Another product construction for large sets of resolvable directed triple systems

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Abstract

A large set of resolvable directed triple systems of order v, denoted by $\mathrm{LRDTS}(v)$, is a collection of $3(v-2)$ $\mathrm{RDTS}(v)$s based on v-set X, such that every transitive triple of X occurs as a block in exactly one of the $3(v-2)$ $\mathrm{RDTS}(v)$s. In this paper, we use DTRIQ and LR-design to present a new product construction for $\mathrm{LRDTS}(v)$s. This provides some new infinite families of $\mathrm{LRDTS}(v)$s.

1 Introduction

Let X be a v-set. In what follows, an *ordered pair* of X is always an ordered pair (x, y), where $x \neq y \in X$. A *transitive triple* on X is a set of three ordered pairs (x, y), (y, z) and (x, z) of X, which is denoted by (x, y, z).

A *directed triple system* of order v, denoted by $\mathrm{DTS}(v)$, is a pair $(X, \mathcal{B})$ where $\mathcal{B}$ is a collection of transitive triples on X, called *blocks*, such that each ordered pair of X occurs in exactly one block of $\mathcal{B}$. A $\mathrm{DTS}(v)$ is called *resolvable* and is denoted by $\mathrm{RDTS}(v)$ if its blocks can be partitioned into subsets (called *parallel classes*), each containing every element of X exactly once.

A *large set* of directed triple systems of order v, denoted by $\mathrm{LDTS}(v)$, is a collection of $3(v-2)$ $\mathrm{DTS}(v)$s based on X such that every transitive triple from X occurs as a block in exactly one of the $3(v-2)$ $\mathrm{DTS}(v)$s. Existence results for LDTSs and RDTSs are well known from [1, 9].

Theorem 1.1 (1) *There exists an* $\mathrm{LDTS}(v)$ *if and only if* $v \equiv 0, 1 \pmod 3$ *and* $v \geqslant 3$.
(2) *There exists an* $\mathrm{RDTS}(v)$ *if and only if* $v \equiv 0 \pmod 3$, $v \geqslant 3$ *and* $v \neq 6$.

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A large set of disjoint RDTS(v)s is denoted by LRDTS(v). The existence of LRDTS(v)s has been investigated by Kang [8], Kang and Lei [10], Kang and Tian [11], Kang and Xu [12], Kang and Zhao [13], Xu and Kang [17] and Zhou and Chang [22, 23]. By their research and related results about large sets of Kirkman triple systems [3, 4, 5, 6, 14, 15, 18, 19, 20, 21], we can list the known results as follows.

Theorem 1.2 *There exists an* LRDTS(v) *for the following orders* v :

(1) $v = 3^k m$, *where* $k \geqslant 1$ *and* $m \in \{1, 4, 5, 7, 11, 13, 17, 23, 25, 35, 37, 41, 43, 47, 53, 55, 57, 61, 65, 67, 91, 123\} \cup \{2^{2r+1}25^s + 1 : r \geqslant 0, s \geqslant 0\}$.

(2) $v = 7^k + 2,\ 13^k + 2,\ 25^k + 2,\ 2^{4k} + 2$ *and* $2^{6k} + 2$, *where* $k \geqslant 0$.

(3) $v = 12(t + 1)$, *where* $t \in \{0, 1, 2, 3, 4, 6, 7, 8, 9, 14, 16, 18, 20, 22, 24\}$.

(4) $v = 6t + 3$, *where* $t \in \{35, 38, 46, 47, 48, 51, 56, 60\}$.

(5) $v = (3\prod_{i=1}^{p}(2q_i^{r_i} + 1)\prod_{j=1}^{q}(4^{s_j} - 1))$, *where* $p + q \geqslant 1, r_i, s_j \geqslant 1$ *and prime power* $q_i \equiv 7 \ (mod\ 12)$.

Also, if there exists an LRDTS(v), *then there exists an* LRDTS($(2 \cdot s^k + 1)v$) *for any* $k \geqslant 0$, $s = 7, 13$ *and* $v \equiv 0, 3, 9 \ (mod\ 12)$.

A *group-divisible design* (briefly GDD) is a triple $(X, \mathcal{G}, \mathcal{B})$ with the following properties: (i) X is a finite set of points; (ii) $\mathcal{G}$ is a partition of X into subsets called groups; (iii) $\mathcal{B}$ is a set of subsets of X (called blocks) such that a group and a block contain at most one common point, and any pair of points from distinct groups occur in exactly one block of $\mathcal{B}$. A GDD $(X, \mathcal{G}, \mathcal{B})$ is called *resolvable*, denoted by RGDD, if there exists a partition $\Gamma = \{P_1, P_2, \cdots, P_r\}$ of $\mathcal{B}$ such that each part P_i (called *parallel classes*) is a partition of X.

A GDD is called a *transversal design* if it has exactly k groups of size n and every block has size k. We denoted such a GDD by TD(k, n). A TD is called *resolvable* (denoted by RTD) if it is a RGDD.

A GDD $(X, \mathcal{G}, \mathcal{B})$ is called a *Steiner triple system* if $|X| = v$ and it has v groups of size 1 and every block has size 3. Such a GDD is denoted briefly by STS(v) $(X, \mathcal{B})$. A resolvable STS(v) is called a *Kirkman triple system* and denoted by KTS(v).

A large set of Kirkman triple system of order v, denoted by LKTS(v), is a collection of $v - 2$ KTS(v)s based on a v-set X, such that each triple from X occurs in exactly one of the $v - 2$ KTS(v)s. In a KTS(v), if we replace any triple $\{x, y, z\}$ by three collections of transitive triples $\{(x, y, z), (z, y, x)\}$, $\{(y, z, x), (x, z, y)\}$ and $\{(z, x, y), (y, x, z)\}$, then we obtain three RDTS(v)s. It is obvious that the existence of an LKTS(v) implies the existence of an LRDTS(v). However, this approach can provide only odd orders of v since the existence of a KTS(v) implies $v \equiv 3 \pmod 6$. The existence of LKTS(v)s, known as the general Sylvester's problem of the 15 schoolgirls, has a long history [3]. Some orders in Theorem 1.2 come from the existence of LKTS(v)s.

The main result of this paper is to give a new product construction for LRDTSs. This provides some new infinite families of LRDTS(v)s. In Section 2, we give some concepts such as TRIQ(v), DTRIQ(v) and LR(u), etc. In Section 3, we make use of DTRIQ(v) and LR(u) to present a new product construction. In Section 4, we give new orders for LRDTS(v)s.

2 Definitions

A quasigroup is a pair $(X, \circ)$, where X is a set and $(\circ)$ is a binary operation on X such that the equation $a \circ x = b$ and $y \circ a = b$ are uniquely solvable for every pair of elements a, b in X. The order of a quasigroup $(X, \circ)$ is the size of X.

A quasigroup of order v is called *idempotent* if the identity $x \circ x = x$ holds for all x in X. An idempotent quasigroup of order v is denoted by IQ(v). A quasigroup of order v is called *symmetric* if the identity $x \circ y = y \circ x$ holds for every pair of elements x, y in X. An symmetric quasigroup of order v is denoted by SQ(v).

A quasigroup $(X, \circ)$ is called *resolvable* if all $v(v-1)$ pairs of distinct elements can be partitioned into subsets $T_i, 1 \leqslant i \leqslant 3(v-1)$, such that every $\{(x, y, x \circ y) : (x, y) \in T_i\}$ is a partition of X. An idempotent quasigroup IQ(v) is called *first transitive* if there exists a group of order v acting transitively on X which forms an automorphism group of the IQ(v). A first transitive resolvable IQ(v) is denoted by TRIQ(v). A first transitive resolvable symmetric IQ(v) is denoted by TRISQ(v).

For an idempotent quasigroup $(Y, \circ)$ and for each ordered pair (i, j), $i \neq j \in \{0, 1, 2\}$, define a collection of transitive triples from $\{i, j\} \times Y$ as follows.

$T(i,j) = \bigcup_{x \neq y \in Y} t(x, y, x \circ y)$, where

$t(x,y,x\circ y) = \{((i,x),(i,y),(j,x\circ y)),((i,x),(j,x\circ y),(i,y)),((j,x\circ y),(i,x),(i,y))\}$.

An idempotent quasigroup $(Y, \circ)$ is called *second transitive* provided that $T(i, j)$ can be partitioned into three sets $T_0(i, j)$, $T_1(i, j)$ and $T_2(i, j)$ such that

i) the three transitive triples in $t(x, y, x \circ y)$ belong to T_0, T_1 and T_2, respectively;

ii) if $a \neq b \in Y$, each of the ordered pairs $((i, a), (j, b))$ and $((j, b), (i, a))$ belongs to exactly one transitive triple in each of $T_0(i, j)$, $T_1(i, j)$ and $T_2(i, j)$.

An IQ(v) with both first and second transitivity is called *doubly transitive* and is denoted by DTRIQ(v). In [22], Zhou and Chang gave the following existence result.

Lemma 2.1 *There exists a* DTRIQ(v) *for any positive integer* $v \equiv 0, 3, 9 \pmod{12}$.

Transitive IQ has been used to give a tripling construction for large sets of STSs in Teirlinck [16]. To consider the similar problem for large sets of KTSs and large sets of RDTSs, we demand that the transitive IQ must have certain property of resolvability. TRISQ(v) was used to construct LKTSs [20]. DTRIQ(v) was used to construct LRDTSs [22, 23].

In [14], Lei introduce a kind of combinatorial design named LR-design, denoted by LR(u). An LR(u) is a collection $\{(X, \mathcal{A}_k^j) : 1 \leqslant k \leqslant \frac{u-1}{2}, j = 0, 1\}$, where each $(X, \mathcal{A}_k^j)$ is a KTS(u) based on u-set X and $\{A_k^j(h); 1 \leqslant h \leqslant \frac{u-1}{2}\}$ is a *resolution* (collection of parallel classes) of $\mathcal{A}_k^j$ with the properties.

i) $\bigcup_{k=1}^{\frac{u-1}{2}} A_k^0(1) = \bigcup_{k=1}^{\frac{u-1}{2}} A_k^1(1) = \mathcal{A}$ forms a KTS(u) over X too;

ii) Any triple from X is contained in $\bigcup_{k=1}^{\frac{u-1}{2}} \bigcup_{j=0}^{1} \mathcal{A}_k^j$.

Lei [14] and Ji and Lei [7] obtained some existence results for LR(u).

Lemma 2.2 *[14, 7] There exists an* $\mathrm{LR}(u)$ *for* $u = 3^n,\ 2\cdot 13^n+1$ *and* $2\cdot 7^n+1$, *where* $n \geqslant 1$.

Recently, using these auxiliary designs and their existence, Chang et al. [22, 23] proved the following conclusions.

Lemma 2.3 *[22] If there exist both a* $\mathrm{DTRIQ}(v)$ *and an* $\mathrm{LRDTS}(v)$, *then there exists an* $\mathrm{LRDTS}(3v)$.

Lemma 2.4 *[23] If there exist an* $\mathrm{LRDTS}(v)$, *a* $\mathrm{DTRIQ}(v)$ *and an* $\mathrm{LR}(u)$, *then there exists an* $\mathrm{LRDTS}(uv)$.

Next, we introduce the concept of complete mapping in a finite group. We follow the definition in Denes and Keedwell [2].

A complete mapping of a group $(G,\cdot)$, is a bijection mapping $x \to \theta(x)$ of G upon G, such that the mapping $\eta(x) = x\cdot\theta(x)$ is also a bijection mapping of G upon G. The following existence results were stated in [2].

Lemma 2.5 *[2] If* G *is an arbitrary group of order* $n = 4k+2$, *then* G *has no complete mapping. If* G *is an abelian group of order* $n \neq 4k+2$, *then* G *does have a complete mapping.*

Let $X = \{0,1,\cdots,v-1\}$ and $(X,\circ)$ be an idempotent quasigroup with a sharply transitive automorphism group G written multiplicatively. It is easy to see that there is a unique $g\in G$ such that $g(x)=y$ for every pair of elements x,y in X. Let the first row of $(X,\circ)$ be of the following ordered triples:

$$(0,h(0),h^*(0)),\ h\in G.$$

Then $h\mapsto h^*$ is a bijection between G, denoted by Φ. Hence, $(g(0),gh(0),gh^*(0)), g,h\in G$ forms the quasigroup $(X,\circ)$.

Then $(g,gh,gh^*), g,h\in G$ is a latin square on G, which implies that

$$\{(gh,gh^*) : g,h\in G\} = G\times G.$$

So, we have

$$\{h(h^*)^{-1} : h\in G\} = G \tag{1}$$

Note that the mapping $\overline{\Phi} : h\mapsto (h^*)^{-1}$ is also a bijection between G. By the definition of complete mapping and formula (1), $\overline{\Phi}$ is a complete mapping of G. Next we record the result as follows.

Lemma 2.6 *If there exists a transitive* IQ *with* G *as a sharply transitive automorphism group, then* G *has a complete mapping.*

3 A new product construction for LRDTS

Let $X = \{0, 1, \cdots, v-1\}$ and $(X, \circ)$ be an idempotent quasigroup with a sharply transitive automorphism group $G = \{\sigma_0, \sigma_1, \cdots, \sigma_{v-1}\}$. By Lemma 2.6, G has a complete mapping, say, Φ^{-1}. Let $\sigma^* = \Phi(\sigma)$ for $\sigma \in G$. Then, by the definition of complete mapping, we have

$$\{\sigma(\sigma^*)^{-1} : \sigma \in G\} = G \qquad (2)$$

Theorem 3.1 *If there exist an* LRDTS$(3v)$*, a* DTRIQ(v) *and an* LR(u)*, then there exists an* LRDTS(uv)*.*

Proof. Suppose that X is a set of size u with a linear order " $<$ " (i.e. for any $x \neq y$, $x, y \in X$, either $x < y$ or $y < x$). We have an LR(u) over X with the following collection of $u-1$ KTS(u)

$$\{(X, \mathcal{A}_k^l) : \ 1 \leqslant k \leqslant \frac{u-1}{2}, \ l = 0, 1\}$$

which with following properties:

(i) Let the resolution of $\mathcal{A}_k^l$ be $\Gamma_k^l = \{A_k^l(h) : 1 \leqslant h \leqslant \frac{u-1}{2}\}$, and

$$\bigcup_{k=1}^{\frac{u-1}{2}} A_k^0(1) = \bigcup_{k=1}^{\frac{u-1}{2}} A_k^1(1) = \mathcal{A},$$

$(X, \mathcal{A})$ is a KTS(u).

(ii) For any triple $T = \{x, y, z\} \subset X$, $x \neq y \neq z \neq x$, there exist k, l such that $T \in \mathcal{A}_k^l$.

Furthermore, suppose that Y is a set of size v. So we have a DTRIQ(v) over Y. Let $(Y, \circ)$ be a DTRIQ(v), $G = \{\sigma_0, \sigma_1, \cdots, \sigma_{v-1}\}$ be the transitive automorphism group of $(Y, \circ)$. We will construct an LRDTS(uv) on the point set $X \times Y$. The construction proceeds in 2 steps.

Step 1: For any $\{x, y, z\} \subseteq X$, $\{x, y, z\} \in \mathcal{A} = \bigcup\limits_{k=1}^{\frac{u-1}{2}} A_k^0(1)$.

(1). If $\{x, y, z\} \in A_1^0(1)$, we have an LRDTS$(3v)$ on the point set $\{x, y, z\} \times Y$. Let its block set be $\{\mathcal{B}_{i,m}^{\{x,y,z\}} : \ 1 \leqslant i \leqslant v-2, m = 0, 1, 2\} \bigcup \{\mathcal{B}_{j,m}^l(\{x, y, z\}) : \ 0 \leqslant j \leqslant v-1, l = 0, 1, m = 0, 1, 2\}$, and each $\mathcal{B}_{i,m}^{\{x,y,z\}}$ can be partitioned into parallel classes $B_{i,m}^{\{x,y,z\}}(n)$, $1 \leqslant n \leqslant 3v-1$, each $\mathcal{B}_{j,m}^l(\{x, y, z\})$ can be partitioned into parallel classes $B_{j,m}^l(\{x, y, z\}, n)$, $1 \leqslant n \leqslant 3v-1$.

(2). If $\{x, y, z\} \notin A_1^0(1)$, i.e. $\{x, y, z\} \in A_k^0(1)$ for some k, $2 \leqslant k \leqslant \frac{u-1}{2}$, $x < y < z$, let

$$P_{j,s}^{\{x,y,z\}} = \{(x, a), (y, \sigma_s(a)), (z, \sigma_j \sigma_s^*(a)) : a \in Y\},$$

$$P_{0,j,s}^{\{x,y,z\}} = \{(u, v, w), (w, v, u) : \{u, v, w\} \in P_{j,s}^{\{x,y,z\}}\},$$

$$P_{1,j,s}^{\{x,y,z\}} = \{(u, w, v), (v, w, u) : \{u, v, w\} \in P_{j,s}^{\{x,y,z\}}\},$$

$$P_{2,j,s}^{\{x,y,z\}} = \{(w, u, v), (v, u, w) : \{u, v, w\} \in P_{j,s}^{\{x,y,z\}}\},$$

where $\sigma_s, \sigma_j \in G$ and let

$$\mathcal{A}_{m,j}^{\{x,y,z\}} = \bigcup_{\sigma_s \in G} P_{m,j,s}^{\{x,y,z\}}, m = 0, 1, 2.$$

So we have: (1) For $x' \neq y' \in \{x, y, z\}, a, b \in Y$, each of the order pair $((x, a), (y, b))$ and $((y, b), (x, a))$ belongs to exactly one triple of $\mathcal{A}_{m,j}^{\{x,y,z\}}$; (2) $\mathcal{A}_{m,j}^{\{x,y,z\}}$ and $\mathcal{A}_{m',j'}^{\{x,y,z\}}$ are disjoint for $(m, j) \neq (m', j')$.

Since $(Y, \circ)$ is a DTRIQ(v), for any ordered pair $(a, b) \in Y \times Y$ $(a \neq b)$ and any $\sigma \in G$, we get an element $a \circ b$ in Y such that $\sigma(a) \circ \sigma(b) = \sigma(a \circ b)$. For $0 \leqslant j \leqslant v - 1$, define six permutations on Y, namely $\alpha_j^{(s)}, \beta_j^{(s)}$ $(s \in Z_3)$ as follows:

$$\alpha_j^{(0)} = \sigma_j, \quad \alpha_j^{(1)} = \sigma_0 \sigma_j^* \sigma_j^{-1}, \quad \alpha_j^{(2)} = (\sigma_0 \sigma_j^*)^{-1} = (\alpha_j^{(1)} \alpha_j^{(0)})^{-1},$$

$$\beta_j^{(0)} = \sigma_{v-1} \sigma_j^*, \quad \beta_j^{(1)} = \sigma_j (\sigma_{v-1} \sigma_j^*)^{-1}, \quad \beta_j^{(2)} = \sigma_j^{-1} = (\beta_j^{(1)} \beta_j^{(0)})^{-1}.$$

Here, if π is a permutation of Y, we denote by $\pi T_m(u, v)$ the transitive triples obtained by replacing each occurrence of (u, a) with $(u, \pi(a))$ (and keeping those occurrences with the first component "u" unchanged). Using the six permutations defined above, for each $m \in \{0, 1, 2\}$ and $j \in \{0, 1, \cdots, v - 1\}$, define

$$\mathcal{C}_{j,m}^0 = \alpha_j^{(0)} T_m(x, y) \cup \alpha_j^{(1)} T_m(y, z) \cup \alpha_j^{(2)} T_m(z, x),$$

$$\mathcal{C}_{j,m}^1 = \beta_j^{(0)} T_m(x, y) \cup \beta_j^{(1)} T_m(y, z) \cup \beta_j^{(2)} T_m(z, x),$$

and

$$\mathcal{B}_{j,m}^l(\{x, y, z\}) = P_{m,v-l,j}^{\{x,y,z\}} \bigcup \mathcal{C}_{j,m}^l,$$

where $0 \leqslant j \leqslant v - 1, m = 0, 1, 2, l = 0, 1$ and $v - l = 0, v - 1$.

Furthermore, $(\{x, y, z\} \times Y, \mathcal{B}_j^l(\{x, y, z\}))$, $0 \leqslant j \leqslant v - 1, l = 0, 1$, is an RDTS($3v$). Let each $\mathcal{B}_{j,m}^l(\{x, y, z\})$ can be partitioned into parallel classes $B_{j,m}^l(\{x, y, z\}, n)$, $1 \leqslant n \leqslant 3v - 1$.

(For any triple T of $X \times Y$, T is form as $((x, a), (x, b), (x, c))$ or $((x, a), (x, b), (y, c))$ or $((x, a), (y, b), (x, c))$ or $((y, a), (x, b), (x, c))$ or $((x, a), (y, b), (z, c))$ with $\{x, y, z\} \in \mathcal{A}$, then T appears in *Step* 1.)

Step 2: For any $\{x, y, z\} \subseteq X$, $x < y < z$, $\{x, y, z\} \notin \mathcal{A}$, (i.e. there exist k, l such that $\{x, y, z\} \in \mathcal{A}_k^l \setminus A_k^l(1)$) define $\mathcal{A}_{m,j}^{\{x,y,z\}}$ like *Step* 1.

Define

$$\mathcal{C}_{m,i} = (\bigcup_{\{x,y,z\} \in \mathcal{A} \setminus A_1^0(1)} \mathcal{A}_{m,i}^{\{x,y,z\}}) \bigcup (\bigcup_{\{x,y,z\} \in A_1^0(1)} \mathcal{B}_{i,m}^{\{x,y,z\}}).$$

It is not difficult to check that each $(X \times Y, \mathcal{C}_{m,i})$, $1 \leqslant i \leqslant v - 2, m = 0, 1, 2$, is an RDTS($uv$) with the following parallel classes:

$$C_{m,i}(n) = \bigcup_{\{x,y,z\} \in A_1^0(1)} B_{i,m}^{\{x,y,z\}}(n), 1 \leqslant n \leqslant 3v - 1;$$

$$C_{m,i}(k,s) = \bigcup_{\{x,y,z\}\in A_k^0(1)} \{(u,v,w) : \{u,v,w\} \in P_{m,i,s}^{\{x,y,z\}}\}, 2 \leqslant k \leqslant \frac{u-1}{2}, 0 \leqslant s \leqslant v-1.$$

$$\overline{C}_{m,i}(k,s) = \bigcup_{\{x,y,z\}\in A_k^0(1)} \{(w,v,u) : \{u,v,w\} \in P_{m,i,s}^{\{x,y,z\}}\}, 2 \leqslant k \leqslant \frac{u-1}{2}, 0 \leqslant s \leqslant v-1.$$

Furthermore, these $3(v-2)$ RDTSs are obviously disjoint.

Define

$$\mathcal{D}_{m,k,j}^l = \Big(\bigcup_{\{x,y,z\}\in A_k^l(1)} \mathcal{B}_{j,m}^l(\{x,y,z\})\Big) \bigcup \Big(\bigcup_{\{x,y,z\}\in \mathcal{A}_k^l \setminus A_k^l(1)} \mathcal{A}_{m,j}^{\{x,y,z\}}\Big),$$

where $1 \leqslant k \leqslant \frac{u-1}{2}, 0 \leqslant j \leqslant v-1, m = 0,1,2, l = 0,1$. It is not difficult to check that each $(X \times Y, \mathcal{D}_{m,k,j}^l)$ is an RDTS(uv) with the following parallel classes:

$$\mathcal{D}_{m,k,j}^l(n) = \bigcup_{\{x,y,z\}\in A_k^l(1)} \mathcal{B}_{j,m}^l(\{x,y,z\}, n), 1 \leqslant n \leqslant 3v-1,$$

$$\mathcal{D}_{m,k,j}^l(h,s) = \bigcup_{\{x,y,z\}\in A_k^l(h)} \{(u,v,w) : \{u,v,w\} \in P_{m,j,s}^{\{x,y,z\}}\},\ 2 \leqslant h \leqslant \frac{u-1}{2},\ 0 \leqslant s \leqslant v-1.$$

$$\overline{\mathcal{D}}_{m,k,j}^l(h,s) = \bigcup_{\{x,y,z\}\in A_k^l(h)} \{(w,v,u) : \{u,v,w\} \in P_{m,j,s}^{\{x,y,z\}}\},\ 2 \leqslant h \leqslant \frac{u-1}{2},\ 0 \leqslant s \leqslant v-1.$$

And these $3(u-1)v$ RDTSs are disjoint. We obtain a total of $3(uv-2)$ disjoint RDTS(uv), a large set. This completes the proof. $\square$

4 New orders

From Lemma 2.1, 2.2 and Theorem 3.1, we can obtain the following conclusion.

Theorem 4.1 *For $v \equiv 0,3,9 \pmod{12}$, if there exists an* LRDTS$(3v)$*, then there exists an* LRDTS$(v \cdot \prod_{m_i\geqslant 0}(2\cdot 7^{m_i}+1)\prod_{n_i\geqslant 0}(2\cdot 13^{n_j}+1))$*, where m_i and n_j are non-negative integers.*

For example, from Theorem 1.2, for $s \in \{57, 93, 132, 240, 255\}$, the existence of LRDTS (s) is unknown. But the existence of LRDTS$(3s)$ is known. And from Lemma 2.1, there exists a DTRIQ(s). Thus, from Theorem 4.1, we get the following result.

Theorem 4.2 *There exists an* LRDTS(v) *for $v = s \cdot \prod_{m_i\geqslant 0}(2\cdot 7^{m_i}+1)\prod_{n_j\geqslant 0}(2\cdot 13^{n_j}+1)$, where $s \in \{57, 93, 132, 240, 255\}$, m_i and n_j are non-negative integers.*

Remark: The smallest order of v (unknown before this paper) obtained from Theorem 4.2 is $1395, 1980, 3600, 3825, \cdots$ in turn.

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