Tiling tripartite graphs with 3-colorable graphs

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Abstract

For any positive real number γ and any positive integer h, there is N_0 such that the following holds. Let $N \geqslant N_0$ be such that N is divisible by h. If G is a tripartite graph with N vertices in each vertex class such that every vertex is adjacent to at least $(2/3+\gamma)N$ vertices in each of the other classes, then G can be tiled perfectly by copies of $K_{h,h,h}$. This extends the work in [Discrete Math. **254** (2002), 289-308] and also gives a sufficient condition for tiling by any fixed 3-colorable graph. Furthermore, we show that the minimum-degree $(2/3+\gamma)N$ in our result cannot be replaced by $2N/3+h-2$.

1 Introduction

Let H be a graph on h vertices, and let G be a graph on n vertices. *Tiling* (or *packing*) problems in extremal graph theory are investigations of conditions under which G must contain many vertex disjoint copies of H (as subgraphs), where minimum degree conditions are studied the most. An H-tiling of G is a subgraph of G which consists of vertex-disjoint copies of H. A *perfect H-tiling*, or *H-factor*, of G is an H-tiling consisting of $\lfloor n/h \rfloor$ copies of H. A very early tiling result is implied by Dirac's theorem on Hamilton cycles [6], which implies that every n-vertex graph G with minimum degree $\delta(G) \geqslant n/2$ contains a perfect matching (usually called 1-factor, instead of K_2-factor). Later Corrádi and Hajnal [4] studied the minimum degree of G that guarantees a K_3-factor. Hajnal and Szemerédi [9] settled the tiling problem for any complete graph K_h by showing that

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every n-vertex graph G with $\delta(G) \geqslant (h-1)n/h$ contains a K_r-factor (it is easy to see that this is sharp). Using the celebrated Regularity Lemma of Szemerédi [23], Alon and Yuster [1, 2] generalized the above tiling results for arbitrary H. Their theorems were later sharpened by various researchers [14, 12, 22, 17]. Results and methods for tiling problems can be found in a recent survey of Kühn and Osthus [18].

In this paper, we consider multipartite tiling, which restricts G to be an r-partite graph. When $r = 2$, The König-Hall Theorem (e.g. see [3]) provides necessary and sufficient conditions to solve the 1-factor problem for bipartite graphs. Wang [24] considered $K_{s,s}$-factors in bipartite graphs for all $s > 1$, the second author [25] gave the best possible minimum degree condition for this problem. Recently Hladký and Schacht [10] determined the minimum degree threshold for $K_{s,t}$-factors with $s < t$.

Let $\mathcal{G}_r(N)$ denote the family of r-partite graphs with N vertices in each of its partition sets. In an r-partite graph G, we use $\bar{\delta}(G)$ for the minimum degree from a vertex in one partition set to any other partition set. Fischer [8] proved almost perfect K_3-tilings in $\mathcal{G}_3(N)$ with $\bar{\delta}(G) \geqslant 2N/3$ and Johansson [11] gives a K_3-factor with the less stringent degree condition $\bar{\delta}(G) \geqslant 2N/3 + O(\sqrt{N})$.

For general $r > 2$, Fischer [8] conjectured the following r-partite version of the Hajnal–Szemerédi Theorem: if $G \in \mathcal{G}_r(N)$ satisfies $\bar{\delta}(G) \geqslant (r-1)N/r$, then G contains a K_r-factor. The first author and Szemerédi [20] proved this conjecture for $r = 4$. Csaba and Mydlarz [5] recently proved that the conclusion in Fischer's conjecture holds if $\bar{\delta}(G) \geqslant \frac{k_r}{k_r+1}n$, where $k_r = r + O(\log r)$. On the other hand, Magyar and the first author [19] showed that Fischer's conjecture is false for all odd $r \geqslant 3$: they constructed r-partite graphs $\Gamma(N) \in \mathcal{G}_r(N)$ for infinitely many N such that $\bar{\delta}(\Gamma(N)) = (r-1)N/r$ and yet $\Gamma(N)$ contains no K_r-factor. Nevertheless, Magyar and the first author proved a theorem (Theorem 1.2 in [19]) which implies the following Corrádi-Hajnal-type theorem.

Theorem 1.1 ([19]) *If $G \in \mathcal{G}_3(N)$ satisfies $\bar{\delta}(G) \geqslant (2/3)N + 1$, then G contains a K_3-factor.*

In this paper we extend this result to all 3-colorable graphs. Our main result is on $K_{h,h,h}$-tiling.

Theorem 1.2 *For any positive real number γ and any positive integer h, there is N_0 such that the following holds. Given an integer $N \geqslant N_0$ such that N is divisible by h, if G is a tripartite graph with N vertices in each vertex class such that every vertex is adjacent to at least $(2/3+\gamma)N$ vertices in each of the other classes, then G contains a $K_{h,h,h}$-factor.*

Since the complete tripartite graph $K_{h,h,h}$ can be perfectly tiled by any 3-colorable graph on h vertices, we have the following corollary.

Corollary 1.3 *Let H be a 3-colorable graph of order h. For any $\gamma > 0$ there exists a positive integer N_0 such that if $N \geqslant N_0$ and N is divisible by h, then every $G \in \mathcal{G}_3(N)$ with $\bar{\delta}(G) \geqslant (2/3+\gamma)N$ contains an H-factor.*

The Alon–Yuster theorem [2] says that for any $\gamma > 0$ and any r-colorable graph H there exists n_0 such that every graph G of order $n \geqslant n_0$ contains an H-factor if n divisible by h and $\delta(G) \geqslant (1-1/r)n + \gamma n$ (Komlós, Sárközy and Szemerédi [14] later reduced γn to a constant that depends only on H). Corollary 1.3 gives another proof of this theorem for $r = 3$ as follows. Let G be a graph of order $n = 3N$ with $\delta(G) \geqslant 2n/3 + 2\gamma n$. A random balanced partition of $V(G)$ yields a subgraph $G' \in \mathcal{G}_3(N)$ with $\bar{\delta}(G') \geqslant \delta(G)/3 - o(n) \geqslant (2/3+\gamma)N$. We then apply Corollary 1.3 to G' obtaining an H-factor in G', hence in G.

Instead of proving Theorem 1.2, we actually prove the stronger Theorem 1.4 below. Given $\gamma > 0$, we say that $G = \big(V^{(1)}, V^{(2)}, V^{(3)}; E\big) \in \mathcal{G}_3(N)$ is in *the extreme case with parameter* γ if there are three sets A_1, A_2, A_3 such that $A_i \subseteq V^{(i)}$, $|A_i| = \lfloor N/3 \rfloor$ for all i and

$$d(A_i, A_j) := \frac{e(A_i, A_j)}{|A_i||A_j|} \leqslant \gamma$$

for $i \neq j$. If $G \in \mathcal{G}_3(N)$ satisfies $\bar{\delta}(G) \geqslant (2/3+\gamma)N$, then G is *not* in the extreme case with parameter γ. In fact, any two sets A and B of size $\lfloor N/3 \rfloor$ from two different vertex classes satisfy $\deg(a, B) \geqslant \gamma N$, for all $a \in A$, and consequently $d(A, B) > \gamma$. Theorem 1.2 thus follows from Theorem 1.4, which is even stronger because of its weaker assumption $\bar{\delta}(G) \geqslant (2/3 - \varepsilon)N$.

Theorem 1.4 *Given any positive integer h and any $\gamma > 0$, there exists an $\varepsilon > 0$ and an integer N_0 such that whenever $N \geqslant N_0$, and h divides N, the following holds: If $G \in \mathcal{G}_3(N)$ satisfies $\bar{\delta}(G) \geqslant (2/3-\varepsilon)N$, then either G contains a $K_{h,h,h}$-factor or G is in the extreme case with parameter γ.*

The following proposition shows that the minimum degree $\bar{\delta}(G) \geqslant (2/3+\gamma)N$ in Theorem 1.2 cannot be replaced by $2N/3 + h - 2$.

Proposition 1.5 *Given any positive integer $h \geqslant 2$, there exists an integer q_0 such that for any $q \geqslant q_0$, there exists a tripartite graph $G_0 \in \mathcal{G}_3(N)$ with $N = 3qh$ such that $\bar{\delta}(G_0) = 2qh + (h-2)$ and G_0 has no $K_{h,h,h}$-factor.*

The structure of the paper is as follows. We first prove Proposition 1.5 in Section 2. After stating the Regularity Lemma and Blow-up Lemma in Section 3, we prove Theorem 1.4 in Section 4. We give concluding remarks and open problems in Section 5.

2 Proof of Proposition 1.5

In a tripartite graph $G = (A, B, C; E)$, the graphs induced by (A, B), (A, C) and (B, C) are called the *natural bipartite subgraphs* of G. First we need to construct a balanced tripartite K_3-free graph in which all natural bipartite graphs are regular and C_4-free. Our construction below is based a construction in [25] of sparse regular bipartite graphs with no C_4.

Lemma 2.1 *For each integer $d \geqslant 0$, there exists an n_0 such that, if $n \geqslant n_0$, there exists a balanced tripartite graph, $Q(n, d)$ on $3n$ vertices such that each of the 3 natural bipartite subgraphs are d-regular, C_4-free and triangle-free.*

Proof. A Sidon set is a set of integers such that sums $i + j$ are distinct for distinct pairs i, j from the set. Let $[n] = \{1, \ldots, n\}$. It is well known (*e.g.*, [7]) that $[n]$ contains a Sidon set of size about $\sqrt{n}$ for large n. Suppose that n is sufficiently large. Let S be a d-element Sidon subset of $[\frac{n}{3} - 1]$. Given two copies of $[n]$, A and B, we construct a bipartite graph $P(A, B)$ on (A, B) whose edges are (ordered) pairs ab, $a \in A$, $b \in B$ such that $b - a \pmod{n} \in S$. It is shown in [25] (in the proof of Proposition 1.3) that $P(A, B)$ is d-regular with no C_4. Given three copies of $[n]$, A, B and C, let Q be the union of $P(A, B)$, $P(B, C)$ and $P(C, A)$. In order to show that Q is the desired graph $Q(n, d)$, we need to verify that Q is K_3-free. In fact, if $a \in A$, $b \in B$, and $c \in C$ form a K_3, then there exist $i, j, k \in S$ such that

$$b \equiv a + i, \quad c \equiv b + j, \quad a \equiv c + k \pmod{n},$$

which implies that $i + j + k \equiv 0 \mod n$. But this is impossible for $i, j, k \in [\frac{n}{3} - 1]$. $\square$

Proof of Proposition 1.5. We will construct 9 disjoint sets $A_j^{(i)}$ with $i, j \in \{1, 2, 3\}$. The union $A_1^{(i)} \cup A_2^{(i)} \cup A_3^{(i)}$ defines the i^{th} *vertex-class*, while the triple $(A_j^{(1)}, A_j^{(2)}, A_j^{(3)})$ defines the j^{th} *column*.

Construct G_0 as follows: For $i = 1, 2, 3$, let $|A_1^{(i)}| = qh - 1$, $|A_2^{(i)}| = qh$ and $|A_3^{(i)}| = qh + 1$. Let the graph in column 1 be $Q(qh - 1, h - 3)$ (as given by Lemma 2.1), the graph in column 2 be $Q(qh, h - 2)$ and the graph in column 3 be $Q(qh + 1, h - 1)$. If two vertices are in different columns and different vertex-classes, then they are also adjacent. It is easy to verify that $\bar{\delta}(G_0) = 2qh + (h - 2)$.

Suppose, by way of contradiction, that G_0 has a $K_{h,h,h}$-factor. Since there are no triangles and no C_4's in any column, the intersection of a copy of $K_{h,h,h}$ with a column is either a star, with all leaves in the same vertex-class, or a set of vertices in the same vertex-class. So, each copy of $K_{h,h,h}$ has at most h vertices in column 3. A $K_{h,h,h}$-factor has exactly $3q$ copies of $K_{h,h,h}$ and so the factor has at most $3qh$ vertices in column 3. But there are $3qh + 3$ vertices in column 3, a contradiction. $\square$

3 The Regularity Lemma and Blow-up Lemma

The Regularity Lemma and the Blow-up Lemma are main tools in the proof of Theorem 1.4. Let us first define ε-regularity and (ε, δ)-super-regularity.

Definition 3.1 *Let $\varepsilon > 0$. Suppose that a graph G contains disjoint vertex-sets A and B.*

1. *The pair (A, B) is* ε-regular *if for every $X \subseteq A$ and $Y \subseteq B$, satisfying $|X| > \varepsilon|A|$, $|Y| > \varepsilon|B|$, we have $|d(X, Y) - d(A, B)| < \varepsilon$.*
2. *The pair (A, B) is* (ε, δ)-super-regular *if (A, B) is ε-regular and $\deg(a, B) > \delta|B|$ for all $a \in A$ and $\deg(b, A) > \delta|A|$ for all $b \in B$.*

The celebrated Regularity Lemma of Szemerédi [23] has a multipartite version (see survey papers [15, 16]), which guarantees that when applying the lemma to a multipartite graph, every resulting cluster is from one partition set. Given a vertex v and a vertex set S in a graph G, we define $\deg(v, S)$ as the number of neighbors of v in S.

Lemma 3.2 (Regularity Lemma - Tripartite Version) *For every positive ε there is an $M = M(\varepsilon)$ such that if $G = (V, E)$ is any tripartite graph with partition sets $V^{(1)}$, $V^{(2)}$, $V^{(3)}$ of size N, and $d \in [0, 1]$ is any real number, then there are partitions of $V^{(i)}$ into clusters $V_0^{(i)}, V_1^{(i)}, \ldots, V_k^{(i)}$ for $i = 1, 2, 3$ and a subgraph $G' = (V, E')$ with the following properties:*

- *$k \leqslant M$,*
- *$|V_0^{(i)}| \leqslant \varepsilon n$ for $i = 1, 2, 3$,*
- *$|V_j^{(i)}| = L \leqslant \varepsilon n$ for all $i = 1, 2, 3$ and $j \geqslant 1$,*
- *$\deg_{G'}(v, V^{(i')}) > \deg_G(v, V^{(i')}) - (d + \varepsilon)N$ for all $v \in V^{(i)}$ and $i \neq i'$,*
- *all pairs $(V_j^{(i)}, V_{j'}^{(i')})$, $i \neq i'$, $1 \leqslant j, j' \leqslant k$, are ε-regular in G', each with density either 0 or exceeding d.*

We will also need the Blow-up Lemma of Komlós, Sárközy and Szemerédi [13].

Lemma 3.3 (Blow-up Lemma) *Given a graph R of order r and positive parameters δ, Δ, there exists an $\varepsilon > 0$ such that the following holds: Let N be an arbitrary positive integer, and let us replace the vertices of R with pairwise disjoint N-sets $V_1, V_2, \ldots, V_r$. We construct two graphs on the same vertex-set $V = \bigcup V_i$. The graph $R(N)$ is obtained by replacing all edges of R with copies of the complete bipartite graph $K_{N,N}$ and a sparser graph G is constructed by replacing the edges of R with some (ε, δ)-super-regular pairs. If a graph H with maximum degree $\Delta(H) \leqslant \Delta$ can be embedded into $R(N)$, then it can be embedded into G.*

4 Proof of Theorem 1.4

In this section we prove Theorem 1.4. First we sketch the proof.

We begin by applying the Regularity Lemma to G, partitioning each vertex class into ℓ clusters and an exceptional set. Next we define the cluster graph G_r (whose vertices are the clusters of G and where clusters from different partition classes are adjacent if the pair is regular with positive density), which is 3-partite and such that $\bar{\delta}(G_r)$ is almost $2\ell/3$. In Step 1, we use the so-called fuzzy tripartite theorem of [19], which states that either G_r is in the extreme case (hence G is in the extreme case) or G_r has a K_3-factor. Having assumed that G_r has a K_3-factor $\mathcal{S} = \{S_1, \ldots, S_\ell\}$, in Step 2 we move a small amount of vertices from each cluster to the exceptional sets such that in each S_j, all three pairs are super-regular and the three clusters have the same size, which is a multiple of h. If we now were to apply the Blow-up Lemma to each S_j, then we would obtain a $K_{h,h,h}$-factor covering all the non-exceptional vertices of G.

So we need to handle the exceptional sets before applying the Blow-up Lemma. Step 3 is a step of preprocessing: we set some copies of $K_{h,h,h}$ aside such that in Step 5 we can modify them by replacing $5h$ vertices from S_1 with $5h$ vertices from an S_j, $j \geqslant 2$. The vertices in these copies of $K_{h,h,h}$ are not now in their original clusters. Since these copies of $K_{h,h,h}$ are from triangles of G_r that are not necessarily in $\mathcal{S}$, we may need to move vertices from other clusters to the exceptional sets to keep the balance of the three clusters in each S_j. For each exceptional vertex v, we will remove a copy of $K_{h,h,h}$ which contains v and $2h-1$ vertices from some cluster-triangle S_j (we call this inserting v into S_j). If this is done arbitrarily, the remaining vertices of some S_j may not induce a $K_{h,h,h}$-factor. In Step 4, we group exceptional vertices into h-element sets such that all h vertices in one h-element set can be inserted into the same S_j. As a result, two clusters in some S_j may have sizes that differ by a multiple of h. We then remove a few more copies of $K_{h,h,h}$ such that the sizes of the three clusters of each S_j are the same and divisible by h. Unfortunately up to $5h$ vertices in each exceptional set may not be removed by this approach. In Step 5 we first insert the remaining exceptional vertices into an arbitrary S_j, $j \geqslant 2$, and then transfer extra vertices from S_j to S_1. As a result, three clusters in all S_j, $j \geqslant 1$ have the same size, which is divisible by h. At the end of Step 5, we apply the Blow-up Lemma to each S_j to complete the $K_{h,h,h}$-factor of G. This ends the proof sketch.

Note that our proof follows the approach in [19], which has a different way of handling exceptional vertices from the bipartite case [25]. Although a $K_{h,h,h}$-tiling is more complex than a K_3-tiling, our proof is not longer than the non-extreme case in [19] because we take advantage of results from [19].

Let us now start the proof. We assume that N is large, and without loss of generality, assume that $\gamma \ll \frac{1}{h}$. We find small constants d_1, ε, and ε_1 such that (actual dependencies result from Lemmas 4.1, 4.4, 4.7, and 3.3):

$$\varepsilon_1 \ll 2\varepsilon = d_1 \ll \gamma. \tag{1}$$

For simplicity, we will refrain from using floor or ceiling functions when they are not crucial.

Begin with a tripartite graph $G = \left(V^{(1)}, V^{(2)}, V^{(3)}; E\right)$ with $\left|V^{(1)}\right| = \left|V^{(2)}\right| = \left|V^{(3)}\right| = N$ such that $\bar{\delta}(G) \geqslant (2/3-\varepsilon)N$. Apply the Regularity Lemma (Lemma 3.2) with ε_1 and d_1, partitioning each $V^{(i)}$ into ℓ clusters $V_1^{(i)}, \dots, V_\ell^{(i)}$ of size $L \leqslant \varepsilon_1 N$ and an *exceptional set* $V_0^{(i)}$ of size at most $\varepsilon_1 N$. Later in the proof, the exceptional sets may grow in size, but will always remain of size $O(\varepsilon_1 N)$. We call the vertices in the exceptional sets *exceptional vertices.*

Let G' be the subgraph of G defined in the Regularity Lemma. We define the reduced graph (or cluster graph) G_r to be the 3-partite graph whose vertices are clusters $V_j^{(i)}$ $j \geqslant 1$, $i = 1, 2, 3$, and two clusters are adjacent if and only if they form an ε_1-regular pair of density at least d_1 in G'. We will use the same notation $V_j^{(i)}$ for a set in G and a vertex in G_r. Let $U^{(1)}, U^{(2)}, U^{(3)}$ denote three partition sets of G_r. We know that $|U^{(i)}| = \ell$. We observe that $\bar{\delta}(G_r) \geqslant (2/3 - 2d_1)\ell$. In fact, consider a cluster $C \in U^{(i)}$ and a vertex $x \in C$, the number m of clusters in $U^{(i')}$ $(i' \neq i)$ that are adjacent to C satisfies

$$\left(\frac{2}{3} - \varepsilon\right) N - (d_1 + \varepsilon_1)N \leqslant \deg_G(v, V^{(i')}) - (d_1 + \varepsilon_1)N \leqslant \deg_{G'}(x, V^{(i')}) \leqslant mL.$$

Since $N \geqslant L\ell$ and $\varepsilon + \varepsilon_1 \leqslant d_1$, we have $m \geqslant (2/3 - \varepsilon - d_1 - \varepsilon_1)\,\ell \geqslant (2/3 - 2d_1)\ell$.

Assume that G is *not* in the extreme case with parameter γ. We claim that G_r is not in the extreme case with parameter $\gamma/3$. Suppose instead, that there are subsets $S_i \subset U^{(i)}$, $i = 1, 2, 3$, of size $\ell/3$ with density at most $\gamma/3$. Let A_i denote the set of all vertices of G contained in a cluster of S_i. Then $N(1 - \varepsilon_1)/3 \leqslant |A_i| = L\ell/3 \leqslant N/3$ because $L\ell \geqslant (1 - \varepsilon_1)N$. The number of edges of G between A_i and $A_{i'}$, $i \neq i'$, is at most

$$e_G(A_i, A_{i'}) \leqslant e_{G'}(A_i, A_{i'}) + |A_i|(d_1 + \varepsilon_1)N \leqslant \frac{\gamma}{3}\left(\frac{\ell}{3}\right)^2 L^2 + (d_1 + \varepsilon_1)\frac{N^2}{3} \leqslant \frac{2\gamma}{3}\left(\frac{N}{3}\right)^2,$$

provided that $9(d_1 + \varepsilon_1) \leqslant \gamma$. After adding at most $\varepsilon_1 N/3$ vertices to each A_i, we obtain three subsets of $V^{(1)}, V^{(2)}, V^{(3)}$ of size $N/3$ with pairwise density at most $(2\gamma/3 + \varepsilon_1) \leqslant \gamma$ in G.

Step 1: Find a K_3-factor in G_r

We apply the following result (Theorem 2.1 in [19]) to the reduced graph G_r with $\alpha = \gamma/3$ and $\beta = 2d_1$.

Lemma 4.1 (Fuzzy tripartite theorem [19]) *For any $\alpha > 0$, there exist $\beta > 0$ and ℓ_0, such that the follows holds for all $\ell \geqslant \ell_0$. Every balanced 3-partite graph $R \in \mathcal{G}_3(\ell)$ with $\bar{\delta}(R) \geqslant (2/3 - \beta)\ell$ either contains a K_3-factor or is in the extreme case with parameter α.*

Since G_r is not in the extreme case with parameter $\gamma/3$, it must contain a K_3-factor $\mathcal{S} = \{S_1, S_2, \ldots, S_\ell\}$. After relabeling, we assume that $S_j = \left\{V_j^{(1)}, V_j^{(2)}, V_j^{(3)}\right\}$ for all j. In G_r, we call these fixed triangles $S_1, \ldots, S_\ell$ *columns* and consider $U^{(1)}, U^{(2)}, U^{(3)}$ as rows.

Step 2: Make pairs in S_j super-regular

For each S_j, remove a vertex v from a cluster in S_j and place it in the exceptional set if v has fewer than $(d_1 - \varepsilon_1)L$ neighbors in one of the other clusters of S_j. By ε_1-regularity, there are at most $2\varepsilon_1 L$ such vertices in each cluster. Remove more vertices if necessary to ensure that each non-exceptional cluster is of the same size and the size is divisible by h.

The Slicing Lemma states the well-known fact that regularity is maintained when small modifications are made to the clusters:

Proposition 4.2 (Slicing Lemma, Fact 1.5 in [19]) *Let (A, B) be an ε-regular pair with density d, and, for some $\alpha > \varepsilon$, let $A' \subset A$, $|A'| \geqslant \alpha|A|$, $B' \subset B$, $|B'| \geqslant \alpha|B|$. Then (A', B') is an ε'-regular pair with $\varepsilon' = \max\{\varepsilon/\alpha, 2\varepsilon\}$, and for its density d', we have $|d' - d| < \varepsilon$.*

Applying Proposition 4.2 with $\alpha = 1 - 2\varepsilon_1$, any pair of clusters which was ε_1-regular with density at least d_1 is now $(2\varepsilon_1)$-regular with density at least $d_1 - \varepsilon_1$ (because $\varepsilon_1 < 1/4$). Furthermore, each pair in the cluster-triangles S_j is $(2\varepsilon_1, d_1 - 3\varepsilon_1)$-super-regular. Each of the three exceptional sets are now of size at most $\varepsilon_1 N + \ell(2\varepsilon_1 L) \leqslant 3\varepsilon_1 N$.

Remark: Because all the pairs in S_j are super-regular and the complete tripartite graph on $\left(V_i^{(1)}, V_i^{(2)}, V_i^{(3)}\right)$ contains a $K_{h,h,h}$-factor, the Blow-up Lemma says that S_j also contains a $K_{h,h,h}$-factor.

Step 3: Create red copies of $K_{h,h,h}$

In this step we show that certain triangles exist in G_r which link each cluster to the one in S_1 from the same partition class. The purpose of this linking is to be able to handle a small discrepancy of sizes among the three clusters that comprise S_j in Step 5.

Definition 4.3 *In a tripartite graph $R = \left(U^{(1)}, U^{(2)}, U^{(3)}; E\right)$, one vertex $x \in U^{(1)}$ (the cases of $x \in U^{(2)}$ or $U^{(3)}$ are defined accordingly) is* **reachable** *from another vertex $y \in U^{(1)}$ in R by using at most $2k$ triangles, if there is a chain of triangles $T_1, \ldots, T_{2k}$ with $T_j = \left\{T_j^{(1)}, T_j^{(2)}, T_j^{(3)}\right\}$ and $T_j^{(i)} \in U^{(i)}$ for $i = 1, 2, 3$ such that the following occurs:*

1. $x = T_1^{(1)}$ *and* $y = T_{2k}^{(1)}$,

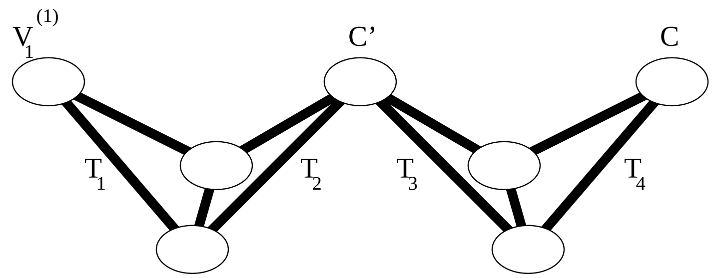

Figure 1: An illustration of how cluster $V_1^{(1)}$ is reachable from a cluster C.

2. $T_{2j-1}^{(2)} = T_{2j}^{(2)}$ *and* $T_{2j-1}^{(3)} = T_{2j}^{(3)}$*, for* $j = 1, \ldots, k$*, and*
3. $T_{2j}^{(1)} = T_{2j+1}^{(1)}$*, for* $j = 1, \ldots, k-1$.

Figure 1 illustrates that $V_1^{(1)}$ is reachable from another cluster C by using four triangles. The Reachability Lemma (Lemma 2.6 in [19]) says that every cluster of S_1 is reachable from any other cluster in the same class by using at most four triangles in G_r. Note that these triangles are not necessarily the fixed triangles S_j. The statement of the Reachability Lemma in [19] refers to the reduced graph, but its proof, in fact, proves the following general statement:

Lemma 4.4 (Reachability Lemma) *For any* $\alpha > 0$*, there exist* $\beta > 0$ *and* ℓ_0*, such that the following holds for all* $\ell \geqslant \ell_0$*. Let* $R \in \mathcal{G}_3(\ell)$ *be a balanced* 3*-partite graph with* $\bar{\delta}(R) \geqslant (2/3 - \beta)\ell$*. Then either each vertex is reachable from every other vertex in the same class by using at most* four *triangles or* R *is in the extreme case with parameter* α*.*

Let $C \neq V_1^{(1)}$ be a cluster in $U^{(1)}$ and let T_1, T_2 or T_1, T_2, T_3, T_4 be cluster-triangles which witness that $V_1^{(1)}$ is reachable from C by using at most $2k$ triangles for some $k \in \{1, 2\}$. Note that $T_1 \cap U^{(1)} = S_1^{(1)}$ and either both $k = 1$ and $T_2 \cap U^{(1)} = C$ or $k = 2$, $T_2 \cap U^{(1)} = T_3 \cap U^{(1)} = C'$ and $T_4 \cap U^{(1)} = C$.

We need a special case of a well-known embedding lemma in [15], which says that three reasonably large subsets of three clusters that form a triangle induce a copy of $K_{h,h,h}$.

Proposition 4.5 (Key Lemma, Theorem 2.1 in [15]) *Let* ε, d *be positive real numbers and* h, L *be positive integers such that* $(d-\varepsilon)^{2h} > \varepsilon$ *and* $\varepsilon(d-\varepsilon)L \geqslant h$*. Suppose that* X_1, X_2, X_3 *are clusters of size* L *and any pair of them is* ε*-regular with density at least* d*. Let* $A_i \subseteq X_i$*,* $i = 1, 2, 3$ *be three subsets of size at least* $(d-\varepsilon)L$*. Then* (A_1, A_2, A_3) *contains a copy of* $K_{h,h,h}$*.*

If $k = 1$, then we pick a vertex $v \in C$ and apply Proposition 4.5 to find a copy of $K_{h,h,h}$, called H', in the cluster triangle T_1 such that $H' \cap V^{(2)}$ and $H' \cap V^{(3)}$ are in the neighborhood of v. If $k = 2$, then we first pick a vertex $v \in C$ and apply Proposition 4.5 to find a copy of $K_{h,h,h}$, called H'', in the cluster triangle T_3 such that $H'' \cap V^{(2)}$ and $H'' \cap V^{(3)}$ are in the neighborhood of v. Next we pick a vertex $v' \in H'' \cap V^{(1)}$ (call it *special*) and apply Proposition 4.5 to find a copy of $K_{h,h,h}$, called H', in the cluster triangle T_1 such that $H' \cap V^{(2)}$ and $H' \cap V^{(3)}$ are in the neighborhood of v'.

Color all of the vertices in H' and in H'' (if it exists) *red* and the vertex in C *orange*. Note that the special vertex in H'' (if existent) is colored red. If a vertex is not colored, we will heretofore call it *uncolored*. Repeat this $5h$ times for each cluster not in S_1. In this process all but a constant number of vertices in each cluster remain uncolored since h is a constant and G_r consists of a constant number (that is, 3ℓ) of clusters. This is why we can repeatedly apply Proposition 4.5 ensuring that all the red copies of $K_{h,h,h}$ and orange vertices are vertex-disjoint.

At the end, each cluster not in S_1 has $5h$ orange vertices (the clusters in S_1 have no orange vertex). Each cluster has at most $3(\ell-1)(5h)(h) < 15\ell h^2$ red vertices because there are $3(\ell-1)$ clusters not in S_1, the process is iterated $5h$ times for each of them and a cluster gets at most h vertices colored red with each iteration.

Remark: This preprocessing ensures that we may later transfer at most $5h$ vertices from any cluster C to S_1 in the following sense: Without loss of generality, suppose C is a cluster in $V^{(1)}$. In the case when $k = 2$ (see Figure 1), identify an orange vertex $v \in C$ and its corresponding red subgraphs H' and H'', including the special vertex $v' \in C'$. (The case where $k = 1$ is similar but simpler.) Recolor v red and uncolor a vertex $u \in H' \cap V_1^{(1)}$. The red vertices still form two copies of $K_{h,h,h}$, one is $H' - \{u\} + \{v'\}$, and the other one is $H'' - \{v'\} + \{v\}$. The number of non-red vertices is decreased by one in C but is increased by one in $V_1^{(1)}$. We will do this in Step 5.

We now move some uncolored vertices from clusters to the corresponding exceptional set such that the three clusters in the same column (some S_j) have the same number of uncolored vertices. In other words, three clusters in any S_j are balanced in terms of uncolored vertices. (Note that this number is always divisible by h because the numbers of red vertices and orange vertices are divisible by h.) Thus, at most $15\ell h^2$ vertices can be removed from a cluster. The three exceptional sets have the same size, at most $3\varepsilon_1 N + 15\ell^2 h^2 \leqslant 4\varepsilon_1 N$. Each cluster still has at least $(1-2\varepsilon_1)L - 15\ell h^2 > (1-3\varepsilon_1)L$ uncolored vertices.

Step 4: Reduce the sizes of exceptional sets

At present the exceptional sets $V_0^{(i)}$, $i = 1, 2, 3$, are all of the same size, which is at most $4\varepsilon_1 N$ and divisible by h. Suppose this size is at least $6h$. We will remove some copies of $K_{h,h,h}$ from G such that $|V_0^{(i)}| \leqslant 5h$ eventually.

First, we say a vertex $v \in V_0^{(i)}$ *belongs to* a cluster $V_j^{(i)}$ if $\deg(v, V_j^{(i')}) \geqslant d_1 L$ for all $i' \neq i$. Using the minimum-degree condition, for fixed $i' \neq i$, the number of clusters $V_j^{(i')}$ such that $\deg(v, V_j^{(i')}) < d_1 L$ is at most

$$\frac{(1/3+\varepsilon)N}{(1-3\varepsilon_1)L - d_1 L} \leqslant \frac{(1/3+\varepsilon)\ell}{(1-3\varepsilon_1-d_1)(1-\varepsilon_1)}. \tag{2}$$

Using (1), the expression in (2) is at most $(1/3+d_1)\ell$. Thus, v is adjacent to at least $d_1 L$ uncolored vertices in at least $(2/3-d_1)\ell$ clusters in $V^{(i')}$ for some $i' \neq i$. Hence, each vertex in $V_0^{(i)}$ belongs to at least $(1/3-2d_1)\ell$ clusters.

If a vertex $v \in V_0^{(i)}$ belongs to a cluster $V_j^{(i)}$, then we may *insert v into $V_j^{(i)}$* (or loosely speaking, insert v into S_j) in the following sense. We permanently remove a copy of $K_{h,h,h}$ from G which consists of v, $h-1$ vertices from $V_j^{(i)}$ and h vertices from each of $V_j^{(k)}$, $k \neq i$. Proposition 4.5 guarantees the existence of this $K_{h,h,h}$.

In order to maintain the size of each cluster as a multiple of h, we will bundle exceptional vertices into h-element sets and handle all h vertices from an h-element set at a time as follows.

Claim 4.6 *Given a subset $Y \subseteq V_0^{(i)}$ of at least $3h$ vertices and a subset $U' \subseteq U^{(i)}$ of at least $(1-d_1)\ell$ clusters, there are h vertices of Y that belong to the same cluster C from U'.*

Proof. Suppose instead, that at most $h-1$ vertices of Y belongs to each cluster $C \in U'$. From earlier calculations and the assumption $|U'| \geqslant (1-d_1)\ell$, we know that each vertex of Y belongs to at least $(1/3-3d_1)\ell$ clusters. By double counting the number of pairs (v, C) such that $v \in Y$ belongs to a cluster $C \in U'$, we have

$$3h\left(\frac{1}{3} - 3d_1\right)\ell \leqslant (h-1)\ell, \tag{3}$$

which implies that $9hd_1 \geqslant 1$, contradicting $d_1 \ll 1$. □

Starting from $Y = V_0^{(i)}$ and $U' = U^{(i)}$, we apply Claim 4.6 four times to find four disjoint h-element subsets $W_1^{(i)}, \ldots, W_4^{(i)}$ of $V_0^{(i)}$ whose vertices belong to clusters $C_1^{(i)}, \ldots, C_4^{(i)}$, respectively. The reason why we need *four* h-element sets can be seen below when we apply Lemma 4.7. We can ensure that $C_1^{(i)}, \ldots, C_4^{(i)}$ are different by letting $U' = U^{(i)} \setminus \{C_{j'}^{(i)} : j' < j\}$ when we select $C_j^{(i)}$.

We now insert $W_j^{(i)}$ into $C_j^{(i)}$ for $i = 1, 2, 3$ and $j = 1, 2, 3, 4$ by removing in total $12h$ copies of $K_{h,h,h}$. All of these copies of $K_{h,h,h}$ are removed permanently, they will be a part of the final $K_{h,h,h}$-factor of G. As a result, each $C_j^{(i)}$ has h more vertices than the other two clusters in the same column (unless accidentally more than one $C_j^{(i)}$ fall into the same column).

The Almost-covering Lemma (Lemma 2.2 in [19]) can help us to balance the sizes of each column:

Lemma 4.7 (Almost-covering Lemma [19]) *For any $\alpha > 0$, there exist $\beta > 0$ and m_0, such that the following holds for all $m \geqslant m_0$. Let $R \in \mathcal{G}_3(m)$ be a balanced 3-partite graph with $\bar{\delta}(R) \geqslant (2/3-\beta)m$. Suppose that $\mathcal{T}_0$ is a partial K_3-tiling in R with $|\mathcal{T}| < m-3$. Then, either*

1. *there exists a partial K_3-tiling $\mathcal{T}'$ with $|\mathcal{T}'| > |\mathcal{T}|$ but $|\mathcal{T}' \setminus \mathcal{T}| \leqslant 15$, or*
2. *R is in the extreme case with parameter most α.*

Let $\tilde{G}$ be a new 3-partite graph obtained from adding four new vertices to each vertex class of G_r. The new 12 vertices are clones of the clusters $C_j^{(i)}$ for $i = 1, 2, 3$, $j = 1, 2, 3, 4$, and we denote them by $\tilde{C}_j^{(i)}$. The clones have the same adjacency in G_r as their originals. Let $m = \ell + 4$ be the size of vertex classes in $\tilde{G}$. We have $\bar{\delta}(\tilde{G}) \geqslant (1/3 - 3d_1)m$ following from $\bar{\delta}(G_r) \geqslant (1/3 - 2d_1)\ell$.

We apply Lemma 4.7 to $\tilde{G}$ with $\alpha = \gamma/3$, $\beta = 3d_1$, and $\mathcal{T} = \{S_1, \dots, S_\ell\}$ (then $|\mathcal{T}| < m-3$). The new graph $\tilde{G}$ is almost the same as G_r, provided ℓ is large enough, which we guaranteed when we applied the Regularity Lemma. Thus, $\tilde{G}$ is not in the extreme case (otherwise G_r is in the extreme case). Lemma 4.7 thus provides a larger partial triangle-cover $\mathcal{T}'$ with $|\mathcal{T}' \setminus \mathcal{T}| \leqslant 15$. For each triangle $T \in \mathcal{T}' \setminus \mathcal{T}$, we permanently remove a copy of $K_{h,h,h}$ from the uncolored vertices of T. For each cluster C that is not covered by the larger $\mathcal{T}'$, take an arbitrary set of h uncolored vertices from C and place it into the exceptional set. As result, all the clusters covered by $\mathcal{T}' \cap \mathcal{T}$ experience no changes while all other clusters lose h uncolored vertices; therefore the three clusters in each S_j remain balanced. The net change in each $V_0^{(i)}$ is the same for all i and each loses at least h vertices because $|\mathcal{T}'| > |\mathcal{T}|$.

We repeat the process of creating $W_j^{(i)}$, $C_j^{(i)}$, $\tilde{G}$, and enlarging $\mathcal{T} = \{S_1, \dots, S_\ell\}$ in $\tilde{G}$ by Lemma 4.7 until the number of vertices remaining in each exceptional set is less than $6h$. There is one caveat: If too many vertices are removed from the clusters of S_j, then we will not be able to apply the Blow-up Lemma later. Therefore, we introduce the following notion: If in the entire process, at least $d_1 L/3$ (uncolored) vertices are removed from a cluster C of S_j, then both C and S_j are called *dead* (otherwise *live*). The dead clusters will be not considered until Step 5, after all the exceptional vertices have been removed.

The number of dead cluster-triangles is not very large. To see this, there are three ways for vertices to leave a cluster. First, they are placed in a $K_{h,h,h}$ with a vertex from the exceptional set, so each vertex class $V^{(i)}$ loses at most $\sum_{i=1}^{3} |V_0^{(i)}|h$ vertices in this way. Second, each time when we apply Lemma 4.7, there are at most 15 triangles in $\mathcal{T}' \setminus \mathcal{T}$ and there are a total of $15h$ vertices lost to 15 copies of $K_{h,h,h}$. Third, there are at most 3 clusters not covered by $\mathcal{T}'$ and they could lose $3h$ vertices to the exceptional sets. Since we apply Lemma 4.7 at most $|V_0^{(i)}|/h$ times, the total number of vertices that leave clusters is at most

$$3|V_0^{(i)}|h + \left(|V_0^{(i)}|/h\right)(15h + 3h) = |V_0^{(i)}|(3h + 18) \leqslant 4\varepsilon_1 N(3h + 18).$$

The number of dead cluster triangles is at most

$$\frac{4\varepsilon_1 N(3h+18)}{(d_1/3)L} \leqslant \frac{36(h+6)\varepsilon_1}{d_1(1-\varepsilon_1)}\ell < \frac{d_1}{2}\ell.$$

because $\varepsilon_1 \ll d_1$.

Because the number of dead clusters is not large, in the subgraph induced by live clusters, each cluster is still reachable from every other cluster in the same partition class. Each vertex in $V_0^{(i)}$ belongs to at least $(1/3 - 3d_1)\ell$ live clusters. By letting U' be the set of available live clusters, we still have $|U'| \geqslant (1-d_1)\ell$ when applying Claim 4.6. After removing the edges incident with dead S_j's, the minimum-degree condition in $\tilde{G}$ is still $\bar{\delta}(\tilde{G}) \geqslant (2/3 - 3d_1)m$ and Lemma 4.7 can still be applied.

At the end each cluster (live or dead) has at least $(1-3\varepsilon_1)L - d_1L/3$ uncolored vertices. Each of the three clusters in any S_j has the same number of uncolored vertices, and this number is always divisible by h.

Step 5: Insert the remaining exceptional vertices and apply the Blow-up Lemma

At this stage, the exceptional sets $V_0^{(i)}$, $i = 1,2,3$ are all of the same size, divisible by h and at most $5h$ (because it is less than $6h$). Consider a vertex $x \in V_0^{(1)}$ and insert x into a live cluster $V_j^{(1)}$ to which x belongs (as shown in Step 4, x belongs to at least $(1/3-3d_1)\ell$ live clusters). As a result, $V_j^{(1)}$ loses $h-1$ vertices while $V_j^{(2)}$ and $V_j^{(3)}$ each loses h vertices. To balance S_j, we move a vertex from $V_j^{(1)}$ to $V_1^{(1)}$ following the remark in Step 3. As a result, $V_j^{(1)}$ loses one orange vertex, and $V_1^{(1)}$ gains an extra uncolored vertex. Repeat this to all the vertices in $V_0^{(1)} \cup V_0^{(2)} \cup V_0^{(3)}$. All S_j, $j>1$, have the same number of non-red vertices among its three clusters. The same holds for S_1 because $|V_0^{(1)}| = |V_0^{(2)}| = |V_0^{(3)}|$. In addition, the number of non-red vertices in each cluster is at least $(1-d_1/2)L$, and always a multiple of h.

Then, uncolor all the remaining orange vertices and remove all red copies of $K_{h,h,h}$ from G. Since each cluster now has at least $(1-d_1/2)L$ vertices, by the Slicing Lemma, any pair of clusters in S_j is $(\varepsilon_1/2)$-regular. Furthermore, each vertex in one cluster of S_j is adjacent to at least $(d_1-\varepsilon_1)L - d_1L/2$ vertices in any other cluster of S_j. Hence all pairs in S_j are $(\varepsilon_1/2, d_1/3)$-super-regular. We finally apply the Blow-up Lemma to each S_j to complete the $K_{h,h,h}$-factor of G.

5 Concluding Remarks

- We could reduce the error term γN in Theorem 1.2 to a constant $C = C(h)$ by showing that if $G \in \mathcal{G}_3(N)$ is in the extreme case with sufficiently small γ and

$\bar{\delta}(G) \geqslant 2N/3 + C$, then G contains a $K_{h,h,h}$-factor. Unfortunately, the methods involve a detailed case analysis which is too long to be included in this paper. However, we can summarize them as follows. Given a positive integer h, let $f(h)$ be the smallest m for which there exists an N_0 such that every balanced tripartite graph $G \in \mathcal{G}_3(N)$ with $N \geqslant N_0$, h divides N, and $\bar{\delta}(G) \geqslant m$ contains a $K_{h,h,h}$-factor. Suppose that $N = (6q + r)h$ with $0 \leqslant r \leqslant 5$. Then, from Proposition 1.5 and a manuscript [21] which details the proof of the extreme case:

$$\begin{array}{rcccll} & & f(h) & = & \frac{2N}{3} + h - 1, & \text{if } r = 0; \\ h\left\lceil \frac{2N}{3h} \right\rceil + h - 2 & \leqslant & f(h) & \leqslant & h\left\lceil \frac{2N}{3h} \right\rceil + h - 1, & \text{if } r = 1, 2, 4, 5; \\ \frac{2N}{3} + h - 1 & \leqslant & f(h) & \leqslant & \frac{2N}{3} + 2h - 1, & \text{if } r = 3. \end{array}$$

We have no conjecture as to whether the upper or lower bound is correct.

- The task of obtaining a tight minimum pairwise degree condition for K_r-factors in $\mathcal{G}_r(N)$ becomes more challenging for larger r. The $r = 2$ case is very easy – we either consider a maximum matching or apply the König-Hall theorem. The $r = 3, 4$ cases become hard – [19] and [20] both applied the Regularity Lemma. At present a tight Hajnal–Szemerédi-type result is out of reach (though an approximate version was given by Csaba and Mydlarz [5]).

- We believe one can prove a similar result as Theorem 1.2 for tiling 4-colorable graphs in 4-partite graphs by adopting the approach of [20] and the techniques in this paper. In general, suppose that we know that every r-partite graph $G \in \mathcal{G}_r(n)$ with $\bar{\delta}(G) \geqslant cn$ contains a K_r-factor. Then applying the Regularity Lemma, one can easily prove that for any $\varepsilon > 0$ and any r-colorable H, every $G \in \mathcal{G}_r(n)$ with $\bar{\delta}(G) \geqslant (c + \varepsilon)n$ contains an H-tiling that covers all but εn vertices (this is similar to an early result of Alon and Yuster [1]). However, it is not clear how to reduce the number of leftover vertices to a constant, or zero (to get an H-factor). As seen from the present manuscript, a minimum degree condition for K_r-factors does not immediately gives a similar degree condition for $K_r(h)$-factors, where $K_r(h)$ is the complete r-partite graph with h vertices in each partition set.

- Theorem 1.2 gives a near tight minimum degree condition $\bar{\delta} \geqslant (2/3 + o(1))N$ for $K_{h,h,h}$-tilings. However, the coefficient $2/3$ may not be best possible for other 3-colorable graphs, *e.g.*, $K_{1,2,3}$. In fact, when tiling a general (instead of 3-partite) graph with certain 3-colorable H, the minimum degree threshold given by Kühn and Osthus [17] has coefficient $1 - 1/\chi_{cr}(H)$ instead of $2/3$, where $\chi_{cr}(H)$ is the so-called critical chromatic number. It would be interesting to see if something similar holds for tripartite tiling.

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