Skew Spectra of Oriented Graphs

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Abstract

An oriented graph G^σ is a simple undirected graph G with an orientation σ, which assigns to each edge a direction so that G^σ becomes a directed graph. G is called the underlying graph of G^σ, and we denote by $Sp(G)$ the adjacency spectrum of G. Skew-adjacency matrix $S(G^\sigma)$ of G^σ is introduced, and its spectrum $Sp_S(G^\sigma)$ is called the skew-spectrum of G^σ. The relationship between $Sp_S(G^\sigma)$ and $Sp(G)$ is studied. In particular, we prove that (i) $Sp_S(G^\sigma) = \mathbf{i}Sp(G)$ for **some** orientation σ if and only if G is bipartite, (ii) $Sp_S(G^\sigma) = \mathbf{i}Sp(G)$ for **any** orientation σ if and only if G is a forest, where $\mathbf{i} = \sqrt{-1}$.

1 Introduction

Let G be a simple graph. With respect to a labeling, the *adjacency matrix* $A(G)$ is the symmetric matrix $[a_{ij}]$ where $a_{ij} = a_{ji} = 1$ if $\{i, j\}$ is an edge of G, otherwise $a_{ij} = a_{ji} = 0$. The *spectrum* $Sp(G)$ of G is defined as the spectrum of $A(G)$. Note that the definition is well defined because symmetric matrices with respect to different labelings are permutationally similar, and so have same spectra. Also note that $Sp(G)$ consists of only real eigenvalues because $A(G)$ is real symmetric.

Example 1.1. Consider the path graph P_4 on 4 vertices. With respect to two different labelings, $A(P_4)$ takes the form

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

And the spectrum $Sp(P_4)$ is $\{\pm\frac{\sqrt{5}+1}{2}, \pm\frac{\sqrt{5}-1}{2}\}$.

Example 1.2. Consider the star graph ST_5 on 5 vertices. With respect to two different labelings, $A(ST_5)$ takes the form

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

And the spectrum $Sp(ST_5)$ is $\{-2, 0^{(3)}, 2\}$.

Example 1.3. Consider the cycle graph C_4 on 4 vertices. With respect to two different labelings, $A(C_4)$ takes the form

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

And the spectrum $Sp(C_4)$ is $\{-2, 0^{(2)}, 2\}$.

Let G^σ be a simple graph with an orientation σ, which assigns to each edge a direction so that G^σ becomes a directed graph. With respect to a labeling, the *skew-adjacency matrix* $S(G^\sigma)$ is the real skew symmetric matrix $[s_{ij}]$ where $s_{ij} = 1$ and $s_{ji} = -1$ if $i \to j$ is an arc of G^σ, otherwise $s_{ij} = s_{ji} = 0$. The *skew spectrum* $Sp_S(G^\sigma)$ of G^σ is defined as the spectrum of $S(G^\sigma)$. Note that the definition is well defined because real skew symmetric matrices with respect to different labelings are permutationally similar, and so have same spectra. Also note that $Sp_S(G^\sigma)$ consists of only purely imaginary eigenvalues because $S(G^\sigma)$ is real skew symmetric.

Example 1.4. Consider the directed path graph P_4^σ on 4 vertices. With respect to two different labelings, $S(P_4^\sigma)$ takes the form

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

And the skew spectrum $Sp_S(P_4^\sigma)$ is $\{\pm\frac{\sqrt{5}+1}{2}\mathbf{i}, \pm\frac{\sqrt{5}-1}{2}\mathbf{i}\}$.

Example 1.5. Consider the oriented star graph ST_5^σ on 5 vertices with the center as a sink. With respect to two different labelings, $S(ST_5^\sigma)$ takes the form

$$\begin{bmatrix} 0 & -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

And the skew spectrum $Sp_S(ST_5^\sigma)$ is $\{-2\mathbf{i}, 0^{(3)}, 2\mathbf{i}\}$.

Example 1.6. Consider two different orientations on the cycle graph C_4 (with the same labeling) such that their skew adjacency matrices are:

$$S(C_4^{\sigma_1}) = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}, \quad S(C_4^{\sigma_2}) = \begin{bmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{bmatrix}$$

respectively. And the skew spectra are

$$Sp_S(C_4^{\sigma_1}) = \{-2\mathbf{i}, 0^{(2)}, 2\mathbf{i}\}, \qquad Sp_S(C_4^{\sigma_2}) = \{-\sqrt{2}\mathbf{i}^{(2)}, \sqrt{2}\mathbf{i}^{(2)}\}$$

respectively.

Examples 1.1, 1.2, 1.4, and 1.5 suggest that $Sp_S(G^\sigma) = \mathbf{i}Sp(G)$. Indeed, it is proved in [1] that $Sp_S(T^\sigma) = \mathbf{i}Sp(T)$ for any tree T and any orientation σ. However Examples 1.3 and 1.6 show that it is not true in general because $Sp_S(C_4^{\sigma_1}) \neq Sp_S(C_4^{\sigma_2}) \neq \mathbf{i}Sp(C_4)$, even though $Sp_S(C_4^{\sigma_1}) = \mathbf{i}Sp(C_4)$. The goal of this short note is to show that trees are the only connected graphs with such property.

2 Main Results

Throughout this section, notation and terminology are as in [3]. First we need a lemma which is an extension of Theorem 7.3.7 in [3].

Lemma 2.1. Let $A = \begin{bmatrix} 0 & X \\ X^T & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & X \\ -X^T & 0 \end{bmatrix}$ be two real matrices. Then $Sp(B) = \mathbf{i}Sp(A)$.

Proof. W.L.O.G. let X be $m \times n$ ($m \leqslant n$) with the singular value decomposition $X = P\Sigma Q^T$ where P and Q are orthogonal matrices, and Σ is diagonal. Then

$$A = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} 0 & \Sigma \\ \Sigma^T & 0 \end{bmatrix} \begin{bmatrix} P^T & 0 \\ 0 & Q^T \end{bmatrix}$$

and

$$B = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} 0 & \Sigma \\ -\Sigma^T & 0 \end{bmatrix} \begin{bmatrix} P^T & 0 \\ 0 & Q^T \end{bmatrix}.$$

Write $\Sigma = Diag(a_1, a_2, \ldots, a_m)$, and so $Sp(A) = \{\pm a_1, \ldots, \pm a_m, 0^{(n-m)}\}$, $Sp(B) = \{\pm a_1\mathbf{i}, \ldots, \pm a_m\mathbf{i}, 0^{(n-m)}\}$. Consequently, $Sp(B) = \mathbf{i}Sp(A)$. ■

Theorem 2.2. G is a bipartite graph if and only if there is an orientation σ such that $Sp_S(G^\sigma) = \mathbf{i}Sp(G)$.

Proof. (Necessity) If G is bipartite, then there is a labeling such that the adjacency matrix of G is of the form

$$A(G) = \begin{bmatrix} 0 & X \\ X^T & 0 \end{bmatrix}.$$

Let σ be the orientation such that the skew-adjacency matrix of G^σ is of the form

$$S(G^\sigma) = \begin{bmatrix} 0 & X \\ -X^T & 0 \end{bmatrix}.$$

By Lemma 2.1, $Sp_S(G^\sigma) = \mathbf{i}Sp(G)$.

(Sufficiency) Suppose that $Sp_S(G^\sigma) = \mathbf{i}Sp(G)$ for some orientation σ. Since $S(G^\sigma)$ is a real skew symmetric matrix, $Sp_S(G^\sigma)$ has only pure imaginary eigenvalues and so is symmetric about the real axis. Then $Sp(G) = -\mathbf{i}Sp_S(G^\sigma)$ is symmetric about the imaginary axis. Hence G is bipartite, see Theorem 3.11 in [2]. ■

Let $|X|$ denote the matrix whose entries are the absolute values of the corresponding entries in X. For real matrices X and Y, $X \leqslant Y$ means that $Y - X$ has nonnegative entries. $\rho(X)$ denotes the spectral radius of a square matrix X. The next lemma is a special case of Theorem 8.4.5 in [3]. We provide here a shorter proof.

Lemma 2.3. Let A be an irreducible nonnegative matrix and B be a real positive semi-definite matrix such that $|B| \leqslant A$ (entry-wise) and $\rho(A) = \rho(B)$. Then $A = DBD$ for some real matrix D such that $|D| = I$, the identity matrix.

Proof. Since B is real positive semi-definite, there exists a real vector x such that $Bx = \rho(B)x$. Write $x = D|x|$ for some real matrix D such that $|D| = I$. Moreover, $DBD \leqslant |B| \leqslant A$ and $\rho(DBD) = \rho(B)$. Since A is irreducible nonnegative, so is A^T. By Perron-Frobenius theory [3], there is a positive vector y such that $A^Ty = \rho(A^T)y$, and so $y^TA = \rho(A^T)y^T = \rho(A)y$. Now we have $y^T(A - DBD)|x| = y^TA|x| - y^TDBD|x| = \rho(A)y^T|x| - y^TDBx = \rho(A)y^T|x| - y^TD\rho(B)x = \rho(A)y^T|x| - y^T\rho(B)|x| = 0$ because $\rho(A) = \rho(B)$. Consequently, $A|x| = DBD|x|$ because $A - DBD \geqslant 0$ and $|x| \geqslant 0$. It follows that $A|x| = DBD|x| = \rho(B)|x| = \rho(A)|x|$, which means that $|x|$ is a multiple of the Perron vector of A. In particular, $|x| > 0$. Finally we have $A = DBD$ because of $A|x| = DBD|x|$ and $A \geqslant DBD$. ■

Theorem 2.4. Let $X = \begin{bmatrix} C & * \\ * & * \end{bmatrix}$ be a (0,1)-matrix where C is a $k \times k$ ($k \geqslant 2$) circulant matrix with the first row as $[1, 0, \ldots, 0, 1]$. Let Y be obtained from X by changing the (1,1) entry to -1. If $X^T X$ is irreducible then $\rho(X^T X) > \rho(Y^T Y)$.

Proof. Note that $|Y^T Y| \leqslant X^T X$ (entry-wise), and so $\rho(Y^T Y) \leqslant \rho(X^T X)$ by Perron-Frobenius theory [3]. Now suppose that $\rho(X^T X) = \rho(Y^T Y)$. Since $X^T X$ is irreducible, by Lemma 2.3, there exists a signature matrix $D = Diag(d_1, d_2, \ldots, d_n)$ such that $X^T X = DY^T Y D$. Therefore $[X^T X]_{ij} = d_i d_j [Y^T Y]_{ij}$ for all i, j. Note that the first k columns of X are $\begin{bmatrix} 1 & 0 & \cdots & 1 \\ 1 & 1 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ a_1 & a_2 & \cdots & a_k \end{bmatrix}$ and the first k columns of Y are $\begin{bmatrix} -1 & 0 & \cdots & 1 \\ 1 & 1 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ a_1 & a_2 & \cdots & a_k \end{bmatrix}$. Now, for $i = 1, \ldots, k-1$, $[X^T X]_{i,i+1} = 1 + a_i^T a_{i+1}$ and $[Y^T Y]_{i,i+1} = 1 + a_i^T a_{i+1}$. Using $d_i d_j [Y^T Y]_{ij} = [X^T X]_{ij}$, we have $d_i d_{i+1} = 1$ for $i = 1, \ldots, k-1$. Hence $d_1 d_k = 1$. On the other hand, $-1 + a_1^T a_k = d_1 d_k [Y^T Y]_{1k} = [X^T X]_{1k} = 1 + a_1^T a_k$, which is impossible. ■

Theorem 2.5 Let G be a connected graph. Then G is a tree if and only if $Sp_S(G^\sigma) = \mathbf{i}Sp(G)$ for any orientation σ.

Proof. (Necessity) See the proof of Theorem 3.3 in [1].

(Sufficiency) Suppose $Sp_S(G^\sigma) = \mathbf{i}Sp(G)$ for any orientation σ. By Theorem 2.2, G is a bipartite graph. And so there is a labeling of G such that

$$A(G) = \begin{bmatrix} 0 & X \\ X^T & 0 \end{bmatrix}$$

where X is an $m \times n$ (0,1)- matrix with $m \leqslant n$. Since G is connected, $X^T X$ is indeed a positive matrix and so irreducible. Now assume that G is NOT a tree. Then G has at least an even cycle because G is bipartite. W.L.O.G. X has the form $\begin{bmatrix} C & * \\ * & * \end{bmatrix}$ where C is a $k \times k$ ($k \geqslant 2$) circulant matrix with the first row as $[1, 0, \ldots, 0, 1]$. Let Y be obtained from X by changing the (1,1) entry to -1. Consider the orientation σ of G such that

$$S(G^\sigma) = \begin{bmatrix} 0 & Y \\ -Y^T & 0 \end{bmatrix}.$$

By hypothesis, $Sp(G^\sigma) = \mathbf{i}Sp(G)$ and hence X and Y have the same singular values. It follows that $\rho(X^T X) = \rho(Y^T Y)$, which contradicts Theorem 2.4. ■

Corollary 2.6 G is a forest if and only if $Sp_S(G^\sigma) = \mathbf{i}Sp(G)$ for any orientation σ.

Proof. (Necessity) Let $G = G_1 \cup \cdots \cup G_r$ where G_j's are trees. Then $G^\sigma = G_1^{\sigma_1} \cup \cdots \cup G_r^{\sigma_r}$. By Theorem 2.5, $Sp_S(G_j^{\sigma_j}) = \mathbf{i}Sp(G_j)$ for all $j = 1, 2, \ldots, r$. Hence $Sp_S(G^\sigma) = Sp_S(G_1^{\sigma_1}) \cup \cdots \cup Sp_S(G_j^{\sigma_j}) = \mathbf{i}Sp(G_1) \cup \cdots \cup \mathbf{i}Sp(G_r) = \mathbf{i}Sp(G_1 \cup \cdots \cup G_r) = \mathbf{i}Sp(G)$.

(Sufficiency) Suppose that G is NOT a forest. Then $G = G_1 \cup \cdots \cup G_r$ where G_1, $\ldots$, G_t are connected, but not trees, and $G_{t+1}, \ldots, G_r$ are trees. By Theorem 2.2, G is a bipartite graph. And so there is a labeling of G such that

$$A(G) = \begin{bmatrix} 0 & X \\ X^T & 0 \end{bmatrix}$$

where $X = X_1 \oplus \cdots \oplus X_r$ and the $(1,1)$-entry of each X_j is 1. Let Y_j be obtained from X_j by changing the (1,1) entry to -1. Consider an orientation σ of G such that

$$S(G^\sigma) = \begin{bmatrix} 0 & Y \\ -Y^T & 0 \end{bmatrix}.$$

where $Y = Y_1 \oplus \cdots \oplus Y_r$. By Lemma 2.1, $Sp_S(G^\sigma) = \mathbf{i}Sp(G)$ implies that the singular values of X coincide with the singular values of Y. Since $G_{t+1}, \ldots, G_r$ are trees, the singular values of X_j coincide with the singular values Y_j for $j = t+1, \ldots, r$. Hence the singular values of $X_1 \oplus \cdots \oplus X_t$ coincide with the singular values of $Y_1 \oplus \cdots \oplus Y_t$. Since $G_1, \ldots, G_t$ are not trees, we have $\rho(X_j^T X_j) > \rho(Y_j^T Y_j)$ for $j = 1, \ldots, t$. Consequently,

$$\max_{1 \leqslant j \leqslant n} \rho(X_j^T X_j) = \max_{1 \leqslant j \leqslant n} \rho(Y_j^T Y_j) = \rho(Y_{j_0}^T Y_{j_0}) < \rho(X_{j_0}^T X_{j_0}) \leqslant \max_{1 \leqslant j \leqslant n} \rho(X_j^T X_j),$$

a contradiction. ■

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References

[1] C. Adiga, R. Balakrishnan and Wasin So, The Skew Energy of a Digraph, *Preprint*, 2009.

[2] D. Cvetkovic, M. Doob and H. Sachs, *Spectra of Graphs Theory and Application, Third Ed.*, Johann Ambosius Barth, Heidelberg, Leipzig, 1995.

[3] R. Horn and C. Johnson, *Matrix Analysis*, Cambridge University Press, 1987.