x_2, \ldots, x_n$ so that $x_i \in A_i$ for $i = 1, 2, \ldots, n$. The transversal S is an *independent transversal* of $\mathcal{A}$ provided that S is an independent set of the matroid $\mathbf{M}$.

Theorem 2.2 (Rado's theorem) *The family $\mathcal{A} = (A_1, A_2, \ldots, A_n)$ of subsets of the set X on which a matroid $\mathbf{M}$ is defined has an independent transversal if and only if*

$$\rho(\cup_{i \in K} A_i) \geq |K| \quad (K \subseteq \{1, 2, \ldots, n\}).$$

Proof of Landau's theorem using Rado's theorem. The necessity of (1) is obvious. Now assume that (1) holds. Let $X = \{(i, j); 1 \leq i, j \leq n, i \neq j\}$. Consider the matroid $\mathbf{M}$ on X whose circuits are the $\binom{n}{2}$ disjoint sets $\{(i, j), (j, i)\}$ of two pairs in X with $i \neq j$. Thus, a subset E of X is independent if and only if it does not contain a symmetric pair $(i, j), (j, i)$ with $i \neq j$. We have $\rho(X) = \binom{n}{2}$. Let $\mathcal{A} = (A_1, A_2, \ldots, A_n)$ be the family of subsets of X where

$$A_i = \{(i, j) : 1 \leq j \leq n, j \neq i\} \quad (i = 1, 2, \ldots, n). \tag{3}$$

Let $r_1, r_2, \ldots, r_n$ be a sequence of nonnegative integers with $r_1 + r_2 + \cdots + r_n = \binom{n}{2}$. There exists a tournament with score sequence $r_1, r_2, \ldots, r_n$ if and only if there exists $P_1, P_2, \ldots, P_n$, with $P_i \subseteq A_i$ and $|P_i| = r_i$ $(1 \leq i \leq n)$, such that $P = P_1 \cup P_2 \cup \cdots \cup P_n$ is an independent set of $\mathbf{M}$, equivalently, if and only if the family

$$\mathcal{A}' = (\underbrace{A_1, \ldots, A_1}_{r_1}, \underbrace{A_2, \ldots, A_2}_{r_2}, \ldots, \underbrace{A_n, \ldots, A_n}_{r_n})$$

has an independent transversal: The desired tournament has vertices $1, 2, \ldots, n$ and an edge from i to j if and only (i, j) is in P_i. The independence of P then implies that there is no edge from j to i.

It follows from Rado's theorem that $\mathcal{A}'$ has an independent transversal provided that

$$\rho(\cup_{i\in K} A_i) \geq \sum_{i\in K} r_i \quad (K \subseteq \{1, 2, \ldots, n\}). \tag{4}$$

From the definition of $\mathbf{M}$ we see that

$$\rho(\cup_{i\in K} A_i) = \binom{k}{2} + k(n-k), \tag{5}$$

where $k = |K|$. By (5), the rank of $\cup_{i\in K} A_i$ depends only on $k = |K|$. By the monotonicity assumption on the r_i, $\sum_{i\in K} r_i$ is largest when $K = \{n-k+1, \ldots, n\}$. Thus, (4) is equivalent to

$$\binom{k}{2} + k(n-k) \geq \sum_{i=n-k+1}^{n} r_i. \tag{6}$$

Since $\sum_{i=1}^{n} r_i = \binom{n}{2}$, (6) becomes

$$\sum_{i=1}^{n-k} r_i \geq \binom{n}{2} - \binom{k}{2} - k(n-k). \tag{7}$$

It follows that (4) is equivalent to

$$\sum_{i=1}^{p} r_i \geq \binom{n}{2} - \binom{n-p}{2} - p(n-p) \quad (p = 1, 2, \ldots, n). \tag{8}$$

A simple calculation shows that

$$\binom{n}{2} - \binom{n-p}{2} - p(n-p) = \binom{p}{2},$$

and Landau's theorem follows from (8). $\square$

3 Completions of partial tournaments

Let $G \subseteq K_n$ be a graph on n vertices. A digraph obtained from G by giving a direction to each of its edges is called an *oriented graph* or a *partial tournament of order n*. Given a partial tournament T' and a sequence of nonnegative integers $r_1, r_2, \ldots, r_n$, it is possible to use Rado's theorem to establish necessary and sufficient conditions for T' to be extendable to a tournament T with score sequence $r_1, r_2, \ldots, r_n$. Thus we seek to complete the partial tournament T' to a tournament T with a prescribed score sequence. Rado's theorem can also be used to characterize when such a completion is possible.

Let T' be a partial tournament of order n with outdegree sequence $s_1, s_2, \ldots, s_n$. Let $r_1, r_2, \ldots, r_n$ be a sequence of nonnegative integers with $\sum_{i=1}^{n} r_i = \binom{n}{2}$. (Now we make no monotone assumption on the r_i or the s_i.) An obvious necessary condtion for T' to be completed to a tournament with score sequence $r_1, r_2, \ldots, r_n$ is that $s_i \leq r_i$ for $i = 1, 2, \ldots, n$, and we assume these inequalities hold. There are two ways to determine when a completion of T' to a tournament with score sequence $r_1, r_2, \ldots, r_n$ is possible.

The first way is to take $X = \{(i,j) : 1 \leq i, j \leq n, i \neq j\}$ as before, and to consider the matroid $\mathbf{M}'$ whose circuits are the singleton pairs $\{(i,j)\}$ and $\{(j,i)\}$ if there is an edge from i to j in T' (thus an edge in T determines two loops of $\mathbf{M}'$), and the pairs $\{(i,j),(j,i)\}$ for all distinct i and j such that there is no edge in T' between i and j (in either of the two possible directions). We note that in this matroid $\mathbf{M}'$,

$$\rho'(X) = \binom{n}{2} - \sum_{i=1}^{n} s_i.$$

Define the family $\mathcal{A} = (A_1, A_2, \ldots, A_n)$ as in (3) and the family

$$\mathcal{A}'' = (\underbrace{A_1, \ldots, A_1}_{r_1 - s_1}, \underbrace{A_2, \ldots, A_2}_{r_2 - s_2}, \ldots, \underbrace{A_n, \ldots, A_n}_{r_n - s_n}).$$

We have

$$\sum_{i=1}^{n} (r_i - s_i) = \binom{n}{2} - \sum_{i=1}^{n} s_i.$$

The partial tournament T' can be completed to a tournament with score sequence $r_1, r_2, \ldots, r_n$ if and only if the family $\mathcal{A}''$ has an independent transversal. It follows from Rado's theorem that $\mathcal{A}''$ has an independent transversal if and only if

$$\rho'(\cup_{i \in K} A_i) \geq \sum_{i \in K} (r_i - s_i) \quad (K \subseteq \{1, 2, \ldots, n\}). \tag{9}$$

For $K \subseteq \{1, 2, \ldots, n\}$, let $\gamma(K)$ equal the number of edges of T' at least one of whose vertices belongs to K. We easily calculate that

$$\rho'(\cup_{i \in K} A_i) = \binom{|K|}{2} + |K|(n - |K|) - \gamma(K).$$

We thus obtain the following generalization of Landau's theorem.[1]

Theorem 3.1 *Let T' be a partial tournament with outdegree sequence $s_1, s_2, \ldots, s_n$. Let $r_1, r_2, \ldots, r_n$ be a sequence of nonnegative integers with $s_i \leq r_i$ for $i = 1, 2, \ldots, n$. Then T' can be completed to a tournament with score sequence $r_1, r_2, \ldots, r_n$ if and only if*

$$\binom{|K|}{2} + |K|(n - |K|) - \gamma(K) \geq \sum_{i \in K} (r_i - s_i) \quad (K \subseteq \{1, 2, \ldots, n\}. \tag{10}$$

[1] Landau's theorem is the special case where T' has no edges.

As a referee observed, because of the presence of the quantity $\gamma(K)$, whether or not the inequalities (10) in Theorem 3.1 are satisfied depends on the initial labeling of the vertices of T'. These conditions may not be satisfied according to one labeling but satisfied according to another.

A second, but basically equivalent, way to approach the proof of Theorem 3.1 is to start with the set

$$Y = X \setminus \{(i,j) : \ (i,j) \text{ or } (j,i) \text{ is an edge of } T'\},$$

and the matroid $\mathbf{M}|_Y$ on Y obtained by restricting $\mathbf{M}$ to Y. If we define the family $\mathcal{B} = (B_1, B_2, \ldots, B_n)$ of subsets of Y by $B_i = A_i \cap Y$ for $i = 1, 2, \ldots, n$, and then apply Rado's theorem to

$$\mathcal{B}' = (\underbrace{B_1, \ldots, B_1}_{r_1 - s_1}, \underbrace{B_2, \ldots, B_2}_{r_2 - s_2}, \ldots, \underbrace{B_n, \ldots, B_n}_{r_n - s_n}),$$

we again obtain a proof of Theorem 3.1.

As a corollary of Theorem 3.1 we obtain the main results in [1]. If n is an odd integer, a *regular tournament of order* n is a tournament with score sequence

$$\underbrace{\frac{n-1}{2}, \frac{n-1}{2}, \ldots, \frac{n-1}{2}}_{n}.$$

If n is an even integer, a *nearly regular tournament of order* n is a tournament with score sequence

$$\underbrace{\frac{n}{2}, \ldots, \frac{n}{2}}_{\frac{n}{2}}, \underbrace{\frac{n}{2} - 1, \ldots, \frac{n}{2} - 1}_{\frac{n}{2}}.$$

Corollary 3.2 *Let T' be a partial tournament with outdegree sequence $s_1, s_2, \ldots, s_n$ where $s_1 \geq s_2 \geq \cdots \geq s_n$. If n is odd, then T' can be completed to a regular tournament provided that*

$$s_i \leq \frac{n+1}{2} - i, \quad \left(i = 1, 2, \ldots, \frac{n+1}{2}\right). \tag{11}$$

If n is even, then T' can be completed to a nearly regular tournament of order n provided that

$$s_i \leq \frac{n}{2} - i + 1, \quad \left(i = 1, 2, \ldots, \frac{n}{2}\right). \tag{12}$$

Proof. First suppose that n is odd and that (11) holds. Then $s_i = 0$ for $i = (n+1)/2, (n+3)/2, \ldots, n$. Hence, there are no edges in T' from a vertex in $\{(n+1)/2, (n+3)/2, \ldots, n\}$ to $\{1, 2, \ldots, (n-1)/2\}$. It follows from Theorem 3.1 that T' can be completed to a regular tournament provided that

$$\binom{|K|}{2} + |K|(n - |K|) - \gamma(K) \geq |K|\left(\frac{n-1}{2}\right) - \sum_{i \in K} s_i \quad (K \subseteq \{1, 2, \ldots, n\},$$

that is, provided that

$$\binom{|K|}{2} + |K|(n-|K|) - \left(\gamma(K) - \sum_{i\in K} s_i\right) \geq |K|\binom{n-1}{2} \quad (K \subseteq \{1,2,\ldots,n\}). \quad (13)$$

The quantity $\gamma^*(K) := \gamma(K) - \sum_{i\in K} s_i$ equals the number of edges of T' with initial vertex in the complement $\overline{K}$ of K and terminal vertex in K. Simplifying (13), we get

$$\frac{|K||\overline{K}|}{2} \geq \gamma^*(K). \quad (14)$$

Since the lefthand side of (14) is symmetric in K and $\overline{K}$, we need only verify it for $|K| \leq (n+1)/2$. It follows from (11) that for $|K| \leq (n+1)/2$,

$$\gamma^*(K) \leq \sum_{i=1}^{|K|}\left(\frac{n+1}{2} - i\right) = \frac{|K|(n-|K|)}{2}.$$

Hence, T' can be completed to a regular tournament.

A similar proof works when n is even. □

References

[1] L. Beasley, D. Brown, and K. B. Reid, Extending partial tournaments, *Mathematical and Computer Modelling*, to appear.

[2] R. A. Brualdi, *Combinatorial Matrix Classes*, Cambridge U. Press, Cambridge, 2006, 34–35.

[3] J.R. Griggs and K.B. Reid, Landau's theorem revisited, *Australasian J. Combinatorics*, 20 (1999), 19–24.

[4] E. S. Mahmoodian, A critical case method of proof in combinatorial mathematics, *Bull. Iranian Math Soc.*, No. 8 (1978),1L-26L.

[5] L. Mirsky, *Transversal Theory*, Oxford University Press, Oxford, 1971, 93–95.

[6] J. Oxley, *Matroid Theory*, The Clarendon Press, Oxford University Press, New York, 1992.

[7] K.B. Reid, Tournaments: scores, kings, generalizations and special topics, *Congressus Numerantium*, 115 (1996), 171–211.

[8] C. Thomassen, Landau's characterization of tournament score sequences, *The Theory and Application of Graphs (Kalamazoo, Michigan 1980)*, Wiley, New York, 1963, 589–591.