# A stability property for coefficients in Kronecker products of complex $S_n$ characters


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**Abstract**

In this note we make explicit a stability property for Kronecker coefficients that is implicit in a theorem of Y. Dvir. Even in the simplest nontrivial case this property was overlooked despite of the work of several authors. As applications we give a new vanishing result and a new formula for some Kronecker coefficients.


## 1 Introduction

Let $\lambda$, $\mu$, $\nu$ be partitions of a positive integer $m$ and let $\chi^\lambda$, $\chi^\mu$, $\chi^\nu$ be their corresponding complex irreducible characters of the symmetric group $\mathsf{S}_m$. It is a long standing problem to give a satisfactory method for computing the multiplicity

$$\mathsf{k}(\lambda,\mu,\nu) := \langle \chi^\lambda \otimes \chi^\mu, \chi^\nu \rangle \tag{1}$$

of $\chi^\nu$ in the Kronecker product $\chi^\lambda \otimes \chi^\mu$ of $\chi^\lambda$ and $\chi^\mu$ (here $\langle \cdot,\cdot \rangle$ denotes the inner product of complex characters). Via the Frobenius map, $\mathsf{k}(\lambda,\mu,\nu)$ is equal to the multiplicity of the Schur function $s_\nu$ in the internal product of Schur functions $s_\lambda * s_\mu$, namely

$$\mathsf{k}(\lambda,\mu,\nu) = \langle s_\lambda * s_\mu, s_\nu \rangle \,,$$

where $\langle \cdot,\cdot \rangle$ denotes the scalar product of symmetric functions.

The first stability property for Kronecker coefficients was observed by F. Murnaghan without proof in [8]. This property can be stated in the following way: Let $\overline{\lambda}$, $\overline{\mu}$, $\overline{\nu}$


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be partitions of $a$, $b$, $c$, respectively. Define $\lambda(n) := (n-a, \overline{\lambda})$, $\mu(n) := (n-b, \overline{\mu})$, $\nu(n) := (n-c, \overline{\nu})$. Then the coefficient $\mathsf{k}(\lambda(n), \mu(n), \nu(n))$ is constant for all $n$ bigger than some integer $N(\overline{\lambda}, \overline{\mu}, \overline{\nu})$. Complete proofs of this property were given by M. Brion [3] using algebraic geometry and E. Vallejo [13] using combinatorics of Young tableaux. Both proofs give different lower bounds $N(\overline{\lambda}, \overline{\mu}, \overline{\nu})$ for the stability of $\mathsf{k}(\lambda(n), \mu(n), \nu(n))$, for all partitions $\overline{\lambda}$, $\overline{\mu}$, $\overline{\nu}$. C. Ballantine and R. Orellana [1] gave an improvement of one of these lower bounds for a particular case.

Here we make explicit another stability property for Kronecker coefficients that is implicit in the work of Y. Dvir (Theorem 2.4′ in [5]). This property can be stated as follows: Let $p$, $q$ and $r$ be positive integers such that $p = qr$. Let $\lambda = (\lambda_1, \ldots, \lambda_p)$, $\mu = (\mu_1, \ldots, \mu_q)$, $\nu = (\nu_1, \ldots, \nu_r)$ be partitions of some nonnegative integer $m$ satisfying $\ell(\lambda) \leq p$, $\ell(\mu) \leq q$, $\ell(\nu) \leq r$, that is, some parts of $\lambda$, $\mu$ and $\nu$ could be zero. For any positive integers $t$ and $n$ let $(t)^n$ denote the vector $(t, \ldots, t) \in \mathbb{N}^n$; and for any partition $\lambda = (\lambda_1, \ldots, \lambda_p)$ of length at most $p$ let $\lambda + (t)^p$ denote the partition $(\lambda_1 + t, \ldots, \lambda_p + t)$. Then we have

**Theorem 3.1.** With the above notation

$$\mathsf{k}(\lambda, \mu, \nu) = \mathsf{k}(\lambda + (t)^p, \mu + (rt)^q, \nu + (qt)^r)\,.$$

It should be noted that even in the simplest nontrivial case, when $q = 2 = r$ and $p = 4$, this property was overlooked despite of the work of several authors [1, 2, 9, 10]. In this situation Remmel and Whitehead noticed (Theorems 3.1 and 3.2 in [9]) that the coefficient $\mathsf{k}(\lambda, \mu, \nu)$ has a much simpler formula if $\lambda_3 = \lambda_4$. The main theorem provides an explanation for that. We also obtain a new formula for $\mathsf{k}(\lambda, \mu, \nu)$ in this case.

This note is organized as follows. Section 2 contains the definitions and notation about partitions needed in this paper. In Section 3 we give the proof of the main theorem. Section 4 deals with the Kronecker coefficient $\mathsf{k}(\lambda, \mu, \nu)$ when $\ell(\lambda) = \ell(\mu)\ell(\nu)$. In particular, we give, in this case, a new vanishing condition. Finally, in Section 5 we give an application of the main theorem.

# 2 Partitions

In this section we recall the notation about partitions needed in this paper. See for example [6, 7, 11, 12].

For any nonnegative integer $n$ let $[\,n\,] := \{1, \ldots, n\}$. A *partition* is a vector $\lambda = (\lambda_1, \ldots, \lambda_p)$ of nonnegative integers arranged in decreasing order $\lambda_1 \geq \cdots \geq \lambda_p$. We consider two partitions equal if they differ by a string of zeros at the end. For example $(3, 2, 1)$ and $(3, 2, 1, 0, 0)$ represent the same partition. The *length* of $\lambda$, denoted by $\ell(\lambda)$, is the number of positive parts of $\lambda$. The *size* of $\lambda$, denoted by $|\lambda|$, is the sum of its parts; if $|\lambda| = m$, we say that $\lambda$ is a partition of $m$ and denote it by $\lambda \vdash m$. The partition conjugate to $\lambda$ is denoted by $\lambda'$. A *composition* of $m$ is a vector $\pi = (\pi_1, \ldots, \pi_r)$ of positive integers such that $\sum_{i=1}^{r} \pi_i = m$.

The *diagram* of $\lambda = (\lambda_1, \ldots, \lambda_p)$, also denoted by $\lambda$, is the set of pairs of integers

$$\lambda = \{\, (i,j) \mid i \in [\,p\,],\ j \in [\,\lambda_i\,]\,\}.$$

The identification of $\lambda$ with its diagram permits us to use set theoretic notation for partitions. If $\delta$ is another partition and $\delta \subseteq \lambda$, we denote by $\lambda/\delta$ the *skew diagram* consisting of the pairs in $\lambda$ that are not in $\delta$, and by $|\lambda/\delta|$ its cardinality. If $\mu$ is another partition, then $\lambda \cap \mu$ denotes the set theoretic intersection of $\lambda$ and $\mu$.

# 3 Main theorem

**3.1 Theorem.** *Let $\lambda$, $\mu$, $\nu$ be partitions of some integer $m$. Let $p$, $q$, $r$ be integers such that $p \geq \ell(\lambda)$, $q \geq \ell(\mu)$, $r \geq \ell(\nu)$ and $p = qr$. Then for any positive integer $t$ we have*

$$\mathsf{k}(\lambda, \mu, \nu) = \mathsf{k}(\lambda + (t)^p, \mu + (rt)^q, \nu + (qt)^r)\,.$$

The proof of the main theorem will follow from Dvir's theorem

**3.2 Theorem.** *[5, Theorem 2.4′] Let $\lambda$, $\mu$, $\nu$ be partitions of $n$ such that $\ell(\nu) = |\lambda \cap \mu'|$. Let $l = \ell(\nu)$ and $\rho = \nu - (1^l)$. Then*

$$\mathsf{k}(\lambda, \mu, \nu) = \langle \chi^{\lambda/\lambda\cap\mu'} \otimes \chi^{\mu/\lambda'\cap\mu}, \chi^{\rho} \rangle\,.$$

*Proof of theorem 3.1.* It is enough to prove the theorem for $t = 1$. The general case follows by repeated application of the particular case. Let $\alpha = \lambda + (1)^p$, $\beta = \mu + (r)^q$ and $\gamma = \nu + (q)^r$. Then $\beta \cap \gamma' = (r)^q$. In particular, $|\beta \cap \gamma'| = p = \ell(\alpha)$. So, we have $\beta/\beta \cap \gamma' = \mu$ and $\gamma/\beta' \cap \gamma = \nu$. Thus, by Dvir's theorem, we have

$$\mathsf{k}(\beta, \gamma, \alpha) = \mathsf{k}(\mu, \nu, \lambda)\,.$$

The claim follows from the symmetry $\mathsf{k}(\lambda, \mu, \nu) = \mathsf{k}(\mu, \nu, \lambda)$ of Kronecker coefficients. □

**3.3 Example.** To illustrate how Dvir's theorem applies, let $\lambda = (8, 4)$, $\mu = (6, 6)$ and $\nu = (5, 3, 2, 2)$. Then $\lambda \cap \mu' = (2, 2) = \lambda' \cap \mu$, $\lambda/\lambda \cap \mu' = (6, 2)$, $\mu/\lambda' \cap \mu = (4, 4)$ and $\nu - (1^4) = (4, 2, 1, 1)$. After two applications of Dvir's theorem we get

$$\begin{aligned}\mathsf{k}((8,4),(6,6),(5,3,2,2)) &= \mathsf{k}((6,2),(4,4),(4,2,1,1)) \\ &= \mathsf{k}((4),(2,2),(3,1)) = 0\,.\end{aligned}$$

# 4 The case $\ell(\lambda) = \ell(\mu)\ell(\nu)$

In this section we give a general result for the Kronecker coefficient $\mathsf{k}(\lambda, \mu, \nu)$ when $\ell(\lambda) = \ell(\mu)\ell(\nu)$. On the one hand it gives a new vanishing condition. On the other hand, when this vanishing condition does not hold, it reduces the computation of $\mathsf{k}(\lambda, \mu, \nu)$ to the computation of a simpler Kronecker coefficient.

Let $m$ be a positive integer, $\lambda$, $\mu$ be partitions of $m$ and $\pi = (\pi_1, \ldots, \pi_r)$ be a composition of $m$. Let $\rho(i) \vdash \pi_i$ for $i \in [\,r\,]$. A sequence $T = (T_1, \ldots, T_r)$ of tableaux is called a *Littlewood-Richardson multitableau* of *shape* $\lambda$, *content* $(\rho(1), \ldots, \rho(r))$ and *type* $\pi$ if

(1) there exists a sequence of partitions

$$\varnothing = \lambda(0) \subset \lambda(1) \subset \cdots \subset \lambda(r) = \lambda$$

such that $|\lambda(i)/\lambda(i-1)| = \pi_i$ for all $i \in [\,r\,]$, and

(2) $T_i$ is Littlewood-Richardson tableau of shape $\lambda(i)/\lambda(i-1)$ and content $\rho(i)$, for all $i \in [\,r\,]$.

For example,

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **1** | **1** | **1** | **1** | *1* | *1* | *1* | 1 | 1 | 1 |
| **2** | **2** | **2** | **2** | *2* | 2 | 2 | 2 | | |
| **3** | **3** | *2* | *2* | *3* | | | | | |
| *3* | 3 | | | | | | | | |

is a Littlewood-Richardson multitableau of shape $(10, 8, 5, 2)$, type $(10, 8, 7)$ and content $((4, 4, 2), (3, 3, 2), (3, 3, 1))$.

Let $\mathsf{LR}(\lambda, \mu; \pi)$ denote the set of pairs $(S, T)$ of Littlewood-Richardson multitableaux of shape $(\lambda, \mu)$, same content and type $\pi$. This means that $S = (S_1, \ldots, S_r)$ is a Littlewood-Richardson multitableau of shape $\lambda$, $T = (T_1, \ldots, T_r)$ is a Littlewood-Richardson multitableau of shape $\mu$ and both $S_i$ and $T_i$ have the same content $\rho(i)$ for some partition $\rho(i)$ of $\pi_i$, for all $i \in [\,r\,]$. Let $c^{\lambda}_{(\rho(1),\ldots,\rho(r))}$ denote the number of Littlewood-Richardson multitableaux of shape $\lambda$ and content $(\rho(1), \ldots, \rho(r))$ and let $\mathsf{lr}(\lambda, \mu; \pi)$ denote the cardinality of $\mathsf{LR}(\lambda, \mu; \pi)$. Then

$$\mathsf{lr}(\lambda, \mu; \pi) = \sum_{\rho(1)\vdash\pi_1,\ldots,\rho(r)\vdash\pi_r} c^{\lambda}_{(\rho(1),\ldots,\rho(r))} c^{\mu}_{(\rho(1),\ldots,\rho(r))} .$$

Similar numbers have already proved to be useful in the study of minimal components, in the dominance order of partitions, of Kronecker products [14].

The number $\mathsf{lr}(\lambda, \mu; \pi)$ can be described as an inner product of characters. For this description we need the permutation character $\phi^{\pi} := \mathrm{Ind}_{\mathsf{S}_\pi}^{\mathsf{S}_m}(1_\pi)$, namely, the induced character from the trivial character of $\mathsf{S}_\pi = \mathsf{S}_{\pi_1} \times \cdots \times \mathsf{S}_{\pi_r}$. It follows from Frobenius reciprocity and the Littlewood-Richardson rule that (see also [6, 2.9.17])

**4.1 Lemma.** *Let $\lambda$, $\mu$, $\pi$ be as above. Then*

$$\mathsf{lr}(\lambda, \mu; \pi) = \langle \chi^{\lambda} \otimes \chi^{\mu}, \phi^{\pi} \rangle.$$

Since Young's rule and Lemma 4.1 imply that $\mathsf{lr}(\lambda, \mu; \nu) \geq \mathsf{k}(\lambda, \mu, \nu)$, then we have

**4.2 Corollary.** *Let $\lambda$, $\mu$, $\nu$ be partitions of $m$. If $\mathsf{lr}(\lambda, \mu; \nu) = 0$, then $\mathsf{k}(\lambda, \mu, \nu) = 0$.*

**4.3 Lemma.** *Let $\lambda$, $\mu$, $\nu$ be partitions of $m$ of lengths $p$, $q$, $r$, respectively. If $p = qr$, and $\mu_q < r\lambda_p$ or $\nu_r < q\lambda_p$, then $\mathsf{lr}(\lambda, \mu; \nu) = 0$.*

*Proof.* We assume that $\mathsf{lr}(\lambda, \mu; \nu) > 0$ and show that $\mu_q \geq r\lambda_p$ and $\nu_r \geq q\lambda_p$. Let $(S, T)$ be an element in $\mathsf{LR}(\lambda, \mu; \nu)$ having content $(\rho(1), \ldots, \rho(r))$. Since $T_i$ is contained in $\mu$, one has, by elementary properties of Littlewood-Richardson tableaux, that $\ell(\rho(i)) \leq \ell(\mu) = q$. For any $i$, let $n_i$ be the number of squares of $S_i$ that are in column $\lambda_p$ of $\lambda$, then $n_i \leq q$. We conclude that $p = n_1 + \cdots + n_r \leq rq = p$. Therefore $n_i = q = \ell(\rho(i))$ for all $i$. This forces that each $S_i$ contains a $j$ in the squares $(j + (i-1)q, 1), \ldots, (j + (i-1)q, \lambda_p)$ of $\lambda$, for all $j \in [\,q\,]$. So, $\rho(i)_j \geq \lambda_p$ for all $j$. In particular, for $i = r$, since $S_r$ has $\nu_r$ squares, one has $\nu_r \geq q\lambda_p$. Now, since $\ell(\mu) = q$, all entries of $T_i$ equal to $q$ must be in row $q$ of $\mu$. Then $\mu_q \geq \rho(1)_q + \cdots + \rho(r)_q \geq r\lambda_p$. The claim follows. $\square$

**4.4 Example.** To illustrate the idea in the proof of the previous lemma let $\lambda = (8, 5, 4, 3)$ and let $\mu$ and $\nu$ be partitions of 20 length 2. Let $(S, T)$ be any multitableau in $\mathsf{LR}(\lambda, \mu; \nu)$. Then, elementary properties of Littlewood-Richardson tableaux force $S$ to have the form

$$\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
\mathit{1} & \mathit{1} & \mathit{1} & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\mathit{2} & \mathit{2} & \mathit{2} & \cdot & \cdot \\
\cline{1-5}
\mathbf{1} & \mathbf{1} & \mathbf{1} & \cdot \\
\cline{1-4}
\mathbf{2} & \mathbf{2} & \mathbf{2} \\
\cline{1-3}
\end{array}$$

Here $S = (S_1, S_2)$, $S_1$ is formed by *italic* numerals and $S_2$ by **boldface** numerals. The dots indicate entries that can be either in $S_1$ or $S_2$. This partial information on $S$ forces $\mu_2 \geq 6$ and $\nu_2 \geq 6$.

**4.5 Corollary.** *Let $\lambda$, $\mu$, $\nu$ be partitions of $m$ of length $p$, $q$, $r$, respectively. If $p = qr$, and $\mu_q < r\lambda_p$ or $\nu_r < q\lambda_p$, then $\mathsf{k}(\lambda, \mu, \nu) = 0$.*

*Proof.* This follows from Lemma 4.3 and Corollary 4.2. $\square$

Corollary 4.5 and Theorem 3.1 imply the following

**4.6 Theorem.** *Let $\lambda$, $\mu$, $\nu$ be partitions of $m$ of length $p$, $q$, $r$, respectively. Let $t = \lambda_p$ and assume $p = qr$, then we have*

*(1) If $\mu_q < rt$ or $\nu_r < qt$, then $\mathsf{k}(\lambda, \mu, \nu) = 0$.*

*(2) If $\mu_q \geq rt$ and $\nu_r \geq qt$, let $\widetilde{\lambda} = \lambda - (t)^p$, $\widetilde{\mu} = \mu - (rt)^q$ and $\widetilde{\nu} = \nu - (qt)^r$. Then, $\mathsf{k}(\lambda, \mu, \nu) = \mathsf{k}(\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\nu})$.*

# 5 Applications

We conclude this paper with an application to the expansion of $\chi^\mu \otimes \chi^\nu$ when $\ell(\mu) = 2 = \ell(\nu)$. It is well known that any component of $\chi^\mu \otimes \chi^\nu$ corresponds to a partition of length at most $|\mu \cap \nu'| \leq 4$, see Satz 1 in [4], Theorem 1.6 in [5] or Theorem 2.1 in [9].

Even in this simple case a *nice* closed formula seems unlikely to exist. J. Remmel and T. Whitehead (Theorem 2.1 in [9]) gave a close, though intricate, formula for $\mathsf{k}(\lambda,\mu,\nu)$ valid for any $\lambda$ of length at most 4; M. Rosas (Theorem 1 in [10]) gave a formula of combinatorial nature for $\mathsf{k}(\lambda,\mu,\nu)$, which requires taking subtractions, also valid for any $\lambda$ of length at most 4; C. Ballantine and R. Orellana (Proposition 4.12 in [2]) gave a simpler formula for $\mathsf{k}(\lambda,\mu,\nu)$, at the cost of assuming an extra condition on $\lambda$.

Note that when $\ell(\lambda)=1$ the coefficient $\mathsf{k}(\lambda,\mu,\nu)$ is trivial to compute. For $\ell(\lambda)=2$ the Remmel-Whitehead formula for $\mathsf{k}(\lambda,\mu,\nu)$ reduces to a simpler one (Theorem 3.3 in [9]). This formula was recovered by Rosas in a different way (Corollary 1 in [10]). So, the nontrivial cases are those for which $\ell(\lambda)=3,4$. Corollary 5.1 deals with the case of length 4. On the one hand it gives a new vanishing condition. On the other hand, when this vanishing condition does not hold, it reduces the case of length 4 to the case of length 3. Thus, this reduction would help to simplify the proofs of the formulas given by Remmel-Whitehead and Rosas.

The following corollary is a particular case of Theorem 4.6.

**5.1 Corollary.** *Let $\lambda$, $\mu$, $\nu$ be a partitions of $m$ of length 4, 2, 2, respectively. Let $t=\lambda_4$, then we have*

*(1) If $\mu_2<2t$ or $\nu_2<2t$, then $\mathsf{k}(\lambda,\mu,\nu)=0$.*

*(2) If $\mu_2\geq 2t$ and $\nu_2\geq 2t$, let $\widetilde{\lambda}=(\lambda_1-t,\lambda_2-t,\lambda_3-t)$, $\widetilde{\mu}=(\mu_1-2t,\mu_2-2t)$ and $\widetilde{\nu}=(\nu_1-2t,\nu_2-2t)$. Then, $\mathsf{k}(\lambda,\mu,\nu)=\mathsf{k}(\widetilde{\lambda},\widetilde{\mu},\widetilde{\nu})$.*

Another observation of Remmel and Whitehead (Theorems 3.1 and 3.2 in [9]) is that their formula simplifies considerably in the case $\lambda_3=\lambda_4$. Corollary 5.1 explains this phenomenon since, in this case, the computation of $\mathsf{k}(\lambda,\mu,\nu)$ reduces to the computation of a Kronecker coefficient involving only three partitions of length at most 2, which have a simple nice formula (Theorem 3.3 in [9]). In fact, combining our result with this simple formula we obtain a new one. For completeness we record here the Remmel-Whitehead formula in the equivalent version of Rosas.

In the next theorems the notation $(y\geq x)$ means 1 if $y\geq x$ and 0 if $y\ngeq x$.

**5.2 Theorem.** *[9, Theorem 3.3] Let $\lambda$, $\mu$, $\nu$ be partitions of $m$ of length 2. Let $x=\max\left(0,\left\lceil\frac{\nu_2+\mu_2+\lambda_2-m}{2}\right\rceil\right)$ and $y=\left\lceil\frac{\nu_2+\mu_2-\lambda_2+1}{2}\right\rceil$. Assume $\nu_2\leq\mu_2\leq\lambda_2$. Then*

$$\mathsf{k}(\lambda,\mu,\nu)=(y-x)(y\geq x)\,.$$

From Corollary 5.1 and Theorem 5.2 we obtain

**5.3 Theorem.** *Let $\lambda$, $\mu$, $\nu$ be partitions of $m$ of length 4, 2, 2, respectively. Suppose that $\lambda_3=\lambda_4$ and that $2\lambda_3\leq\nu_2\leq\mu_2$. Let $x=\max\left(0,\left\lceil\frac{\nu_2+\mu_2+\lambda_2-\lambda_3-m}{2}\right\rceil\right)$, $y=\left\lceil\frac{\nu_2+\lambda_2-\mu_2-\lambda_3+1}{2}\right\rceil$ and $z=\left\lceil\frac{\nu_2+\mu_2-\lambda_2-3\lambda_3+1}{2}\right\rceil$. We have*

*(1) If $\lambda_2+\lambda_3\leq\mu_2$, then $\mathsf{k}(\lambda,\mu,\nu)=(y-x)(y\geq x)$.*

*(2) If $\lambda_2+\lambda_3>\mu_2$, then $\mathsf{k}(\lambda,\mu,\nu)=(z-x)(z\geq x)$.*

*Proof.* Let $\widetilde{\lambda} = (\lambda_1 - \lambda_3, \lambda_2 - \lambda_3)$, $\widetilde{\mu} = (\mu_1 - 2\lambda_3, \mu_2 - 2\lambda_3)$ and $\widetilde{\nu} = (\nu_1 - 2\lambda_3, \nu_2 - 2\lambda_3)$. These are partitions of $m - 4\lambda_3$. Then, by Corollary 5.1, $\mathsf{k}(\lambda, \mu, \nu) = \mathsf{k}(\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\nu})$. Since $\ell(\widetilde{\lambda}) = \ell(\widetilde{\mu}) = \ell(\widetilde{\nu}) = 2$, we can apply Theorem 5.2. Due to the symmetry of the Kronecker coefficients we are assuming $\nu_2 \leq \mu_2$. We have to consider three cases: (a) $\lambda_2 - \lambda_3 \leq \nu_2 - 2\lambda_3$, (b) $\nu_2 - 2\lambda_3 < \lambda_2 - \lambda_3 \leq \mu_2 - 2\lambda_3$ and (c) $\mu_2 - 2\lambda_3 < \lambda_2 - \lambda_3$. In the first two cases the Remmel-Whitehead formula yields the same formula for $\mathsf{k}(\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\nu})$. So, we have only two cases to consider: (1) $\lambda_2 + \lambda_3 \leq \mu_2$ and (2) $\mu_2 < \lambda_2 + \lambda_3$. In the first case Theorem 5.2 yields

$$\mathsf{k}(\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\nu}) = (y' - x')(y' \geq x')$$

where $x' = \max\left(0, \left\lceil \frac{\nu_2 - 2\lambda_3 + \lambda_2 - \lambda_3 + \mu_2 - 2\lambda_3 - (m - 4\lambda_3)}{2} \right\rceil\right)$ and $y' = \left\lceil \frac{\nu_2 - 2\lambda_3 + \lambda_2 - \lambda_3 - (\mu_2 - 2\lambda_3) + 1}{2} \right\rceil$. It is straightforward to check that $x' = x$ and $y' = y$, so the first claim follows.

The second case is similar. □

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