Balanced online Ramsey games in random graphs

Anupam Prakash*
Department of Computer Science and Engineering
IIT Kharagpur, India
anupam@berkeley.edu

Reto Spöhel†
Institute of Theoretical Computer Science
ETH Zürich, Switzerland
rspoehel@inf.ethz.ch

Henning Thomas
Institute of Theoretical Computer Science
ETH Zürich, Switzerland
hthomas@inf.ethz.ch

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Abstract

Consider the following one-player game. Starting with the empty graph on n vertices, in every step r new edges are drawn uniformly at random and inserted into the current graph. These edges have to be colored immediately with r available colors, subject to the restriction that each color is used for exactly one of these edges. The player's goal is to avoid creating a monochromatic copy of some fixed graph F for as long as possible.

We prove explicit threshold functions for the duration of this game for an arbitrary number of colors r and a large class of graphs F. This extends earlier work for the case $r = 2$ by Marciniszyn, Mitsche, and Stojaković. We also prove a similar threshold result for the vertex-coloring analogue of this game.

1 Introduction

Consider the following one-player game. Starting with the empty graph on n vertices, in every step r new edges are drawn uniformly at random and inserted into the current graph. These edges have to be colored immediately with r available colors, subject to the restriction that each color is used for exactly one of these edges. The player's goal is to avoid creating a monochromatic copy of some fixed graph F for as long as possible. We call this game the *balanced online F-avoidance edge-coloring game (with r colors)*.

*Now at UC Berkeley, USA. Part of this research was done while the author was visiting ETH Zürich as an intern.

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For which functions $N = N(n)$ can the player 'survive' for N steps a.a.s. (asymptotically almost surely, with probability tending to 1 as n tends to infinity), i.e., avoid creating a monochromatic copy in the first N steps? We say that $N_0 = N_0(n)$ is a *threshold function* for this game if on the one hand a.a.s. the player survives for any $N = o(N_0)$ steps if she uses an appropriate strategy, but on the other hand a.a.s. she will not survive for any $N = \omega(N_0)$ steps, regardless of her strategy.

Note that in the special case $r = 1$ this simply asks about the appearance of the first copy of F in the graph process where edges are drawn uniformly at random and revealed one by one. This problem dates back to the pioneering work of Erdős and Rényi [4] and was eventually solved in full generality by Bollobás [2], who proved the following result. For any graph G, let e_G or $e(G)$ denote its number of edges, and similarly v_G or $v(G)$ its number of vertices. Then the threshold for the appearance of a copy of F is $N_0(F, n) = n^{2-1/m(F)}$, where $m(F) := \max_{H \subseteq F} e_H/v_H$.

Our interest is in the case $r \geq 2$. The game was introduced for the case $r = 2$ by Marciniszyn, Mitsche, and Stojaković [10], who derived explicit threshold functions $N_0(F, n)$ for graphs F satisfying certain properties. For example, their result covers cycles of arbitrary size, but is not applicable to cliques of any size larger than 3. We prove a similar result for the general case when r is an arbitrary fixed integer. Since our methods are different (in fact, more elementary) from those used in [10], we also obtain a more general statement for the case $r = 2$. In order to state our result, we need to introduce some notation. For any graph F, let

$$m_2(F) := \max_{H \subseteq F} \frac{e_H - 1}{v_H - 2} \tag{1}$$

(in this and similar definitions, we assume that the maximum is taken over appropriate subgraphs H [here those satisfying $v_H \geq 3$], and do not worry about graphs with maximum degree 0 or 1, which are not always covered by our definitions). For any graph F and for $r \geq 1$, let

$$\overline{m}_{2b}^r(F) := \max_{H \subseteq F} \frac{r(e_H - 1) + 1}{r(v_H - 2) + 2}. \tag{2}$$

While the parameter m_2 is a standard notation and appears in several known results (cf. the paragraph on related work below), the parameter $\overline{m}_{2b}^r$ has not been used before. The overline indicates the online nature of the game (this is in line with [11, 12, 13]), the 2 indicates that the parameter is related to an *edge*-coloring problem, and the b stands for 'balanced'.

Note that the fraction on the right hand side of (2) is the ratio of edges to vertices in a graph formed by r copies of H that intersect in one edge and are pairwise vertex-disjoint otherwise. It is not hard to see that for F fixed, the parameter $\overline{m}_{2b}^r(F)$ is strictly increasing in r and satisfies

$$\lim_{r \to \infty} \overline{m}_{2b}^r(F) = m_2(F).$$

With these notations, our main result reads as follows.

Theorem 1 (Main result). *Let $r \geq 1$ be fixed, and let F be a graph that has a subgraph $F_- \subset F$ with $e_F - 1$ edges satisfying*

$$m_2(F_-) \leq \overline{m}_{2b}^r(F). \qquad (3)$$

Then the threshold for the balanced online F-avoidance edge-coloring game with r colors is

$$N_0(F, r, n) = n^{2-1/\overline{m}_{2b}^r(F)}.$$

The condition (3) is only used in the upper bound proof. In other words, we show that $n^{2-1/\overline{m}_{2b}^r(F)}$ is a general lower bound that is indeed the threshold of the game provided (3) is satisfied.

Let us state explicitly what Theorem 1 implies for the two most prominent special cases. We denote cycles and cliques of size ℓ by C_ℓ and K_ℓ respectively. If F is a cycle, we obtain the following generalization of the main result from [10].

Corollary 2. *For all $\ell \geq 3$ and $r \geq 1$, the threshold for the balanced online C_ℓ-avoidance edge-coloring game with r colors is*

$$N_0(\ell, r, n) = n^{2-\frac{r(\ell-2)+2}{r(\ell-1)+1}}.$$

For the case where F is a clique, Theorem 1 yields only partial results. Perhaps surprisingly, the desired threshold follows readily if e.g. $\ell = r = 100$, but not in the seemingly simpler case $\ell = 4$, $r = 2$. This is due to the fact that for ℓ fixed, inequality (3) is only satisfied if we choose r large enough.

Corollary 3. *For all $\ell \geq 2$ and $r \geq \ell$, the threshold for the balanced online K_ℓ-avoidance edge-coloring game with r colors is*

$$N_0(\ell, r, n) = n^{2-\frac{r(\ell-2)+2}{r\left(\binom{\ell}{2}-1\right)+1}}.$$

Theorem 1 is not applicable to trees. In fact, it has been shown in [10] that for $F = P_4$ (a path on four edges), the threshold is strictly higher than $n^{2-1/\overline{m}_{2b}^2(P_4)} = n^{6/7}$.

Related work The main motivation for the game studied here comes from an 'unbalanced' game in which the edges are presented one by one and can be colored by one of r available colors without any restriction. Again the goal is to avoid creating a monochromatic copy of some fixed graph F for as long as possible. This game was introduced by Friedgut *et al.* in [5] for the case where $r = 2$ and $F = K_3$, and was further investigated in [12, 13]. There a general result similar to Theorem 1 was proved for the game with two colors. In particular, the thresholds for the cases where F is a cycle or a clique of arbitrary size were found. The thresholds given in Corollaries 2 and 3 are strictly lower than these 'unbalanced' thresholds. For example, if $F = K_3$ and $r = 2$, the threshold is $n^{4/3}$ in the unbalanced game and $n^{6/5}$ in the balanced game.

A priori, this could be due to the fact that the corresponding offline problems are not equally hard. However, it turns out that this is not the case: The graph obtained after N steps of the above unbalanced game is uniformly distributed over all graphs on n vertices with exactly N edges. Thus the offline problem corresponding to the unbalanced game is the following: Given a graph drawn uniformly at random from all graphs on n vertices with N edges, is there an r-edge-coloring avoiding monochromatic copies of F? A classical result by Rödl and Ruciński [15, 16] states that for any number of colors $r \geq 2$, the threshold for this property is $N_0(F, n) = n^{2-1/m_2(F)}$, unless F is a star forest. (This is a simplified version of the full result.)

Similarly, the offline problem corresponding to the balanced game considered in this paper is the following: Given a random graph with rN edges and a random partition of these edges into sets of size r, is there an r-edge-coloring avoiding monochromatic copies of F such that every color is used for exactly one edge from each partition class? It can be shown [8] that for 'most' graphs F the threshold for this problem is also $N_0(F, n) = n^{2-1/m_2(F)}$. Thus the difference in the thresholds of the two games is indeed a result of our online setting and not just inherited from the underlying offline problems.

Another closely related problem was studied first by Krivelevich, Loh, and Sudakov in [7], and solved completely in [14]. As in our game, in every step the player is presented r random edges of the complete graph on n vertices. The difference to our scenario is that she has to keep only one of them and is allowed to discard the remaining $r-1$ edges. This is known in the literature as a *(generalized) Achlioptas process*. Again the question is for how long she can avoid creating a copy of some fixed graph F. Note that this setup can be viewed as a relaxation of the balanced Ramsey game studied here, the relaxation being that the player has only to worry about copies of F in one specific color. The general threshold found in [14] coincides with the formula in Corollary 3 whenever the corollary is applicable. It is an interesting open question whether the two problems have in fact the same threshold for all nonforests F (it is not hard to see that the two thresholds differ if F is e.g. a star). We hope to address this in future work.

The vertex case We now present our results for the vertex case, which has not been studied before. As usual, we denote by $G_{n,p}$ a random graph on n vertices obtained by including each of the $\binom{n}{2}$ possible edges with probability p independently. The setup is as follows: The vertices of a random graph $G_{n,p}$ are revealed to the player r vertices at a time, along with all edges induced by the vertices revealed so far. The r vertices revealed in each step have to be colored immediately with r available colors subject to the restriction that each color is used for exactly one vertex. Again the goal is to avoid a monochromatic copy of some fixed graph F. We call this game the *balanced online F-avoidance vertex-coloring game*. For which densities $p = p(n)$ of the underlying random graph can the player color all n vertices a.a.s.? We say that $p_0 = p_0(n)$ is a threshold function for this game if for $p = o(p_0)$ the player succeeds in coloring all vertices a.a.s. if she uses an appropriate strategy, but for $p = \omega(p_0)$ she fails to do so a.a.s., regardless of her strategy.

We prove the following vertex-coloring analogue to Theorem 1. For any graph F, let

$$m_1(F) := \max_{H \subseteq F} \frac{e_H}{v_H - 1}. \tag{4}$$

For any graph F and for $r \geq 1$, let

$$\overline{m}_{1b}^r(F) := \max_{H \subseteq F} \frac{r e_H}{r(v_H - 1) + 1}. \tag{5}$$

Theorem 4. *Let $r \geq 1$ be fixed, and let F be a graph that has an induced subgraph $F^\circ \subset F$ on $v_F - 1$ vertices satisfying*

$$m_1(F^\circ) \leq \overline{m}_{1b}^r(F). \tag{6}$$

Then the threshold for the balanced online F-avoidance vertex-coloring game with r colors is

$$p_0(F, r, n) = n^{-1/\overline{m}_{1b}^r(F)}.$$

Again the condition (6) is only needed in the upper bound proof. Theorem 4 is applicable to cycles and cliques of arbitrary size, regardless of the number of colors r.

Corollary 5. *For all $\ell \geq 3$ and $r \geq 1$, the threshold for the balanced online C_ℓ-avoidance vertex-coloring game with r colors is*

$$p_0(\ell, r, n) = n^{-\frac{r(\ell-1)+1}{r\ell}}.$$

Corollary 6. *For all $\ell \geq 2$ and $r \geq 1$, the threshold for the balanced online K_ℓ-avoidance vertex-coloring game with r colors is*

$$p_0(\ell, r, n) = n^{-\frac{r(\ell-1)+1}{r\binom{\ell}{2}}}.$$

For the vertex case, the unbalanced game is better understood than for the edge case. In [11], threshold functions for the game with an arbitrary number of colors and a class of graphs including cycles and cliques of arbitrary size were proved.

Let us compare the thresholds of the two games for a very special case: Setting $F = K_2$, we are dealing with proper r-vertex-colorings in the usual sense. While for the balanced game Corollary 6 yields a threshold of $n^{-1-1/r}$, the threshold in the unbalanced game is $n^{-1-1/(2^r-1)}$ [11]. Both exponents converge to -1, which is indeed the exponent of the threshold for proper r-vertex-colorability in an offline setup (see e.g. [1]). Note that the speed of convergence differs dramatically between the two cases.

The proofs Our lower bound proofs rely on the first moment method. In the edge case, we apply it to the number of copies of (constant-size) *r-matched graphs* in the *random r-matched graph $G^r_{n,m}$*. These notions are elementary generalizations of their well-known non-matched counterparts and have applications beyond the present paper (e.g. [8]).

Following [10], our upper bound proofs proceed by two-round exposure and apply counting versions of known offline results to the first round. In the second round we use standard second moment calculations which do not require F to satisfy any balancedness condition (as is needed by the approach pursued in [10]).

For both the edge and vertex case, the extension of the ideas presented in [10] to more than two colors is an application of Hall's well-known theorem about matchings in bipartite graphs (see e.g. [3]).

Organization of this paper For ease of exposition, we settle the somewhat simpler vertex case first. After giving some general preliminaries in Section 2, we prove Theorem 4 in Section 3 and Theorem 1 in Section 4. Both proofs are preceded by their own case-specific preliminary section.

2 General preliminaries

All graphs are simple and undirected. We write $\cong$ to denote graph isomorphism.

We use standard asymptotic notations. We sometimes write $f \ll g$ for $f = o(g)$, and similarly $f \gg g$ for $f = \omega(g)$, $f \lesssim g$ for $f = \mathcal{O}(g)$, $f \gtrsim g$ for $f = \Omega(g)$, and $f \asymp g$ for $f = \Theta(g)$. This is particularly useful in long chains of asymptotic (in-)equalities.

As already mentioned, by $G_{n,p}$ we denote a random graph on n vertices obtained by including each of the $\binom{n}{2}$ possible edges with probability p independently. We denote the underlying vertex set by $\{v_1, \ldots, v_n\}$. By $G_{n,m}$ we denote a graph drawn uniformly at random from all graphs on n vertices with m edges. It is well-known that these two models are asymptotically equivalent if $m = \left\lfloor \binom{n}{2} p \right\rfloor \asymp n^2 p$. We will need the following lemma.

Lemma 7. *Let F be a fixed graph. The expected number of copies of F in $G_{n,p}$ (or $G_{n,m}$ with $m \gg 1$) is of order $n^{v_F} p^{e_F}$ (where $p := mn^{-2}$).*

Proof. Let $\mathrm{Aut}(F)$ denote the number of automorphisms of F. There are

$$\binom{n}{v_F} \frac{v_F!}{\mathrm{Aut}(F)} \asymp n^{v_F}$$

possible copies of F in K_n, and each of them is present in $G_{n,p}$ with probability p^{e_F}, resp. in $G_{n,m}$ with probability

$$\frac{\binom{\binom{n}{2}-e_F}{m-e_F}}{\binom{\binom{n}{2}}{m}} = \frac{m(m-1)\ldots(m-e_F+1)}{\binom{n}{2}(\binom{n}{2}-1)\ldots(\binom{n}{2}-e_F+1)} \asymp (mn^{-2})^{e_F}.$$

□

We state the following proposition for further reference.

Proposition 8. *For a, c, $C \in \mathbb{R}$ and $b > d > 0$, we have*

$$\frac{a}{b} \geq C \ \wedge \ \frac{c}{d} \leq C \quad \Longrightarrow \quad \frac{a-c}{b-d} \geq C.$$

3 Proof of Theorem 4

In this section we prove our results for the vertex case. We start by fixing our notation and stating two results we will need in the upper bound proof.

3.1 Preliminaries

We denote the set of vertices that are revealed in step k, $1 \leq k \leq n/r$, by $S_k := \{v_{(k-1)r+1}, \dots, v_{kr}\}$. Furthermore, we let G_k denote the graph that is visible to the player after step k, i.e., the subgraph of $G_{n,p}$ induced by $\cup_{1 \leq i \leq k} S_i := \{v_1, \dots, v_{kr}\}$. Thus the player's task in step k is to extend the coloring of G_{k-1} to a coloring of G_k without creating a monochromatic copy of F and using each color exactly once.

Janson's inequality is a very useful tool in probabilistic combinatorics. In many cases, it yields an exponential bound on lower tails where the second moment method only gives a bound of $o(1)$. Here we formulate a version tailored to random graphs.

Theorem 9 ([6]). *Consider a family (potentially a multi-set) $\mathcal{F} = \{H_i \mid i \in I\}$ of graphs on the vertex set $\{v_1, \dots, v_n\}$. For each $H_i \in \mathcal{F}$, let X_i denote the indicator random variable for the event $H_i \subseteq G_{n,p}$, and for each pair $H_i, H_j \in \mathcal{F}$, $i \neq j$, write $H_i \sim H_j$ if H_i and H_j are not edge-disjoint. Let*

$$\begin{aligned} X &= \sum_{H_i \in \mathcal{F}} X_i, \\ \mu &= \mathbb{E}[X] = \sum_{H_i \in \mathcal{F}} p^{e(H_i)}, \\ \Delta &= \sum_{\substack{H_i, H_j \in \mathcal{F} \\ H_i \sim H_j}} \mathbb{E}[X_i X_j] = \sum_{\substack{H_i, H_j \in \mathcal{F} \\ H_i \sim H_j}} p^{e(H_i) + e(H_j) - e(H_i \cap H_j)}. \end{aligned}$$

Then for all $0 \leq \delta \leq 1$ we have

$$\Pr[X \leq (1-\delta)\mu] \leq \mathrm{e}^{-\frac{\delta^2 \mu^2}{2(\mu + \Delta)}}.$$

In particular, Janson's inequality yields the following strengthening of a result from [9]. The proof is essentially the one given there.

Theorem 10 ([9]). *Let $r \geq 2$ and F be a nonempty graph. Then there exist positive constants $C = C(F, r)$ and $a = a(F, r)$ such that for $p(n) \geq C n^{-1/m_1(F)}$, where $m_1(F)$ is defined as in (4), the random graph $G_{n,p}$ a.a.s. has the property that in every r-vertex-coloring there are at least $a n^{v_F} p^{e_F}$ monochromatic copies of F.*

3.2 Lower Bound

In this section we show that a simple greedy strategy allows the player to color all vertices without creating a monochromatic copy of F a.a.s. if $p \ll n^{-1/\overline{m}_{1b}^{r}(F)}$. Throughout this section, we fix F and r, and let

$$H = H(F, r) := \arg\max_{H' \subseteq F} \frac{re(H')}{r(v(H') - 1) + 1} \tag{7}$$

(cf. (5)). The *greedy H-avoidance strategy* tries, in every step $k > 0$, to extend the coloring of G_{k-1} to a coloring of G_k without creating a monochromatic copy of H. Any balanced coloring of S_k that avoids monochromatic copies of H is acceptable. If no such coloring exists, the greedy H-avoidance strategy simply gives up (possibly prematurely, as it might still be able to avoid monochromatic copies of F for some time). Clearly, if the greedy H-avoidance strategy is successful, it yields a coloring which contains no monochromatic copy of H and therefore also no monochromatic copy of F.

We now derive a necessary condition for the greedy H-avoidance strategy to fail. This condition is 'static' in the sense that we can decide whether it holds simply by looking at the random graph $G_{n,p}$ on which the game is played before the actual game starts. Recall that $S_k := \{v_{(k-1)r+1}, \ldots, v_{kr}\}$. For $1 \leq k \leq n/r$, we define the event $\mathcal{E}_k$ as follows:

$$\begin{aligned}\mathcal{E}_k :=\{\exists H_1, H_2, \ldots, H_r \subset G_{n,p} : \quad & H_1, H_2, \ldots, H_r \cong H \\ \wedge \quad & |V(H_i) \cap S_k| = 1, \quad 1 \leq i \leq r \\ \wedge \quad & V(H_i) \cap S_k = V(H_j) \cap S_k \Longrightarrow |V(H_i) \cap V(H_j)| = 1, \quad 1 \leq i < j \leq r\}.\end{aligned} \tag{8}$$

In words, the event $\mathcal{E}_k$ occurs if there are at least r copies of H intersecting with S_k in exactly one vertex, such that if two of these copies intersect S_k in the same vertex, they are otherwise disjoint.

Let X_k be the indicator variable for the event $\mathcal{E}_k$, and set $X := \sum_{1 \leq k \leq n/r} X_k$.

Claim 11. *If the greedy H-avoidance strategy fails then $X > 0$.*

Proof. The greedy H-avoidance strategy fails if and only if there is an integer $1 \leq k \leq n/r$ such that in step k the set S_k cannot be colored without creating a monochromatic copy of H. We shall prove that this implies that the event $\mathcal{E}_k$ occurs. For a fixed k, assume that G_{k-1} has already been colored successfully, and consider the bipartite graph B_k with S_k as one partition class and the set $\{1, \ldots, r\}$ of available colors as the other partition class, where a vertex $v \in S_k$ is connected to a color $s \in \{1, \ldots, r\}$ by an edge if and only if assigning color s to v does not create a monochromatic copy of H. By definition, each valid coloring of S_k corresponds to a perfect matching in the bipartite graph B_k.

Hall's Theorem (see e.g. [3]) states that in any bipartite graph $G = (V_1 \dot\cup V_2, E)$, $|V_1| = |V_2|$, a perfect matching exists if and only if the neighborhood of every set $C \subseteq V_1$ has size at least $|C|$. It follows that the graph B_k does not contain a perfect matching if and only if there is a set $C \subseteq S_k$ such that more than $r - |C|$ colors are excluded for all of the vertices in C. That is, each vertex $v \in C$ is contained in $r - |C| + 1$ different copies of H, which pairwise intersect only in v since each of these copies is in a different color. Thus, there are at least $|C| \cdot (r - |C| + 1)$ many copies of H with the properties specified in (8). Since $|C| \cdot (r - |C| + 1) \geq r$ for $1 \leq |C| \leq r$, it follows that $\mathcal{E}_k$ occurs if the greedy H-avoidance strategy fails in step k. $\square$

Next, we show that $\Pr[\mathcal{E}_1] = o(n^{-1})$ if $p \ll n^{-1/\overline{m}^r_{1b}(F)}$. Once this is established, Markov's inequality immediately yields with

$$\mathbb{E}[X] = \sum_{k=1}^{n/r} \mathbb{E}[X_k] = \frac{n}{r} \cdot \Pr[\mathcal{E}_1] = o(1) \tag{9}$$

that $X = 0$ a.a.s., which together with Claim 11 implies that the greedy H-avoidance strategy succeeds a.a.s. This proves the first part of Theorem 4.

Claim 12. *For* $p \ll n^{-1/\overline{m}^r_{1b}(F)}$ *we have* $\Pr[\mathcal{E}_1] = o(n^{-1})$.

Proof. We define the following family of graphs reflecting the definition of $\mathcal{E}_1$ (cf. (8)):

$$\begin{aligned} \mathcal{T} :=\{T = H_1 \cup H_2 \cup \cdots \cup H_r : \quad & H_1, H_2, \ldots, H_r \cong H \\ \wedge \quad & |V(H_i) \cap S_1| = 1, \quad 1 \leq i \leq r \\ \wedge \quad & V(H_i) \cap S_1 = V(H_j) \cap S_1 \Longrightarrow |V(H_i) \cap V(H_j)| = 1, \quad 1 \leq i < j \leq r\}. \end{aligned}$$

Note that this is a family of subgraphs of K_n which in some sense are 'rooted' in the set $S_1 = \{v_1, \ldots, v_r\}$. Clearly, the event $\mathcal{E}_1$ occurs if and only if one of these graphs is present in $G_{n,p}$. For any subgraph G of K_n, we define the set of external vertices of G as $V_{\mathrm{ext}}(G) := V(G) \setminus S_1$ and let $v_{\mathrm{ext}}(G) := |V_{\mathrm{ext}}(G)|$.

Consider a fixed graph $T = H_1 \cup \cdots \cup H_r \in \mathcal{T}$. Here the labeling of the r copies of H is arbitrary but fixed. For $2 \leq i \leq r$, let

$$J'_i := H_i \cap \Big(\bigcup_{j=1}^{i-1} H_j \Big)$$

denote the intersection of the ith copy of H in T with the preceding $i-1$ copies. A standard inductive argument yields that

$$v_{\mathrm{ext}}(T) = \sum_{i=1}^{r} v_{\mathrm{ext}}(H_i) - \sum_{i=2}^{r} v_{\mathrm{ext}}(J'_i) = r(v_H - 1) - \sum_{i=2}^{r} v_{\mathrm{ext}}(J'_i) \tag{10}$$

and analogously

$$e(T) = re_H - \sum_{i=2}^{r} e(J'_i). \tag{11}$$

As H_i contains exactly one vertex from S_1, it follows that J'_i contains at most one vertex from S_1. Moreover, if J'_i indeed contains a vertex $v \in S_1$, then v must be isolated in J'_i: If $\{v, v'\}$ is an edge of J'_i, then it is also an edge of H_j for some $j < i$. However, by definition of T, H_i and H_j cannot have both the vertex $v \in S_1$ and an external vertex v' in common. This proves that v is an isolated vertex in the graph J'_i.

Consequently, we may define for $2 \leq i \leq r$ the graph $J_i := (V_{\mathrm{ext}}(J_i'), E(J_i'))$ obtained by simply removing the vertex $\{v\} = V(J_i') \cap S_1$ from J_i' if it is present. Using that $v(J_i) = v_{\mathrm{ext}}(J_i')$ and $e(J_i) = e(J_i')$, we obtain from (10) and (11) that

$$v_{\mathrm{ext}}(T) = r(v_H - 1) - \sum_{i=2}^{r} v(J_i) \qquad \text{and} \qquad e(T) = re_H - \sum_{i=2}^{r} e(J_i). \tag{12}$$

By our choice of H in (7), and since each J_i is a subgraph of H, we have for $2 \leq i \leq r$ with $v(J_i) > 0$ that

$$\frac{e(J_i)}{v(J_i)} \leq \frac{re(J_i)}{r(v(J_i) - 1) + 1} \leq \frac{re_H}{r(v_H - 1) + 1}. \tag{13}$$

Using (13) and applying Proposition 8 repeatedly, we obtain from (12) that

$$\frac{e(T)}{v_{\mathrm{ext}}(T) + 1} = \frac{re_H - \sum_{i=2}^{r} e(J_i)}{r(v_H - 1) + 1 - \sum_{i=2}^{r} v(J_i)} \geq \frac{re_H}{r(v_H - 1) + 1} = \overline{m}_{1b}^{r}(F). \tag{14}$$

Note that (14) holds with equality if all J_i are empty.

We define the following equivalence relation on $\mathcal{T}$: For $T_1, T_2 \in \mathcal{T}$ we have $T_1 \sim T_2$ if and only if there exists a graph isomorphism $\phi : T_1 \to T_2$ such that the restriction of ϕ to S_1 is the identity. Let $\widetilde{\mathcal{T}} \subseteq \mathcal{T}$ denote a family of representatives for this equivalence relation. Note that the isomorphism class of a given graph $T \in \mathcal{T}$ has size $\Theta(n^{v_{\mathrm{ext}}(T)})$. Moreover, since any member of $\mathcal{T}$ has at most $r(v_H - 1)$ external vertices, the number of isomorphism classes is bounded by a constant only depending on H and r.

Let $X_{\mathcal{T}}$ denote the random variable which counts the number of graphs from $\mathcal{T}$ occurring in $G_{n,p}$. We have

$$\begin{aligned}
\mathbb{E}[X_{\mathcal{T}}] = \sum_{T \in \mathcal{T}} p^{e(T)} &\asymp \sum_{T \in \widetilde{\mathcal{T}}} n^{v_{\mathrm{ext}}(T)} p^{e(T)} \\
&\ll \sum_{T \in \widetilde{\mathcal{T}}} n^{v_{\mathrm{ext}}(T) - e(T)/\overline{m}_{1b}^{r}(F)} \\
&\overset{(14)}{\leq} \sum_{T \in \widetilde{\mathcal{T}}} n^{v_{\mathrm{ext}}(T) - (v_{\mathrm{ext}}(T)+1)} \asymp n^{-1},
\end{aligned}$$

i.e., $\mathbb{E}[X_{\mathcal{T}}] = o(n^{-1})$. Claim 12 now follows from Markov's inequality. □

As already mentioned, the lower bound in Theorem 4 follows with (9) from Claims 11 and 12.

3.3 Upper Bound

In this section we prove that regardless of her strategy the player will a.a.s. be forced to create a monochromatic copy of F if $p \gg n^{-1/\overline{m}_{1b}^{r}(F)}$, provided that there exists an induced subgraph $F^\circ \subset F$ on $v_F - 1$ vertices satisfying (6). We will need this assumption in order to apply Theorem 10 to F°.

Assume that $p \gg n^{-1/\overline{m}_{1b}^r(F)}$ is given. We employ a two-round approach to prove that every strategy a.a.s. fails to color all vertices. We partition the vertex set of $G = G_{n,p}$ into two sets $V_1 := \{v_1, \dots, v_{n/2}\}$ and $V_2 = \{v_{n/2+1}, \dots, v_n\}$ (for simplicity we assume that $2r|n$), and relax the problem to a *two-round offline* problem as follows. In the first round, we generate the random edges induced by V_1 and reveal them all at once. We allow the player to color these edges offline. (In fact, we do not even require this coloring to be balanced or free of monochromatic copies of F.) In the second round, the remaining random edges are generated and revealed, and the vertices of V_2 have to be colored respecting the condition that each of the r colors appears exactly once in each set $S_k = \{v_{(k-1)r+1}, \dots, v_{kr}\}$, $n/2r + 1 \le k \le n/r$. Again we allow the player to see the edges of the second round all at once and color them offline. We will show that a.a.s. the player will create a monochromatic copy of F in this relaxed two-round game.

Suppose now that some coloring of the vertices V_1 has been fixed, and consider the edges between V_1 and V_2. For each color $s \in \{1, \dots, r\}$, this edge set defines a vertex set $\mathrm{Base}(s) \subseteq V_2$ consisting of all vertices in V_2 that complete a copy of F in color s. Obviously, no vertex in $\mathrm{Base}(s)$ may be assigned color s when V_2 is colored. Consequently, if one of the sets S_k, $n/2r + 1 \le k \le n/r$, is contained entirely in $\mathrm{Base}(s)$ for some color s, it is not possible to extend the coloring of V_1 to a coloring of V_2. With this observation at hand, the upper bound in Theorem 4 is an easy consequence of the next claim.

Claim 13. *For F as in Theorem 4 and $p \gg n^{-1/\overline{m}_{1b}^r(F)}$ the following holds. A.a.s. the first round is such that for any fixed coloring of V_1, there exists a color $s_0 \in \{1, \dots, r\}$ such that in the second round we have for every vertex $v \in V_2$ that*

$$\Pr[v \in \mathrm{Base}(s_0)] = \omega(n^{-1/r}).$$

Since for a fixed coloring of V_1, the events $\{u \in \mathrm{Base}(s_0)\}$ and $\{v \in \mathrm{Base}(s_0)\}$ are independent for $u \neq v \in V_2$, Claim 13 yields that for $n/2r + 1 \le k \le n/r$ we have

$$\Pr[S_k \subseteq \mathrm{Base}(s_0)] = \omega(n^{-1/r})^r = \omega(n^{-1}).$$

As moreover the $n/2r$ events $\{S_k \subseteq \mathrm{Base}(s_0)\}$ are mutually independent, the probability that at least one of them occurs is $1 - (1 - \omega(n^{-1}))^{n/2r} = 1 - \mathrm{e}^{-\omega(1)} = 1 - o(1)$. Thus, in the second round the player will create a monochromatic copy of F a.a.s., and Theorem 4 is proved.

It remains to prove Claim 13.

Proof of Claim 13. We will obtain the desired probability for $\{v \in \mathrm{Base}(s_0)\}$ by an application of Theorem 9 to the random edges between V_1 and V_2 generated in the second round. For this calculation to work out we need certain properties to hold for the random graph on V_1 generated in the first round. In the following we specify these properties and prove that they hold a.a.s.

For $p \gg n^{-1/\overline{m}^r_{1b}(F)}$ and any graph $J \subseteq F$ with $v_J \geq 1$, we have

$$n^{v_J} p^{e_J} \gg n^{v_J - e_J/\overline{m}^r_{1b}(F)} \geq n^{v_J - e_J \frac{r(v_J-1)+1}{re_J}} = n^{1-1/r}. \tag{15}$$

Consider a fixed induced subgraph $F^\circ \subset F$ on $v_F - 1$ vertices satisfying (6). Let $s_0 \in \{1, \ldots, r\}$ denote the color for which the number of monochromatic copies of F° in $G[V_1]$ is largest, and let M denote the number of copies in this color. In the following, we label these copies by F_i°, $i = 1, \ldots, M$. By Theorem 10, we a.a.s. have

$$M \gtrsim n^{v(F^\circ)} p^{e(F^\circ)}. \tag{16}$$

For any vertex $v \in V_2$ and $i = 1, \ldots, M$, let $T_{v,i} \subseteq V_1 \times \{v\}$ be a set of potential edges that connect F_i° and v such that they form a copy of F. If there are several such sets, pick one arbitrarily. Thus $|T_{v,i}| = \deg_F(u)$ for all v and i, where u denotes the vertex that was removed from F to obtain F°.

For $v \in V_2$ and any pair of indices i, j, $1 \leq i, j \leq M$, let

$$J_{v,ij} := (F_i^\circ \cup T_{v,i}) \cap (F_j^\circ \cup T_{v,j})$$

denote the graph in which the two potential copies of F intersect. Furthermore, for $J \subseteq F$ let $M_{v,J}$ denote the number of pairs i, j, $i \neq j$, for which $T_{v,i} \cap T_{v,j} \neq \emptyset$ and $J_{v,ij} \cong J$. Note that $M_{v,J}$ is bounded by a constant times the number of subgraphs in $G[V_1]$ formed by two copies of F° intersecting in a copy of $J^\circ := J \cap F^\circ$. (Here the constant accounts for the fact that the same subgraph of $G[V_1]$ might correspond to different overlapping pairs of copies of F°.) By Lemma 7, the expected number of such subgraphs in $G[V_1]$ is of order

$$\begin{aligned} n^{2v(F^\circ)-v(J^\circ)} p^{2e(F^\circ)-e(J^\circ)} &= \left(n^{v(F^\circ)} p^{e(F^\circ)}\right)^2 n^{-v_J+1} p^{-e_J+\deg_J(u)} \\ &\overset{(15)}{\ll} \left(n^{v_F-1} p^{e(F^\circ)}\right)^2 n^{1/r} p^{\deg_J(u)}. \end{aligned}$$

Thus it follows with Markov's inequality that

$$M_{v,J} \ll \left(n^{v_F-1} p^{e(F^\circ)}\right)^2 n^{1/r} p^{\deg_J(u)} \tag{17}$$

a.a.s. Since the number of subgraphs $J^\circ \subseteq F^\circ$ is a constant only depending on F, a.a.s. this bound holds simultaneously for all subgraphs $J \subseteq F$ and all $v \in V_2$ after the first round.

For $v \in V_2$, let

$$\mathcal{F}_v := \{T_{v,1}, \ldots, T_{v,M}\}.$$

Note that $\mathcal{F}_v$ might be a multiset, since the same set of edges may complement distinct monochromatic copies of F° in $G[V_1]$ to a copy of F. For $i = 1, \ldots, M$, let $X_{v,i}$ denote the indicator random variable for the event $T_{v,i} \subseteq G_{n,p}$, and set

$$X_v := \sum_{i=1}^{M} X_{v,i}.$$

Clearly, the vertex v is contained in $\mathrm{Base}(s_0)$ if $X_v \geq 1$. We apply Theorem 9 to the family $\mathcal{F}_v$ in order to obtain a lower bound on the probability of this event. Conditioning on the outcome of the first round as specified, i.e., on (16) and (17), we obtain

$$\mu = \mathbb{E}[X_v] = M \cdot p^{\deg_F(u)} \overset{(16)}{\gtrsim} n^{v_F - 1} p^{e_F} \overset{(15)}{\gg} n^{-1/r}, \tag{18}$$

i.e., $\mu = \omega(n^{-1/r})$, and

$$\begin{aligned}
\Delta &= \sum_{\substack{i,j=1 \\ T_{v,i} \sim T_{v,j}}}^{M} \mathbb{E}[X_{v,i} X_{v,j}] \\
&= \sum_{\substack{J \subseteq F \\ u \in \overline{V}(J)}} \sum_{\substack{i,j=1 \\ T_{v,i} \sim T_{v,j}, J_{v,ij} \cong J}}^{M} \mathbb{E}[X_{v,i} X_{v,j}] \\
&= \sum_{\substack{J \subseteq F \\ u \in \overline{V}(J)}} M_{v,J} \cdot p^{2 \deg_F(u) - \deg_J(u)} \\
&\overset{(17)}{\ll} \left(n^{v_F - 1} p^{e_F}\right)^2 n^{1/r} \overset{(18)}{\lesssim} \mu^2 n^{1/r},
\end{aligned}$$

i.e., $\Delta = o(\mu^2 n^{1/r})$. Therefore, Theorem 9 yields that

$$\Pr[v \notin \mathrm{Base}(s_0)] \leq \Pr\left[X_v = 0\right] \leq \exp\left\{-\frac{\mu^2}{2(\mu + \Delta)}\right\} = \exp\left\{-\omega\left(n^{-1/r}\right)\right\} = 1 - \omega\left(n^{-1/r}\right).$$

□

4 Proof of Theorem 1

In this section we prove our results for the edge case. We start by introducing the notion of r-matched graphs already mentioned in the introduction, and stating a result we will need in the upper bound proof.

4.1 Preliminaries

While in the vertex case, the partition into sets of size r is given by the vertex labels and thus fixed before the game starts, in the edge case it is a random object that emerges over the course of the game. In order to give a convenient formal description we introduce the notion of *r-matched graphs*.

An r-matched graph $F' = (V, \mathcal{K})$ consists of a finite set V of vertices and a set $\mathcal{K}$ of pairwise disjoint sets of edges of cardinality r each. We refer to these as r-sets. We use the notations $V(F')$ and $\mathcal{K}(F')$, and write $E(F') = \bigcup_{K \in \mathcal{K}(F')} K$ to refer to the edge set of the underlying unmatched graph. The graph process that underlies our game can be described by r-matched graphs $(G'_k)_{0\le k\le \binom{n}{2}/r}$, where $G'_0 = (V(K_n), \emptyset)$ and G'_k is obtained from G'_{k-1} by adding an r-set E_k drawn uniformly at random from all remaining edges. Thus, $G'_k = (V(K_n), \{E_1, E_2, \ldots, E_k\})$, and the player's task in step k is to extend the coloring of G'_{k-1} to a coloring of G'_k without creating a monochromatic copy of F. Note that, by symmetry, G'_k is distributed like $G_{n,m}$ with $m = rk$ if we ignore the partition into r-sets (and the order in which the edges appear).

By $G^r_{n,m}$ we denote a random r-matched graph obtained by first generating a normal random graph $G_{n,m}$ and then choosing a random partition of its edge set into sets of size r uniformly at random (w.l.o.g. we assume that m is divisible by r). Again by symmetry, G'_k is distributed like $G^r_{n,m}$ with $m = rk$ if we take into account the partition into r-sets (but ignore the order in which those appear). This allows us to analyze the process $(G'_k)_{0\le k\le \binom{n}{2}/r}$ by studying the 'static' object $G^r_{n,m}$. We will use the following analogue to Lemma 7.

Lemma 14. *Let $r \ge 1$ be a fixed integer, and let $F' = (V, \mathcal{K})$ be a fixed r-matched graph. The expected number of copies of F' in $G^r_{n,m}$ with $m \gg 1$ is of order $n^{|V|}(mn^{-2r})^{|\mathcal{K}|}$.*

Proof. Let $\mathrm{Aut}(F')$ denote the number of automorphisms of F'. There are

$$\binom{n}{|V|}\frac{|V|!}{\mathrm{Aut}(F')} \asymp n^{|V|}$$

possible copies of F' in K_n, and each of them is present in $G^r_{n,m}$ with probability

$$\begin{aligned}
&\frac{\binom{\binom{n}{2}-|\mathcal{K}|r}{m-|\mathcal{K}|r}}{\binom{\binom{n}{2}}{m}} \cdot \frac{1}{\binom{m-1}{r-1}} \cdot \frac{1}{\binom{m-r-1}{r-1}} \cdot \ldots \cdot \frac{1}{\binom{m-(|\mathcal{K}|-1)r-1}{r-1}} \\
&= \frac{(\binom{n}{2} - |\mathcal{K}|r)!}{\binom{n}{2}!} \cdot (r-1)!^{|\mathcal{K}|} \cdot m(m-r)\ldots(m-(|\mathcal{K}|-1)r) \\
&\asymp (n^{-2})^{|\mathcal{K}|r} m^{|\mathcal{K}|} = (mn^{-2r})^{|\mathcal{K}|}.
\end{aligned}$$

□

We will use the following edge-coloring analogue of Theorem 10.

Theorem 15 ([16]). *Let $r \ge 2$ and F be a nonempty graph. Then there exist positive constants $C = C(F,r)$ and $a = a(F,r)$ such that for $m(n) \ge Cn^{2-1/m_2(F)}$, where $m_2(F)$ is defined as in (1), the random graph $G_{n,m}$ a.a.s. has the property that in every r-edge-coloring there are at least $an^{v_F}(m/n^2)^{e_F}$ monochromatic copies of F.*

4.2 Lower Bound

In this section we show that a simple greedy strategy allows the player to survive for any $N \ll n^{2-1/\overline{m}^r_{2b}(F)}$ steps without creating a monochromatic copy of F a.a.s. Throughout this section, we fix F and r, and let

$$H = H(F, r) := \arg\max_{H' \subseteq F} \frac{r(e(H') - 1) + 1}{r(v(H') - 2) + 2} \tag{19}$$

(cf. (2)). Similarly to the vertex case, the *greedy H-avoidance strategy* tries, in every step $k > 0$, to extend the coloring of G'_{k-1} to a coloring of G'_k without creating a monochromatic copy of H. Again it simply gives up if this is not possible. Clearly, if the greedy H-avoidance strategy is successful, it yields a coloring which contains no monochromatic copy of H and therefore also no monochromatic copy of F.

As in the vertex case we define a family of objects (which are r-matched graphs in this case) that represent 'traces of failure' for the H-avoidance strategy, and show that a.a.s. these do not appear in G'_N with N as claimed. This introduces a static view on the problem in the sense that we abstract from the order in which the r-sets appear and only look at the r-matched random graph obtained after N steps of the process. Let

$$\begin{aligned}
\mathcal{T}'_b := \{T'_b = (V, \mathcal{K}) : \exists \widetilde{K} \in \mathcal{K},\ \ T = H_1 \cup H_2 \cup \cdots \cup H_r \subseteq E(T'_b) : \quad & H_1, H_2, \ldots, H_r \cong H \\
\wedge \quad & |E(H_i) \cap \widetilde{K}| = 1, \quad 1 \leq i \leq r \\
\wedge \quad & E(H_i) \cap \widetilde{K} = E(H_j) \cap \widetilde{K}, \quad 1 \leq i < j \leq b \\
\wedge \quad & E(H_k) \cap \widetilde{K} \neq E(H_j) \cap \widetilde{K}, \quad b+1 \leq k \leq r, j \neq k \\
\wedge \quad & |E(H_i) \cap E(H_j)| = 1, \quad 1 \leq i < j \leq b \\
\wedge \quad & T'_b \text{ is inclusion minimal with these properties}\}.
\end{aligned} \tag{20}$$

Every graph $T'_b \in \mathcal{T}'_b$ contains a simple graph T that – similar to the vertex case – consists of r copies of H that intersect with a distinguished r-set $\widetilde{K}$ in exactly one edge. Unlike in the vertex case however, we specify the position of these r copies quite precisely: There is one distinguished edge in $\widetilde{K}$ that intersects with the first b copies of H (which are otherwise edge-disjoint), and $r - b$ edges in $\widetilde{K}$ that close exactly one copy H_k, $b + 1 \leq k \leq r$. The minimality condition excludes the addition of r-sets that do not intersect with T. In particular, it ensures that the class $\mathcal{T}'_b$ is finite.

Let $\mathcal{T}' := \bigcup_{1 \leq b \leq r} \mathcal{T}'_b$, and let X denote the number of copies of r-matched graphs from $\mathcal{T}'$ in G'_N.

Claim 16. *If the greedy H-avoidance strategy fails within N steps then $X > 0$.*

Proof. Suppose that the H-avoidance strategy gives up when coloring E_k. As in the proof of Claim 11 it follows from Hall's Theorem that there exists $C \subseteq E_k$ such that each edge $e \in C$ is contained in $r - |C| + 1$ different copies of H which satisfy $|E(H) \cap E_k| = 1$ and pairwise intersect only in e. Let $b := r - |C| + 1$. It immediately follows that picking all b copies of H containing an arbitary fixed edge $e' \in C$, and one copy of H for each of the remaining $|C| - 1 = r - b$ edges in C, yields a graph T such that, by definition, the union of all r-sets intersecting with T forms a graph in $\mathcal{T}'_b$ in which E_k plays the role of $\widetilde{K}$. $\square$

We show that $\mathbb{E}[X] = o(1)$ provided that $N \ll n^{2-1/\overline{m}^r_{2b}(F)}$. It then follows with the first moment method that $X = 0$ a.a.s. , which implies by Claim 16 that the greedy H-avoidance strategy succeeds a.a.s. This proves the first part of Theorem 1.

Claim 17. *For $N \ll n^{2-1/\overline{m}^r_{2b}(F)}$ we have $\mathbb{E}[X] = o(1)$.*

Proof. Consider a fixed graph $T' \in \mathcal{T}'$. We continue to use the notations introduced above and assume that the numbering of the copies of H is fixed and as in (20) for some $1 \leq b \leq r$. For $2 \leq i \leq r$, we define

$$J_i := H_i \cap \big(\bigcup_{j=1}^{i-1} H_j\big).$$

Recall that the first b copies intersect in one specific edge from the central r-set $\widetilde{K}$ and are edge-disjoint otherwise. In particular, we have for $2 \leq i \leq b$ that $v(J_i) \geq 2$. It follows that

$$|V(T)| = rv_H - \sum_{i=2}^{r} v(J_i) \leq rv_H - 2(b-1) - \sum_{i=b+1}^{r} v(J_i) \tag{21}$$

and

$$|E(T) \setminus \widetilde{K}| = r(e_H - 1) - \sum_{i=b+1}^{r} e(J_i). \tag{22}$$

Recall that T' is the union of all r-sets intersecting $T = H_1 \cup \cdots \cup H_r \subseteq E(T')$. For $K \in \mathcal{K}(T')$ we define the parameter

$$y(K) := |K \cap E(T)|,$$

which counts the number of edges in which K intersects T. Note that

$$y(\widetilde{K}) = 1 + (r - b). \tag{23}$$

In the simplest case, we have $y(K) = 1$ for all other r-sets, but in general T may contain several edges from the same r-set. Setting

$$\begin{aligned} D :&= \sum_{K \in \mathcal{K}(T') \setminus \{\widetilde{K}\}} (y(K) - 1) \\ &= |E(T) \setminus \widetilde{K}| - |\mathcal{K}(T') \setminus \{\widetilde{K}\}| \\ &\overset{(22)}{=} r(e_H - 1) - \sum_{i=b+1}^{r} e(J_i) - (|\mathcal{K}(T')| - 1), \end{aligned} \tag{24}$$

we obtain that

$$|\mathcal{K}(T')| = r(e_H - 1) + 1 - \sum_{i=b+1}^{r} e(J_i) - D. \tag{25}$$

This is essentially the analogue of (12), with D playing the role of a correction term. Note that $D = 0$ if indeed $y(K) = 1$ for all $K \in \mathcal{K}(T') \setminus \{\widetilde{K}\}$.

Next we derive an upper bound on the number of vertices in T'. In addition to the vertices from T, T' contains vertices that are incident to edges outside T. Since each such edge contributes at most 2 vertices, this number is bounded by $\sum_{K\in\mathcal{K}(T')} 2(r - y(K))$. Thus we have

$$\begin{aligned}
|V(T')| &\leq |V(T)| + \sum_{K\in\mathcal{K}(T')} 2(r - y(K)) \\
&\overset{(23)}{=} |V(T)| + 2(b-1) + 2 \sum_{K\in\mathcal{K}(T')\setminus\{\widetilde{K}\}} \big((r-1) - (y(K)-1)\big) \\
&\overset{(24)}{=} |V(T)| + 2(b-1) + (|\mathcal{K}(T')| - 1)\cdot 2(r-1) - 2D \\
&\overset{(21)}{\leq} r(v_H - 2) + 2 - \sum_{i=b+1}^{r} v(J_i) + |\mathcal{K}(T')| \cdot 2(r-1) - 2D.
\end{aligned} \tag{26}$$

For $b+1 \leq i \leq r$ with $v(J_i) > 0$, we have

$$\frac{e(J_i)}{v(J_i)} \leq \overline{m}_{2b}^{r}(F). \tag{27}$$

This is trivially true if $e(J_i)/v(J_i) < 1/2$, and follows from our choice of H in (19) and

$$\frac{e(J_i)}{v(J_i)} \overset{\text{Prop. 8}}{\leq} \frac{re(J_i) - (r-1)}{rv(J_i) - 2(r-1)} = \frac{r(e(J_i)-1)+1}{r(v(J_i)-2)+2} \leq \frac{r(e_H-1)+1}{r(v_H-2)+2} = \overline{m}_{2b}^{r}(F),$$

otherwise. Using (27) and applying Proposition 8 repeatedly, we obtain from (25) and (26) that

$$\begin{aligned}
\frac{|\mathcal{K}(T')|}{|V(T')| - |\mathcal{K}(T')|\cdot 2(r-1)} &\geq \frac{r(e_H-1)+1-\sum_{i=b+1}^{r} e(J_i) - D}{r(v_H-2)+2-\sum_{i=b+1}^{r} v(J_i) - 2D} \\
&\geq \frac{r(e_H-1)+1}{r(v_H-2)+2} = \overline{m}_{2b}^{r}(F).
\end{aligned} \tag{28}$$

for any $T' \in \mathcal{T}'$.

We can now estimate the expectation of X. Observing that G'_N is distributed like

$G^r_{n,m}$ with $m = rN$, we obtain with Lemma 14 that

$$\begin{aligned}
\mathbb{E}[X] &\asymp \sum_{T' \in \mathcal{T}'} n^{|V(T')|} (Nn^{-2r})^{|\mathcal{K}(T')|} \\
&= \sum_{T' \in \mathcal{T}'} n^{|V(T')| - |\mathcal{K}(T')| \cdot 2(r-1)} (N/n^2)^{|\mathcal{K}(T')|} \\
&\ll \sum_{T' \in \mathcal{T}'} n^{|V(T')| - |\mathcal{K}(T')| \cdot 2(r-1) - |\mathcal{K}(T')| / \overline{m}^r_{2b}(F)} \\
&\overset{(28)}{\leq} \sum_{T' \in \mathcal{T}'} n^{|V(T')| - |\mathcal{K}(T')| \cdot 2(r-1) - (|V(T')| - |\mathcal{K}(T')| \cdot 2(r-1))} \asymp 1,
\end{aligned}$$

i.e., $\mathbb{E}[X] = o(1)$. □

As already mentioned, the lower bound in Theorem 1 follows with the first moment method from Claims 16 and 17.

4.3 Upper Bound

In this section we prove that regardless of her strategy the player will be forced to close a monochromatic copy of F within any $N \gg n^{2-1/\overline{m}^r_{2b}(F)}$ steps a.a.s., provided that there exists a subgraph $F_- \subset F$ with $e_F - 1$ edges satisfying (3). We will need this assumption in order to apply Theorem 15 to F_-.

As in the vertex case, we employ a two-round approach to prove that every strategy a.a.s. fails to color N edges. Specifically, we prove an upper bound for the following relaxed two-round offline problem: In the first round, the first $N/2$ random edges are presented all at once and may be colored arbitrarily. In the second round, the remaining random edges appear partitioned in sets E_k of size r uniformly at random, and have to be colored respecting the condition that each of the r colors appears exactly once in each set E_k. Again we allow the player to see these edges all at once. We will show the following:

Claim 18. *For F as in Theorem 1 and $N \gg n^{2-1/\overline{m}^r_{2b}(F)}$ the following holds. A.a.s. the first round is such that for any fixed coloring of its edges, there exists a color $s_0 \in \{1, \ldots, r\}$ such that a.a.s. every coloring of the edges of the second round creates a copy of F in color s_0.*

Proof. Our proof strategy is similar to the vertex case. We show that a.a.s. the first round satisfies a set of properties such that, conditioned on these properties, calculating the variance of a suitably defined random variable in the second round yields the desired result.

Let $N \gg n^{2-1/\overline{m}^r_{2b}(F)}$ be given. W.l.o.g. we assume that $N \ll n^2$. In the following we sometimes use the parameter $p = N/n^2 \gg n^{-1/\overline{m}^r_{2b}(F)}$ to improve readability. Note that for any subgraph $J \subseteq F$ with $v_J \geq 2$ we have

$$n^{v_J} p^{e_J} = \left(n^{r(v_J-2)+2} p^{r(e_J-1)+1} \cdot (n^2 p)^{r-1}\right)^{1/r} \overset{(2)}{\gg} N^{(r-1)/r} = N^{1-1/r}. \tag{29}$$

We write $r \cdot G$ to refer to the union of r disjoint copies of some graph G. Let F_- denote an arbitrary fixed subgraph with $e_F - 1$ edges satisfying (3), and let $s_0 \in \{1, \ldots, r\}$ denote the color for which the number of monochromatic copies of $r \cdot F_-$ in the coloring of the first round is largest. Using that $m_2(r \cdot F_-) = m_2(F_-)$, we obtain from Theorem 15 that a.a.s. there are $\Omega(n^{v(r \cdot F_-)} p^{e(r \cdot F_-)})$ many such copies. Moreover, Lemma 7 yields with Markov's inequality and $p \ll 1$ that a.a.s. all but a negligible fraction of these copies are induced. In particular, the edges that complete them to copies of F have not yet appeared. Thus, denoting the number of induced copies of $r \cdot F_-$ in color s_0 by M we a.a.s. have

$$M \gtrsim n^{v(r \cdot F_-)} p^{e(r \cdot F_-)}. \tag{30}$$

For the second round, we condition on this event and label these copies of $r \cdot F_-$ by $[r \cdot F_-]_1, [r \cdot F_-]_2, \ldots, [r \cdot F_-]_M$. By $T_1, T_2, \ldots, T_M \subseteq E(K_n)$ we denote edge sets that complete them to copies of $r \cdot F$; i.e., for $1 \leq i \leq M$ we have $|T_i| = r$ and

$$\Big(V([r \cdot F_-]_i), E([r \cdot F_-]_i) \,\dot{\cup}\, T_i\Big) \cong r \cdot F.$$

Note that depending on F the T_i's need not be unique. In such a case we fix one of the possible sets arbitrarily. Clearly, if one of the sets $T_1, T_2, \ldots, T_M$ appears as one of the sets E_k of the second round, the player is forced to close a copy of F in color s_0. In the following we prove that this will a.a.s. happen.

For each set T_i we introduce an indicator variable Z_i for the event that T_i appears as an edge set E_k of the second round. Note that there might exist $i \neq j$ with $T_i = T_j$ and thus also $Z_i \equiv Z_j$. We set $Z := \sum_{i=1}^{M} Z_i$ and use the second moment method to show that $Z > 0$ holds a.a.s. Due to our assumption that $N \ll n^2$, we still have $\Theta(n^2)$ edges to choose from in the second round. Since $Z_i = 1$ if and only if the strategy is presented T_i as one of the $N/2$ edge sets of the second round, we obtain

$$\Pr[Z_i = 1] = N/2 \cdot \binom{\Theta(n^2)}{r}^{-1} \asymp N n^{-2r} \tag{31}$$

and, conditioning on (30),

$$\begin{aligned} \mathbb{E}[Z] &\asymp M \cdot N n^{-2r} \\ &\overset{(30)}{\gtrsim} n^{v(r \cdot F_-)} p^{e(r \cdot F_-)} \cdot N n^{-2r} \\ &= (n^{v_F} p^{e_F})^r \cdot p^{-r} \cdot N n^{-2r} \\ &\overset{(29)}{\gg} N^{r-1} \cdot (n^2/N)^r \cdot N n^{-2r} \;\; = \;\; 1 \;\; , \end{aligned} \tag{32}$$

i.e., $\mathbb{E}[Z] = \omega(1)$.

It remains to show that $\mathrm{Var}[Z] = o(\mathbb{E}[Z]^2)$. Since for i, j with $T_i \cap T_j = \emptyset$ the random variables Z_i and Z_j are negatively correlated, we have

$$\mathrm{Var}[Z] \leq \sum_{\substack{i,j=1 \\ T_i \cap T_j \neq \emptyset}}^{M} (\mathbb{E}[Z_i Z_j] - \mathbb{E}[Z_i]\mathbb{E}[Z_j]) \leq \sum_{\substack{i,j=1 \\ T_i \cap T_j \neq \emptyset}}^{M} \mathbb{E}[Z_i Z_j] = \sum_{\substack{i,j=1, \\ T_i = T_j}}^{M} \mathbb{E}[Z_i Z_j]. \tag{33}$$

Here the last equation follows from the observation that $Z_iZ_j \equiv 0$ if T_i and T_j share some but not all edges, since the r-sets appearing in the second round are pairwise disjoint.

We are thus left to deal with pairs i, j for which $T_i = T_j$, which immediately implies $Z_i \equiv Z_j$ and hence $\mathbb{E}[Z_iZ_j] = \Pr[Z_i = 1]$. Note that for such pairs we have $[r \cdot F_-]_i \cap [r \cdot F_-]_j \neq \emptyset$ since the two copies overlap at least in the $2r$ endvertices of the edges of $T_i = T_j$. We let $J_{i,j} := [r \cdot F_-]_i \cap [r \cdot F_-]_j$ denote the intersection graph.

For $J \subseteq r \cdot F_-$ let M_J denote the number of pairs i, j for which $T_i = T_j$ and $J_{i,j} \cong J$. Up to a constant factor, M_J counts the number of s_0-colored subgraphs consisting of two copies of $r \cdot F_-$ which form identical 'threats' and intersect in a copy of J. Note that J contains the endvertices of the $2r$ edges that complete $r \cdot F_-$ to $r \cdot F$, i.e., setting $T = E(r \cdot F) \setminus E(r \cdot F_-)$ we have $V(T) \subseteq V(J)$. This allows us to add the edges of T to J resulting in a graph J^+ with $v(J^+) = v(J)$ and $e(J^+) = e(J) + r$. Moreover, the graph J^+ can be split into r components $J_1^+, J_2^+, \ldots, J_r^+$ such that $J_i^+ \subseteq F$ for all $1 \leq i \leq r$. By Lemma 7, the expected number of subgraphs of the considered type formed by the edges of the first round is of order

$$\begin{aligned} n^{2v(r\cdot F_-)-v_J} p^{2e(r\cdot F_-)-e_J} &= \left(n^{v(r\cdot F_-)} p^{e(r\cdot F_-)}\right)^2 \cdot n^{-v(J^+)} p^{-e(J^+)+r} \\ &= \left(n^{v(r\cdot F_-)} p^{e(r\cdot F_-)}\right)^2 \cdot \prod_{i=1}^{r} \left(n^{-v(J_i^+)} p^{-e(J_i^+)}\right) \cdot p^r \\ &\overset{(29)}{\ll} \left(n^{v(r\cdot F_-)} p^{e(r\cdot F_-)}\right)^2 \cdot N^{-r+1} \cdot (N/n^2)^r \\ &= \left(n^{v(r\cdot F_-)} p^{e(r\cdot F_-)}\right)^2 \cdot N n^{-2r}. \end{aligned}$$

Thus it follows with Markov's inequality that

$$M_J \ll \left(n^{v(r\cdot F_-)} p^{e(r\cdot F_-)}\right)^2 \cdot N n^{-2r} \tag{34}$$

a.a.s. Since the number of subgraphs $J \subseteq r \cdot F_-$ is a constant only depending on F and r, this bound holds a.a.s. for all subgraphs J with $V(T) \subseteq V(J)$ simultaneously. Conditioning on this being the case, we may continue (33) as follows:

$$\begin{aligned} \mathrm{Var}[Z] &\leq \sum_{\substack{i,j=1 \\ T_i = T_j}}^{M} \mathbb{E}[Z_iZ_j] \\ &= \sum_{\substack{J \subseteq r\cdot F_- \\ V(T)\subseteq V(J)}} \sum_{\substack{i,j=1 \\ T_i = T_j, J_{i,j}\cong J}}^{M} \Pr[Z_i = 1] \\ &\overset{(31)}{\asymp} \sum_{\substack{J \subseteq r\cdot F_- \\ V(T)\subseteq V(J)}} M_J \cdot N n^{-2r} \\ &\overset{(34)}{\ll} \left(n^{v(r\cdot F_-)} p^{e(r\cdot F_-)}\right)^2 \cdot (N n^{-2r})^2 \overset{(32)}{\lesssim} \mathbb{E}[Z]^2, \end{aligned} \tag{35}$$

i.e., $\mathrm{Var}[Z] = o(\mathbb{E}[Z]^2)$. Conditioning on the outcome of the first round as specified (i.e., M and all M_J satisfying (30) and (34), respectively), it follows with the second moment method from (32) and (35) that in the second round a.a.s. we have $Z > 0$. In other words, a.a.s. at least one of the edge sets $T_1, T_2, \ldots, T_M$ appears during the second round, and the coloring of the edges of the first round can not be extended. $\square$

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