A note on $K^-_{\Delta+1}$-free precolouring with Δ colours

Tom Rackham

Mathematical Institute
University of Oxford
Oxford OX1 3LB, UK

rackham@maths.ox.ac.uk

Submitted: Jan 30, 2008; Accepted: Sep 13, 2009; Published: Sep 18, 2009
Mathematics Subject Classification: 05C15

Abstract

Let G be a simple graph of maximum degree $\Delta \geqslant 3$, not containing $K_{\Delta+1}$, and $\mathcal{L}$ a list assignment to $V(G)$ such that $|\mathcal{L}(v)| = \Delta$ for all $v \in V(G)$. Given a set $P \subset V(G)$ of pairwise distance at least d then Albertson, Kostochka and West (2004) and Axenovich (2003) have shown that every $\mathcal{L}$-precolouring of P extends to a $\mathcal{L}$-colouring of G provided $d \geqslant 8$.

Let $K^-_{\Delta+1}$ denote the graph $K_{\Delta+1}$ with one edge removed. In this paper, we consider the problem above and the effect on the pairwise distance required when we additionally forbid either $K^-_{\Delta+1}$ or K_Δ as a subgraph of G. We have the corollary that an extra assumption of 3-edge-connectivity in the above result is sufficient to reduce this distance from 8 to 4.

This bound is sharp with respect to both the connectivity and distance. In particular, this corrects the results of Voigt (2007, 2008) for which counterexamples are given.

1 Introduction

Let G be a simple graph of maximum degree $\Delta \geqslant 3$, not containing $K_{\Delta+1}$, then Brooks' theorem [3] states that G is Δ-(vertex-)colourable. Given two vertices x and y of such a graph, which are far apart in terms of a shortest path between them, it is natural to ask whether there exist *two* Δ-colourings, one with x and y coloured the same and another when they are coloured differently. This was answered affirmatively by Sajith and Saxena [6] for the case $\Delta = 3$, who showed that there exists some (large) sufficient distance between x and y. Rackham [5] showed, for any $\Delta \geqslant 3$, the question is affirmative provided x and y are distance at least 6 apart and that this is best possible in each case. (The *distance* between two vertices is the number of edges on a shortest path.)

More generally we have the following *distance-constraint precolouring* problem: given $P \subset V(G)$ of any size in a graph G of maximum degree $\Delta \geqslant 3$, does there exist some sufficient pairwise distance $d(P)$ between vertices of P such that every Δ-colouring of P extends to a Δ-colouring of G? The global bound was given by Albertson, Kostochka and West [1] and Axenovich [2] who showed that $d(P) \geqslant 8$ is sufficient in every case, and this is sharp provided $|P| \geqslant \Delta$. With a small number of precoloured vertices, $2 \leqslant |P| < \Delta$, then Rackham [5] proved that $d(P) \geqslant 6$ is sufficient, and this is sharp in each case. The graphs in Figure 1 provide a lower bound of 5 when $2 \leqslant |P| < \Delta$ and 7 when $|P| \geqslant \Delta$. For each graph, consider a precolouring of all the vertices of P (indicated with a dashed box) with the same colour; such precolourings cannot be extended to any proper Δ-colouring.

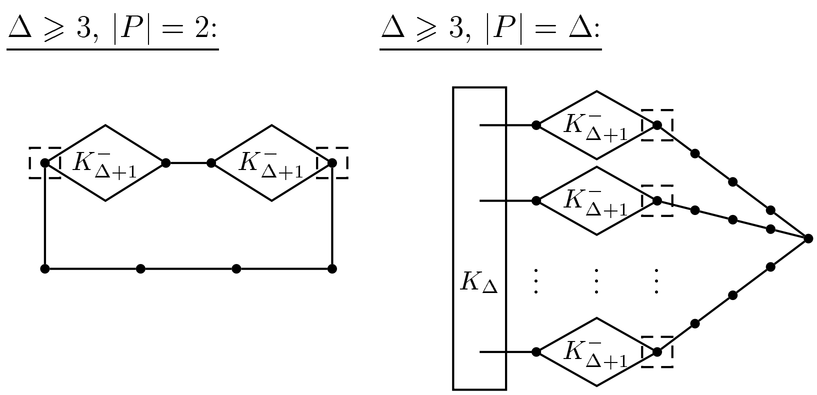

Figure 1: 2-connected graphs & no precolouring extension for distances 5 and 7

Brooks' theorem also has a natural strengthening to list-colourings due to Vizing [7]. Let G be a graph of maximum degree $\Delta \geqslant 3$, not containing $K_{\Delta+1}$, and let $\mathcal{L}$ be a list assignment to $V(G)$ such that $|\mathcal{L}(v)| = \Delta$ for all $v \in V(G)$. Vizing's result gives the existence of a proper $\mathcal{L}$-colouring of G. We can ask the same distance-constraint precolouring question in this list-colouring context. In fact, the result of Albertson, Kostochka and West [1] and Axenovich [2] holds. That is, given a set $P \subset V(G)$ such that $d(P) \geqslant 8$, every precolouring of P extends to a $\mathcal{L}$-colouring of G. List-colouring extension of a set P in a graph G is equivalent to the vertices of P being assigned lists of size 1 and the remainder assigned lists of size Δ. We use the list-colouring formulation of the distance constraint problem throughout this paper.

Voigt [8], [9] considered the question of the effect of an additional connectivity assumption on the distance required. However, the results of both papers are incorrect. The following claims were made:

- (Theorem 2 of [8]:) Let $G = (V, E)$ be a 2-connected graph with $k = \Delta(G) \geqslant 4$ which is not K_{k+1}, $W \subseteq V$ an independent subset of the vertex set, $d(W) \geqslant 4$, and $\mathcal{L}$ a list assignment with $|\mathcal{L}| = k$ for all $v \in V$. Then every precoloring of W extends to a proper $\mathcal{L}$-list coloring of V.

- (Theorem 2 of [9]:) Let $G = (V, E)$ be a 2-connected graph with $k = \Delta(G) = 3$ which is not K_4, $W \subseteq V$ an independent subset of the vertex set, $\mathcal{L}$ a list assignment with $|\mathcal{L}| = 3$ for all $v \in V$ and $d(W) \geqslant 6$. Then every precoloring of W extends to a proper $\mathcal{L}$-list coloring of V.

The graphs shown in Figure 1 are sufficient to provide counterexamples to both statements above. They show that the previously known sufficient distances of 6 and 8 cannot be improved by an additional assumption of 2-connectivity. The error in both proofs is due to a mistaken assumption of connectedness early in the proof.

In Section 2 of this paper we address this question of increased connectivity on the pairwise distance required. Our main result is that 3-edge-connectivity is the correct condition for an improvement in the pairwise distance required:

Theorem 1. *Let G be a 3-edge-connected graph with $\Delta := \Delta(G) \geqslant 3$, and let $P \subset V(G)$. Let $\mathcal{L}$ be a list assignment to $V(G)$ such that $|\mathcal{L}(v)| = \Delta$ for all $v \in G$. If $d(P) \geqslant 4$ then any colouring of P extends to a $\mathcal{L}$-colouring of G.*

This result is sharp, for each $\Delta \geqslant 3$, with respect to both connectivity (as mentioned above, see Figure 1) and distance (see Figure 2). Note that this is a global bound for each $\Delta \geqslant 3$ and any number of precoloured vertices $|P|$.

Let $K^-_{\Delta+1}$ denote the graph $K_{\Delta+1}$ with one edge removed. Our proof method of Theorem 1 does not look directly at the problem with the additional assumption of 3-edge-connectivity, but rather we exclude $K^-_{\Delta+1}$ as a subgraph. (A 3-edge-connected graph of maximum degree Δ cannot contain $K^-_{\Delta+1}$ as a proper subgraph.) Since Brooks' theorem itself requires the exclusion of $K_{\Delta+1}$ components, this would seem like a natural approach to take. In Section 3 we also consider the effect of excluding K_Δ only. In this situation, the sufficient distance required depends on $|P|$ and Δ but there is mostly an improvement from distance 4. (See Theorem 5 for the details.)

Key lemma

Our main tool is an extension of both Brooks' theorem and Vizing's theorem given by Kostochka, Stiebitz and Wirth [4], and the general approach is that of Axenovich [2] . A *block* of a graph is a maximal 2-connected subgraph. A *Gallai tree* is a graph all of whose blocks are either complete graphs, odd cycles or single edges. A *leaf block* of a Gallai tree is a block containing at most one cut-vertex. Then:

Theorem 2 (Kostochka, Stiebitz and Wirth [4])**.** *Let G be a connected and let $\mathcal{L}$ be a list-assignment of $V(G)$ such that $|\mathcal{L}(v)| \geqslant d(v)$ for each $v \in V(G)$. If G is not $\mathcal{L}$-colourable then it is a Gallai tree and $|\mathcal{L}(v)| = d(v)$ for each $v \in V(G)$.*

This gives the following useful corollary:

Lemma 3. *Let G be a connected graph with $\Delta(G) := \Delta \geqslant 3$, $P \subset V(G)$, $\mathcal{L}$ be a list assignment to $V(G)$ such that*

- $|\mathcal{L}(v)| = d(v) = \Delta$ *for all* $v \in V(G)\backslash P$
- $|\mathcal{L}(v)| = 1$ *for all* $v \in P$

and G *cannot be* $\mathcal{L}$*-list coloured. Suppose* $G\backslash P$ *(the graph induced on vertex set* $V(G)\backslash P$*) is connected. Then* $G\backslash P$ *is a Gallai tree* T*.*

Moreover, if $d(P) \geqslant 3$ *then all vertices of* T *have degree* $\Delta - 1$ *(if adjacent to some* v *in* P*) or* Δ *(if not). If* $|P| = 2$*, then every vertex of* T *has degree at least* $\Delta - 2$*.*

Proof. Consider the graph $G\backslash P$ with list assignment $\mathcal{L}'$ defined by: $\mathcal{L}'(v) := \mathcal{L}(v) - c(N_P(v))$, where $c(N_P(v))$ denotes the set of colours of the neighbours of v restricted to the set P. (This is the empty set if v is not adjacent to any vertex of P.) The graph $G\backslash P$ is not $\mathcal{L}'$-colourable since G is not $\mathcal{L}'$-colourable; so by Theorem 2 the graph $G\backslash P$ is a Gallai tree.

The condition $d(P) \geqslant 3$ implies that each vertex of T is adjacent in G to at most one vertex of P, and thus the degree in T of each vertex is either $\Delta - 1$ (if it had been adjacent to some vertex in P) or Δ (if not). If $|P| = 2$, then each vertex of T is adjacent in G to at most two vertices of P, and so the degree in T of each vertex is at least $\Delta - 2$. □

2 Distance 4 extension for $K^-_{\Delta+1}$-free graphs

In this section we consider the distance-constraint precolouring problem for graphs not containing $K^-_{\Delta+1} := K_{\Delta+1} - e$ as a subgraph. We find the following:

Theorem 4. *Let* G *be a connected graph with* $\Delta := \Delta(G) \geqslant 3$ *containing no* $K^-_{\Delta+1}$ *subgraph, and let* $P \subset V(G)$*. Let* $\mathcal{L}$ *be a list assignment to* $V(G)$ *such that* $|\mathcal{L}(v)| = \Delta$ *for all* $v \in G$*. If* $d(P) \geqslant 4$ *then any colouring of* P *extends to an* $\mathcal{L}$*-colouring of* G*.*

Our main theorem (Theorem 1) now follows as a corollary:

Proof of Theorem 1. It is sufficient to observe that a 3-edge-connected graph of maximum degree Δ, which is neither $K^-_{\Delta+1}$ nor K_Δ, cannot contain $K^-_{\Delta+1}$ as a subgraph. This holds since there are at most two edges incident with, but not contained in, a $K^-_{\Delta+1}$-subgraph. □

Theorem 1 and Theorem 4 are both sharp with respect to connectivity and distance, as demonstrated by the graphs in Figures 1 and 2. For each graph, precolour the vertices of P with colour 1 and give the list $\{1, 2, \ldots, \Delta\}$ to all other vertices; such precolourings cannot be extended from the lists.

Proof of Theorem 4. Let G be a counterexample to Theorem 4 with the smallest number of vertices. Let $P \subseteq \mathrm{V(G)}$ with $d(P) \geqslant 4$ and consider a precolouring of P which cannot be extended. By minimality, $G\backslash v$ is not a counterexample for any $v \in G\backslash P$ and hence if there exists a $v \in V\backslash P$ with $d(v) < \Delta$ then we could extend the precolouring first to $G\backslash v$ and then to G, a contradiction. So $d(v) = \Delta$ for all $v \in V\backslash P$ and G satisfies the conditions of Lemma 3.

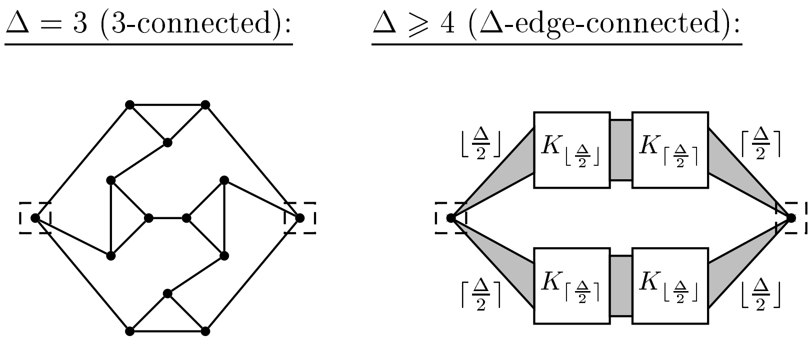

Figure 2: $K^-_{\Delta+1}$-free graphs of high edge-connectivity with no precolouring extension at distance 3

Thus $G \backslash P$ is a connected Gallai tree T with the specified restriction on the degree sequence. We now split the argument based on the nature of a leaf block B of T. (We choose a leaf block B arbitrarily.)

- Suppose B is a complete graph of order Δ. Note that B contains at most 1 cutvertex of T. There is at most one vertex $v \in P$ which is adjacent to a given vertex of B, since $d(P) \geqslant 4$. Vertex v cannot be adjacent to all non-cutvertices of B or else we would have a copy of $K^-_{\Delta+1}$. Therefore some $w \in B$ has degree at most $\Delta - 1$ in G, contradicting $d(w) = \Delta$ for all $w \in G \backslash P$.

- Suppose that B is either:

  - a single edge; or,
  - a complete graph of order at most $\Delta - 1$; or,
  - an odd cycle and that $\Delta \geqslant 4$.

  These conditions each imply that there is a non cut-vertex v in B with degree at most $\Delta - 2$ in $G \backslash P$, which contradicts Lemma 3.

- The leaves only the case when $\Delta = 3$ and B is an odd cycle of length at least 5. Since $d(P) \geqslant 4$, two different vertices of P cannot be incident to adjacent vertices of B and since we have at least 4 consecutive non-cutvertices of B, at least one of these w is not adjacent to some $v \in P$; so w has degree 2 in G, a contradiction.

These contradictions complete the proof of Theorem 4. $\square$

3 Distance 3 extension for K_Δ-free graphs

We now consider the exclusion of K_Δ, to find that we may often further improve the distance required:

Theorem 5. *Let G be a connected graph with $\Delta := \Delta(G) \geqslant 3$ containing no K_Δ subgraph, and let $P \subset V(G)$. Let $\mathcal{L}$ be a list assignment to $V(G)$ such that $|\mathcal{L}(v)| = \Delta$ for all $v \in G$. If either:*

(i) $\Delta \geqslant 4$, $|P| \geqslant 3$ and $d(P) \geqslant 3$; or,

(ii) $\Delta \geqslant 5$, $|P| = 2$ and $d(P) \geqslant 2$; or,

(iii) $\Delta = 3$, $|P| \geqslant 3$ and $d(P) \geqslant 4$; or,

(iv) $\Delta = 3$ or $\Delta = 4$, $|P| = 2$ and $d(P) \geqslant 3$

then any $\mathcal{L}$-colouring of P extends to a $\mathcal{L}$-colouring of G. Moreover, the inequalities concerning $d(P)$ are best possible.

		Δ		
		3	4	$\geqslant 5$
$\lvert P\rvert$	2	$3^{(\mathrm{iv})}$	$3^{(\mathrm{iv})}$	$2^{(\mathrm{ii})}$
	$\geqslant 3$	$4^{(\mathrm{iii})}$	$3^{(\mathrm{i})}$	$3^{(\mathrm{i})}$

Table 1: Summary of distances required for K_Δ-free graphs

Proof of Theorem 5. As with the proof of Theorem 4, let G be a counterexample to Theorem 5 with the smallest number of vertices. It follows that G satisfies the conditions of Lemma 3. Thus $G \backslash P$ is a Gallai tree T and is connected by minimality. For cases (i), (iii) and (iv), P the condition $d(P) \geqslant 3$ implies that each vertex of T has degree $\Delta - 1$ or Δ. For case (ii), the condition that $|P| = 2$ implies that each vertex of T has degree at least $\Delta - 2$.

Let B be an arbitrary leaf block of T. Since G is K_Δ-free, B is either an odd cycle of a complete graph of order at most $\Delta - 1$. We now cover each of the four cases separately:

(i) G is K_Δ-free and so B is either an odd cycle, a single edge, or a complete graph of order less than Δ. Each of these gives a non cut-vertex of T with degree at most $\Delta - 2$, a contradiction.

We find a lower bound on the distance required by considering the following graph:

$V(G) = \{x, y, z, a_1, \ldots, a_{\Delta-1}\}$
$E(G)$ consists of a complete graph on the a_i plus edges $\{xa_i : i = 0 \text{ or } 1 \bmod 3\}$, $\{ya_i : i = 0 \text{ or } 2 \bmod 3\}$, $\{za_i : i = 1 \text{ or } 2 \bmod 3\}$.
$P = \{x, y, z\}$

Then a precolouring giving x, y and z different colours cannot be extended if the list of every vertex a_i consists of the 2 colours of its neighbours in P, plus $\Delta - 2$ additional (fixed) colours.

(ii) The leaf block B has a non-cutvertex of degree at least $\Delta - 2$ and, since $\Delta \geqslant 5$, B cannot be an odd cycle. Thus B is a complete graph and must be of order $\Delta - 1$ since G is K_Δ-free.

If T consists of a unique leaf block B, then every vertex of B is adjacent to both vertices of P since the degree of any vertex is at most $\Delta - 2$ in T but equal to Δ in G. This gives a copy of K_Δ and thus rules out the possibility of T consisting of a single block.

If T contains two or more leaf blocks then the number of vertices of degree $\Delta - 2$ in T (i.e. the non-cutvertices of at least two leaf blocks of T) is at least $2(\Delta - 2)$ which is strictly greater than Δ for $\Delta \geqslant 5$. Since all such vertices must be adjacent to both vertices of P, which have degree at most Δ, we have a contradiction.

Conversely, consider the graph K_2. The endpoints of this edge are at distance 1 and may not be simultaneously precoloured with the same colour. Thus, we trivially see that distance 2 may not be improved.

(iii) There is no improvement on the distance required of 4 so this case follows from Theorem 4, because K_Δ is a subgraph of $K^-_{\Delta+1}$. The graph shown in Figure 3 with the given list-assignment establishes that this distance cannot be improved.

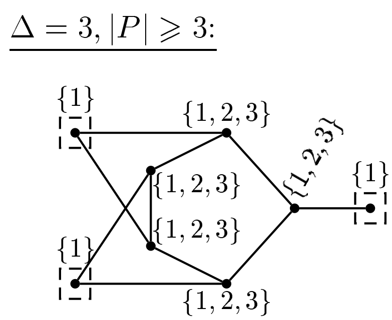

Figure 3: Graph providing lower bound for Theorem 5 part (iii)

(iv) If $\Delta = 3$ then the degree in T of a non-cutvertex of B equals 2. Since G is K_3-free, B must be an odd cycle of length at least 5. If T consists of a unique block B then every vertex of B is adjacent in G to one of the two vertices of P. Since the cycle is odd, this gives a K_3 subgraph in G and a contradiction. Otherwise, T has at least 2 leaf blocks each with at least 4 non cut-vertices, which must all be adjacent to a vertex of P. However, there are at most 6 edges incident with P and we have a contradiction.

If $\Delta = 4$ then B must have a non-cutvertex of degree in T equal to 3. However, since G is K_4-free, we have the final contradiction required.

Graphs and list-assignments establishing a lower bound of distance 3 for part (iv) are shown in Figure 4. This completes the proof of Theorem 5.

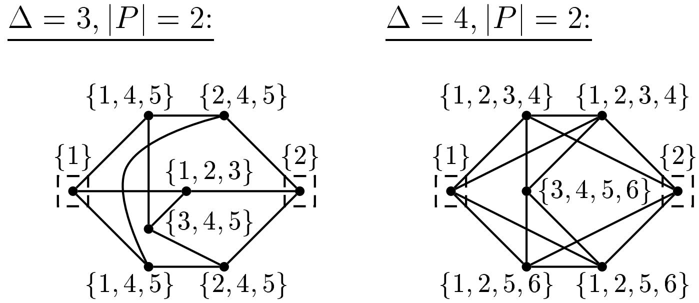

Figure 4: Graphs providing lower bounds for Theorem 5 part (iv)

□

Acknowledgements: Thanks to the referee for finding the second graph in Figure 4 and for other helpful comments.

References

[1] M. O. Albertson, A. V. Kostochka, and D. B. West, *Precolouring extensions of Brooks' theorem*, SIAM J. Discrete Math. **18** (2004), 542–553.

[2] M. Axenovich, *A note on graph colouring extensions and list-colourings*, Electronic J. Comb. **10** (2003), #N1.

[3] R. L. Brooks, *On colouring the nodes of a network*, Proc. Cambridge Philos. Soc. **37** (1941), 194–197.

[4] A. V. Kostochka, M. Stiebitz, and B. Wirth, *The colour theorems of Brooks and Gallai extended*, Discrete Math. **162** (1996), 299–303.

[5] T. Rackham, *A precolouring extension of Brooks' theorem*, submitted.

[6] G. Sajith and S. Saxena, *Local nature of Brooks' colouring for degree 3 graphs*, Graphs Combin. **19** (2003), no. 4, 551–565.

[7] V. G. Vizing, *Coloring the vertices of a graph in prescribed colors*, Diskret. Analiz (1976), no. 29 Metody Diskret. Anal. v Teorii Kodov i Shem, 3–10, 101.

[8] M. Voigt, *Precoloring extension for 2-connected graphs*, SIAM J. Discrete Math. **21** (2007), no. 1, 258–263 (electronic).

[9] ______, *Precoloring extension for 2-connected graphs with maximum degree three*, Discrete Math. (2008), doi:10.1016/j.disc.2008.05.024.