Degree powers in graphs with a forbidden even cycle

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Abstract

Let C_l denote the cycle of length l. For $p \geqslant 2$ and integer $k \geqslant 1$, we prove that the function

$$\phi(k,p,n) = \max\left\{ \sum_{u \in V(G)} d^p(u) : G \text{ is a graph of order } n \text{ containing no } C_{2k+2} \right\}$$

satisfies $\phi(k,p,n) = kn^p(1+o(1))$. This settles a conjecture of Caro and Yuster. Our proof is based on a new sufficient condition for long paths.

1 Introduction

Our notation and terminology follow [1]; in particular, C_l denotes the cycle of length l.

For $p \geqslant 2$ and integer $k \geqslant 1$, Caro and Yuster [3], among other things, studied the function

$$\phi(k,p,n) = \max\left\{ \sum_{u \in V(G)} d_G^p(u) : G \text{ is a graph of order } n \text{ without a } C_{2k+2} \right\}$$

and conjectured that

$$\phi(k,p,n) = kn^p(1+o(1)). \tag{1}$$

The graph $K_k + \overline{K}_{n-k}$, i.e., the join of K_k and $\overline{K}_{n-k}$, gives $\phi(k,p,n) > k(n-1)^p$, so to prove (1), a matching upper bound is necessary. We give such a bound in Corollary 3 below. Our main tool, stated in Lemma 1, is a new sufficient condition for long paths. This result has other applications as well, for instance, the following spectral bound, proved in [5]:

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Let G be a graph of order n and μ be the largest eigenvalue of its adjacency matrix. If G does not contain C_{2k+2}, then

$$\mu^2 - k\mu \leqslant k(n-1).$$

2 Main results

We write $|X|$ for the cardinality of a finite set X. Let G be a graph, and X and Y be disjoint sets of vertices of G. We write:

- $V(G)$ for the vertex set of G and $|G|$ for $|V(G)|$;
- $e_G(X)$ for the number of edges induced by X;
- $e_G(X, Y)$ for the number of edges joining vertices in X to vertices in Y;
- $G - u$ for the graph obtained by removing the vertex $u \in V(G)$;
- $\Gamma_G(u)$ for the set of neighbors of the vertex u and $d_G(u)$ for $|\Gamma_G(u)|$.

The main result of this note is the following lemma.

Lemma 1 *Suppose that $k \geqslant 1$ and let the vertices of a graph G be partitioned into two sets A and B.*

(1) If

$$2e_G(A) + e_G(A, B) > (2k-2)|A| + k|B|, \tag{2}$$

then there exists a path of order $2k$ or $2k+1$ with both ends in A.

(2) If

$$2e_G(A) + e_G(A, B) > (2k-1)|A| + k|B|, \tag{3}$$

then there exists a path of order $2k+1$ with both ends in A.

Note that if we choose the set B to be empty, Lemma 1 amounts to a classical result of Erdős and Gallai:

If a graph of order n has more than $kn/2$ edges, then it contains a path of order $k+2$.

We postpone the proof of Lemma 1 to Section 3 and turn to two consequences.

Theorem 2 *Let G be a graph with n vertices and m edges. If G does not contain a C_{2k+2}, then*

$$\sum_{u \in V(G)} d_G^2(u) \leqslant 2km + k(n-1)n.$$

Proof Let u be any vertex of G. Partition the vertices of the graph $G - u$ into the sets $A = \Gamma_G(u)$ and $B = V(G) \setminus (\Gamma_G(u) \cup \{u\})$. Since G contains no C_{2k+2}, the graph $G - u$ does not contain a path of order $2k+1$ with both ends in A. Applying Lemma 1, part (2), we see that

$$2e_{G-u}(A) + e_{G-u}(A, B) \leqslant (2k-1)|A| + k|B|,$$

and therefore,

$$\sum_{v\in\Gamma_G(u)} (d_G(v)-1) = \sum_{v\in\Gamma_G(u)} d_{G-u}(v) = 2e_{G-u}(A) + e_{G-u}(A,B)$$
$$\leqslant (2k-1)|A| + k|B|$$
$$= (2k-1)\,d_G(u) + k(n-d_G(u)-1).$$

Rearranging both sides, we obtain

$$\sum_{v\in\Gamma_G(u)} d_G(v) \leqslant k d_G(u) + k(n-1).$$

Adding these inequalities for all vertices $u \in V(G)$, we find out that

$$\sum_{u\in V(G)} \sum_{v\in\Gamma_G(u)} d_G(v) \leqslant k \sum_{u\in V(G)} d_G(u) + k(n-1)n = 2km + k(n-1)n.$$

To complete the proof of the theorem note that the term $d_G(v)$ appears in the left-hand sum exactly $d_G(v)$ times, and so

$$\sum_{u\in V(G)} \sum_{v\in\Gamma_G(u)} d_G(v) = \sum_{v\in V(G)} d_G^2(v).$$

□

Here is a corollary of Theorem 2 that gives the upper bound for the proof of (1).

Corollary 3 *Let G be a graph with n vertices. If G does not contain a C_{2k+2}, then for every $p \geqslant 2$,*

$$\sum_{u\in V(G)} d_G^p(u) \leqslant kn^p + O\left(n^{p-1/2}\right).$$

Proof Letting m be the number of edges of G, we first deduce an upper bound on m. Theorem 2 and the AM-QM inequality imply that

$$\frac{4m^2}{n} \leqslant \sum_{u\in V(G)} d_G^2(u) \leqslant 2km + k(n-1)n,$$

and so,

$$m \leqslant -kn + n\sqrt{k(n-1)+k^2} < n\sqrt{kn}. \tag{4}$$

Note that much stronger upper bounds on m are known (e.g., see [2] and [6]), but this one is simple and unconditional.

Now Theorem 2 and inequality (4) imply that

$$\sum_{u\in V(G)} d_G^p(u) < \sum_{u\in V(G)} n^{p-2} d_G^2(u) < kn^p + 2kmn^{p-2} < kn^p + 2(kn)^{3/2} n^{p-2}$$
$$= kn^p + O\left(n^{p-1/2}\right),$$

completing the proof. □

Note that we need only part (2) of Lemma 1 to prove Theorem 2 and Corollary 3. However, part (1) of Lemma 1 may have also applications, as shown in [5].

3 Proof of Lemma 1

To simplify the proof of Lemma 1 we state two routine lemmas whose proofs are omitted.

Lemma 4 *Let $P = (v_1, \ldots, v_p)$ be a path of maximum order in a connected non-Hamiltonian graph G. Then $p \geqslant d_G(v_1) + d_G(v_p) + 1$.*

Lemma 5 *Let $P = (v_1, \ldots, v_p)$ be a path of maximum order in a graph G. Then either v_1 is joined to two consecutive vertices of P or G contains a cycle of order at least $2d_G(v_1)$.*

Proof of Lemma 1 For convenience we shall assume that the set B is independent. Also, we shall call a path with both ends in A an *A-path*.

Claim 6 *If G contains an A-path of order $p > 2$, then G contains an A-path of order $p - 2$.*

Indeed, let $(v_1, \ldots, v_p)$ be an A-path. If $v_2 \in B$, then $v_3 \in A$, and so $(v_3, \ldots, v_p)$ is an A-path of order $p - 2$. If $v_{p-1} \in B$, then $v_{p-2} \in A$, and so $(v_1, \ldots, v_{p-2})$ is an A-path of order $p - 2$. Finally, if both $v_2 \in A$ and $v_{p-1} \in A$, then $(v_2, \ldots, v_{p-1})$ is an A-path of order $p - 2$.

The proofs of the two parts of Lemma 1 are very similar, but since they differ in the details, we shall present them separately.

Proof of part (1)

From Claim 6 we easily obtain the following consequence:

Claim 7 *If G contains an A-path of order $p \geqslant 2k$, then G contains an A-path of order $2k$ or $2k + 1$.*

This in turn implies

Claim 8 *If G contains a cycle C_p for some $p \geqslant 2k + 1$, then G contains an A-path of order $2k$ or $2k + 1$.*

Indeed, let $C = (v_1, \ldots, v_p, v_1)$ be a cycle of order $p \geqslant 2k+1$. The assertion is obvious if C is entirely in A, so let assume that C contains a vertex of B, say $v_1 \in B$. Then $v_2 \in A$ and $v_p \in A$; hence, $(v_2, \ldots, v_p)$ is an A-path of order at least $2k$. In view of Claim 7, this completes the proof of Claim 8.

To complete the proof of part (1) we shall use induction on the order of G. First we show that condition (2) implies that $|G| \geqslant 2k$. Indeed, assume that $|G| \leqslant 2k - 1$. We have

$$|A|^2 - |A| + |A|\,|B| \geqslant 2e_G(A) + e_G(A, B) > (2k - 2)\,|A| + k\,|B|$$

and so,

$$|G|\,(|A| - k) = (|A| + |B|)\,(|A| - k) > (k - 1)\,|A|.$$

Hence, we find that

$$(2k-1)(|A|-k) > (k-1)|A|$$

and so, $|A| > 2k-1$, a contradiction with $|A| \leqslant |G|$.

The conclusion of Lemma 1, part (1) follows when $|G| \leqslant 2k-1$ since then the hypothesis is false. Assume now that $|G| \geqslant 2k$ and that the Lemma holds for graphs with fewer vertices than G. It is easy to see that this assumption implies the assertion if G is disconnected. Indeed, let $G_1, \ldots, G_s$ be the components of G. Assuming that G has no A-path of order $2k+1$, the inductive assumption implies that each component G_i satisfies

$$2e_{G_i}(A_i) + e_{G_i}(A_i, B_i) \leqslant (2k-2)|A_i| + k|B_i|, \tag{5}$$

where

$$A_i = A \cap V(G_i) \quad \text{and} \quad B_i = B \cap V(G_i).$$

Summing (5) for $i = 1, \ldots, s$, we obtain a contradiction to (2).

Thus, to the end of the proof we shall assume that G is connected. Also, we can assume that G is non-Hamiltonian. Indeed, in view of Claim 8, this is obvious when $|G| > 2k$. If $|G| = 2k$ and G is Hamiltonian, then no two consecutive vertices along the Hamiltonian cycle belong to A, and since B is independent, we have $|B| = |A| = k$. Then

$$k(2k-1) \geqslant 2e_G(A) + e_G(A,B) > (2k-2)|A| + k|B| = k(2k-1),$$

contradicting (2). Thus, we shall assume that G is non-Hamiltonian.

The induction step is completed if there is a vertex $u \in B$ such that $d_G(u) \leqslant k$. Indeed the sets A and $B' = B \backslash \{u\}$ partition the vertices of $G-u$ and also

$$\begin{aligned} 2e_{G-u}(A) + e_{G-u}(A,B) &= 2e_G(A) + e_G(A,B) - d_G(u) > (2k-2)|A| + k|B| - k \\ &= (2k-2)|A| + k|B'|; \end{aligned}$$

hence $G-u$ contains an A-path of order at least $2k$, completing the proof. Thus, to the end of the proof we shall assume that

(a) $d_G(u) \geqslant k+1$ *for every vertex* $u \in B$.

For every vertex $u \in A$, write $d'_G(u)$ for its neighbors in A and $d''_G(u)$ for its neighbors in B. The induction step can be completed if there is a vertex $u \in A$ such that $2d'_G(u) + d''_G(u) \leqslant 2k-2$. Indeed, if u is such a vertex, note that the sets $A' = A \backslash \{u\}$ and B partition the vertices of $G-u$ and also

$$\begin{aligned} 2e_{G-u}(A) + e_{G-u}(A,B) &= 2e_G(A) + e_G(A,B) - 2d'_G(u) - d''_G(u) \\ &> (2k-2)|A| + k|B| - 2k + 2 \\ &= (2k-2)|A'| + k|B|; \end{aligned}$$

hence $G-u$ contains an A-path of order at least $2k$, completing the proof. Hence we have $2d'_G(u) + d''_G(u) \geqslant 2k-1$, and so $d_G(u) \geqslant k$. Thus, to the end of the proof, we shall assume that:

(b) $d_G(u) \geqslant k$ *for every vertex* $u \in A$.

Select now a path $P = (v_1, \ldots, v_p)$ of maximum length in G. To complete the induction step we shall consider three cases: *(i)* $v_1 \in B$, $v_p \in B$; *(ii)* $v_1 \in B$, $v_p \in A$, and *(iii)* $v_1 \in A$, $v_p \in A$.

Case (i): $v_1 \in B$, $v_p \in B$

In view of assumption *(a)* we have $d_G(v_1) + d_G(v_p) \geqslant 2k+2$, and Lemma 4 implies that $p \geqslant 2k+3$. We see that $(v_2, \ldots, v_{p-1})$ is an A-path of order at least $2k+1$, completing the proof by Claim 7.

Case (ii): $v_1 \in B$, $v_p \in A$

In view of assumptions *(a)* and *(b)* we have $d_G(v_1) + d_G(v_p) \geqslant 2k+1$, and Lemma 4 implies that $p \geqslant 2k+2$, and so, $(v_2, \ldots, v_p)$ is an A-path of order at least $2k+1$. This completes the proof by Claim 7.

Case (iii): $v_1 \in A$, $v_p \in A$

In view of assumption *(b)* we have $d_G(v_1) + d_G(v_p) \geqslant 2k$, and Lemma 4 implies that $p \geqslant 2k+1$. Since $(v_1, \ldots, v_p)$ is an A-path of order at least $2k+1$, by Claim 7, the proof of part (A) of Lemma 1 is completed.

Proof of part (2)

From Claim 6 we easily obtain the following consequence:

Claim 9 *If* G *contains an* A*-path of odd order* $p \geqslant 2k+1$, *then* G *contains an* A*-path of order exactly* $2k+1$.

From Claim 9 we deduce another consequence:

Claim 10 *If* G *contains a cycle* C_p *for some* $p \geqslant 2k+1$, *then* G *contains an* A*-path of order exactly* $2k+1$.

Indeed, let $C = (v_1, \ldots, v_p, v_1)$ be a cycle of order $p \geqslant 2k+1$. If p is odd, then some two consecutive vertices of C belong to A, say the vertices v_1 and v_2. Then $(v_2, \ldots, v_p, v_1)$ is an A-path of odd order $p \geqslant 2k+1$, and by Claim 9 the assertion follows. If p is even, then $p \geqslant 2k+2$. The assertion is obvious if C is entirely in A, so let assume that C contains a vertex of B, say $v_1 \in B$. Then $v_2 \in A$ and $v_p \in A$; hence $(v_2, \ldots, v_p)$ is an A-path of odd order at least $2k+1$, completing the proof of Claim 10.

To complete the proof of Lemma 1 we shall use induction on the order of G. First we show that condition (3) implies that $|G| \geqslant 2k+1$. Indeed, assume that $|G| \leqslant 2k$. We have

$$|A|^2 - |A| + |A|\,|B| \geqslant 2e_G(A) + e_G(A,B) > (2k-1)\,|A| + k\,|B|$$

and so,

$$|G|\,(|A| - k) = (|A| + |B|)\,(|A| - k) > k\,|A|\,.$$

Hence, we find that $2k\left(|A|-k\right) > k\,|A|$, and $|A| > 2k$, contradicting that $|A| \leqslant |G|$.

The conclusion of Lemma 1, part (2) follows when $|G| \leqslant 2k$ since then the hypothesis is false. Assume now that $|G| \geqslant 2k+1$ and that the assertion holds for graphs with fewer vertices than G. As in part (1), it is easy to see that this assumption implies the assertion if G is disconnected, so to the end of the proof we shall assume that G is connected. Also, in view of Claim 10 and $|G| \geqslant 2k+1$, we shall assume that G is non-Hamiltonian.

The induction step is completed if there is a vertex $u \in B$ such that $d_G(u) \leqslant k$. Indeed the sets A and $B' = B\backslash\{u\}$ partition the vertices of $G-u$ and also

$$\begin{aligned} 2e_{G-u}(A) + e_{G-u}(A,B) &= 2e_G(A) + e_G(A,B) - d_G(u) \\ &> (2k-1)\,|A| + k\,|B| - k \\ &= (2k-1)\,|A| + k\,|B'|; \end{aligned}$$

hence $G-u$ contains an A-path of order $2k+1$, completing the proof. Thus, to the end of the proof we shall assume that:

(a) $d_G(u) \geqslant k+1$ *for every vertex* $u \in B$.

For every vertex $u \in A$, write $d'_G(u)$ for its neighbors in A and $d''_G(u)$ for its neighbors in B. The induction step can be completed if there is a vertex $u \in A$ such that $2d'_G(u) + d''_G(u) \leqslant 2k-1$. Indeed, if u is such a vertex, note that the sets $A' = A\backslash\{u\}$ and B partition the vertices of $G-u$ and also

$$\begin{aligned} 2e_{G-u}(A) + e_{G-u}(A,B) &= 2e_G(A) + e_G(A,B) - 2d'_G(u) - d''_G(u) \\ &> (2k-1)\,|A| + k\,|B| - 2k + 1 \\ &= (2k-1)\,|A'| + k\,|B|; \end{aligned}$$

hence $G-u$ contains an A-path of order $2k+1$, completing the proof. Thus, to the end of the proof, we shall assume that:

(b) $d_G(u) \geqslant k$ *for every vertex* $u \in A$ *and if* u *has neighbors in* B*, then* $d_G(u) \geqslant k+1$.

Select now a path $P = (v_1, \ldots, v_p)$ of maximum length in G. To complete the induction step we shall consider three cases: *(i)* $v_1 \in B$, $v_p \in B$; *(ii)* $v_1 \in B$, $v_p \in A$, and *(iii)* $v_1 \in A$, $v_p \in A$.

Case (i): $v_1 \in B$, $v_p \in B$

In view of assumption *(b)* we have $d_G(v_1) + d_G(v_p) \geqslant 2k+2$, and Lemma 4 implies that $p \geqslant 2k+3$. If p is odd, we see that $(v_2, \ldots, v_{p-1})$ is an A-path of order at least $2k+1$, and by Claim 9, the proof is completed.

Suppose now that p is even. Applying Lemma 5, we see that either G has a cycle of order at least $2d_G(v_1) \geqslant 2k+2$, or v_1 is joined to v_i and v_{i+1} for some $i \in \{2, \ldots, p-2\}$. In the first case we complete the proof by Claim 10; in the second case we see that the sequence

$$(v_2, v_3, \ldots, v_i, v_1, v_{i+1}, v_{i+2}, \ldots, v_{p-1})$$

is an A-path of order $p-1$. Since $p-1$ is odd and $p-1 \geqslant 2k+3$, the proof is completed by Claim 9.

Case (ii): $v_1 \in B$, $v_p \in A$

In view of assumptions *(a)* and *(b)* we have $d_G(v_1) + d_G(v_p) \geqslant 2k+1$, and Lemma 4 implies that $p \geqslant 2k+2$. If p is even, we see that $(v_2, \ldots, v_{p-1})$ is an A-path of order at least $2k+1$, and by Claim 9, the proof is completed.

Suppose now that p is odd. Applying Lemma 5, we see that either G has a cycle of order at least $2d_G(v_1) \geqslant 2k+2$, or v_1 is joined to v_i and v_{i+1} for some $i \in \{2, \ldots, p-1\}$. In the first case we complete the proof by Claim 10; in the second case we see that the sequence

$$(v_2, v_3, \ldots, v_i, v_1, v_{i+1}, v_{i+2}, \ldots, v_p)$$

is an A-path of order p. Since p is odd and $p \geqslant 2k+2$, the proof is completed by Claim 9.

Case (iii): $v_1 \in A$, $v_p \in A$

In view of assumption *(b)* we have $d_G(v_1) + d_G(v_p) \geqslant 2k$, and Lemma 4 implies that $p \geqslant 2k+1$. If p is odd, the proof is completed by Claim 9.

Suppose now that p is even, and therefore, $p \geqslant 2k+2$. If $v_2 \in A$, then the sequence $(v_2, \ldots, v_p)$ is an A-path of odd order $p-1 \geqslant 2k+1$, completing the proof by Claim 9. If $v_2 \in B$, we see that v_1 has a neighbor in B, and so, $d_G(v_1) \geqslant k+1$.

Applying Lemma 5, we see that either G has a cycle of order at least $2d_G(v_1) \geqslant 2k+2$, or v_1 is joined to v_i and v_{i+1} for some $i \in \{2, \ldots, p-2\}$. In the first case we complete the proof by Claim 10. In the second case we shall exhibit an A-path of order $p-1$. Indeed, if $i = 2$, let

$$Q = (v_1, v_3, v_4, \ldots, v_p),$$

and if $i \geqslant 3$, let

$$Q = (v_3, \ldots, v_i, v_1, v_{i+1}, v_{i+2}, \ldots, v_p).$$

In either case Q is an A-path of order $p-1$. Since $p-1$ is odd and $p-1 \geqslant 2k+1$, the proof is completed by Claim 9.

This completes the proof of Lemma 1. □

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