Tetravalent non-normal Cayley graphs of order $4p$

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Abstract

A Cayley graph $\mathrm{Cay}(G, S)$ on a group G is said to be *normal* if the right regular representation $R(G)$ of G is normal in the full automorphism group of $\mathrm{Cay}(G, S)$. In this paper, all connected tetravalent non-normal Cayley graphs of order $4p$ are constructed explicitly for each prime p. As a result, there are fifteen sporadic and eleven infinite families of tetravalent non-normal Cayley graphs of order $4p$.

1 Introduction

For a finite, simple, undirected and connected graph X, we use $V(X)$, $E(X)$, $A(X)$ and $\mathrm{Aut}(X)$ to denote its vertex set, edge set, arc set and full automorphism group, respectively. For $u, v \in V(X)$, denote by $\{u, v\}$ the edge incident to u and v in X. A graph X is said to be *vertex-transitive*, *edge-transitive* and *arc-transitive* (or *symmetric*) if $\mathrm{Aut}(X)$ acts transitively on $V(X)$, $E(X)$ and $A(X)$, respectively. In particular, if $\mathrm{Aut}(X)$ acts regularly on $A(X)$, then X is said to be *1-regular*.

Let G be a permutation group on a set Ω and $\alpha \in \Omega$. Denote by G_α the stabilizer of α in G, that is, the subgroup of G fixing the point α. We say that G is *semiregular* on Ω if $G_\alpha = 1$ for every $\alpha \in \Omega$ and *regular* if G is transitive and semiregular. Given a finite group G and an inverse closed subset $S \subseteq G \setminus \{1\}$, the *Cayley graph* $\mathrm{Cay}(G, S)$ on G with respect to S is defined to have vertex set G and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$. A Cayley graph $\mathrm{Cay}(G, S)$ is connected if and only if S generates G. Given a $g \in G$, define the permutation $R(g)$ on G by $x \mapsto xg, x \in G$. Then $R(G) = \{R(g) \mid g \in G\}$, called the *right regular representation* of G, is a regular permutation group isomorphic to G. It is well-known that $R(G) \leqslant \mathrm{Aut}(\mathrm{Cay}(G, S))$. So, $\mathrm{Cay}(G, S)$ is vertex-transitive. In general, a vertex-transitive graph X is isomorphic to a Cayley graph on a group G if and only if

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its automorphism group has a subgroup isomorphic to G, acting regularly on the vertex set of X (see [3, Lemma 16.3]). A Cayley graph $\mathrm{Cay}(G,S)$ is said to be *normal* if $R(G)$ is normal in $\mathrm{Aut}(\mathrm{Cay}(G,S))$.

For two inverse closed subsets S and T of a group G not containing the identity 1, if there is an $\alpha \in \mathrm{Aut}(G)$ such that $S^{\alpha} = T$ then S and T are said to be *equivalent*, denoted by $S \equiv T$. One may easily show that if S and T are equivalent then $\mathrm{Cay}(G,S) \cong \mathrm{Cay}(G,T)$ and then $\mathrm{Cay}(G,S)$ is normal if and only if $\mathrm{Cay}(G,T)$ is normal.

The concept of normal Cayley graph was first proposed by Xu [24], and following this article, the normality of Cayley graphs have been extensively studied from different perspectives by many authors. Note that Wang et al. [22] obtained all disconnected normal Cayley graphs. For this reason, it suffices to consider the connected ones when one investigates the normality of Cayley graphs. One of the standard problems in the studying of normality of Cayley graphs is to determine the normality of Cayley graphs with specific orders. It is well-known that every transitive permutation group of prime degree p is either 2-transitive or solvable with a regular normal Sylow p-subgroup (see, for example, [5, Corollary 3.5B]). This implies that a Cayley graph of prime order is normal if the graph is neither empty nor complete. The normality of Cayley graphs of order a product of two primes was determined by Dobson et al. [6, 8, 17].

There also has been a lot of interest in the studying of normality of small valent Cayley graphs. For example, Baik et al. [1] determined all non-normal Cayley graphs on abelian groups with valency at most 4, and Fang et al. [9] proved that the vast majority of connected cubic Cayley graphs on non-abelian simple groups are normal. Let $\mathrm{Cay}(G,S)$ be a connected cubic Cayley graph on a non-abelian simple group G. Praeger [20] proved that if $N_{\mathrm{Aut}(\mathrm{Cay}_{(G,S)})}(R(G))$ is transitive on $E(\mathrm{Cay}(G,S))$ then $\mathrm{Cay}(G,S)$ is normal. Let p and q be two primes. In [25, 26, 27], all connected cubic non-normal Cayley graphs of order $2pq$ are determined. Wang and Xu [23] determined all tetravalent non-normal 1-regular Cayley graphs on dihedral groups. Feng and Xu [14] proved that every connected tetravalent Cayley graph on a regular p-group is normal when $p \neq 2, 5$. Li et al. [10, 16] investigated the normality of tetravalent edge-transitive Cayley graphs on G, where G is either a group of odd order or a finite non-abelian simple group. Recently, Kovács [15] classified all connected tetravalent non-normal arc-transitive Cayley graphs on dihedral groups satisfying one additional restriction: the graphs are bipartite, with the two bipartition sets being the two orbits of the cyclic subgroup within the dihedral group. For more results on the normality of Cayley graphs, we refer the reader to [12, 24].

In this article, we classify all connected tetravalent non-normal Cayley graphs of order $4p$ with p a prime. It appears that there are fifteen sporadic and eleven infinite families of tetravalent non-normal Cayley graphs of order $4p$ including two infinite families of Cayley graphs on abelian groups, one infinite family of Cayley graphs on the dicyclic group Q_{4p}, three infinite families of Cayley graphs on the Frobenius group F_{4p} and five infinite families of Cayley graphs on the Dihedral group D_{4p}.

2 Preliminaries

We start by some notational conventions used throughout this paper. For a regular graph X, use $d(X)$ to represent its valency, and for any subset B of $V(X)$, the subgraph of X induced by B will be denoted by $X[B]$. For any $v \in V(X)$, let $N_X(v)$ denote the neighborhood of v in X, that is, the set of vertices adjacent to v in X. Let X be a connected vertex-transitive graph, and let $G \leqslant \mathrm{Aut}(X)$ be vertex-transitive on X. For a G-invariant partition $\mathcal{B}$ of $V(X)$, the *quotient graph* $X_{\mathcal{B}}$ is defined as the graph with vertex set $\mathcal{B}$ such that, for any two vertices $B, C \in \mathcal{B}$, B is adjacent to C if and only if there exist $u \in B$ and $v \in C$ which are adjacent in X. Let N be a normal subgroup of G. Then the set $\mathcal{B}$ of orbits of N in $V(X)$ is a G-invariant partition of $V(X)$. In this case, the symbol $X_{\mathcal{B}}$ will be replaced by X_N.

Let X and Y be two graphs. The *direct product* $X \times Y$ of X and Y is defined as the graph with vertex set $V(X) \times V(Y)$ such that for any two vertices $u = (x_1, y_1)$ and $v = (x_2, y_2)$ in $V(X \times Y)$, u is adjacent to v in $X \times Y$ whenever $x_1 = x_2$ and $\{y_1, y_2\} \in E(Y)$ or $\{x_1, x_2\} \in E(X)$ and $y_1 = y_2$. The *lexicographic product* $X[Y]$ is defined as the graph with vertex set $V(X[Y]) = V(X) \times V(Y)$ such that for any two vertices $u = (x_1, y_1)$ and $v = (x_2, y_2)$ in $V(X[Y])$, u is adjacent to v in $X[Y]$ whenever $\{x_1, x_2\} \in E(X)$ or $x_1 = x_2$ and $\{y_1, y_2\} \in E(Y)$.

Let n be a positive integer. Denote by $\mathbb{Z}_n$ the cyclic group of order n as well as the ring of integers modulo n, by $\mathbb{Z}_n^*$ the multiplicative group of $\mathbb{Z}_n$ consisting of numbers coprime to n, by D_{2n} the dihedral group of order $2n$, and by C_n and K_n the cycle and the complete graph of order n, respectively. We call C_n an *n-cycle*.

For two groups M and N, $N \leqslant M$ means that N is a subgroup of M, $N < M$ means that N is a proper subgroup of M, and $N \rtimes M$ denotes a semidirect product of N by M. For a subgroup H of a group G, denote by $C_G(H)$ the centralizer of H in G and by $N_G(H)$ the normalizer of H in G. Then $C_G(H)$ is normal in $N_G(H)$.

Proposition 2.1 [21, Theorem 1.6.13] *The quotient group $N_G(H)/C_G(H)$ is isomorphic to a subgroup of the automorphism group of H.*

The following proposition is due to Burnside.

Proposition 2.2 [21, Theorem 8.5.3] *Let p and q be primes, and let m and n be non-negative integers. Then any group of order $p^m q^n$ is solvable.*

Let $\mathrm{Cay}(G, S)$ be a Cayley graph on a group G with respect to a subset S of G. Set $A = \mathrm{Aut}(\mathrm{Cay}(G, S))$ and $\mathrm{Aut}(G, S) = \{\alpha \in \mathrm{Aut}(G) \mid S^\alpha = S\}$.

Proposition 2.3 [24, Proposition 1.5] *The Cayley graph $\mathrm{Cay}(G, S)$ is normal if and only if $A_1 = \mathrm{Aut}(G, S)$, where A_1 is the stabilizer of the identity 1 of G in A.*

Combining [1, Theorem 1.2], [7, Theorem 1] and [8, Theorem 1], we have the following.

Proposition 2.4 *Let $X = \mathrm{Cay}(G, S)$ be a connected cubic Cayley graph of order twice an odd prime. Then either X is isomorphic to the complete bipartite graph $K_{3,3}$ or the Heawood graph, or $\mathrm{Aut}(X) = R(G) \rtimes \mathrm{Aut}(G, S)$ with $\mathrm{Aut}(G, S) \cong \mathbb{Z}_t$ with $t \leqslant 3$.*

The following proposition can be deduced from [15, Theorem 1.2].

Proposition 2.5 *Let p be a prime, and let $X = \mathrm{Cay}(D_{4p}, S)$ be a connected tetravalent symmetric non-normal Cayley graph, where $D_{4p} = \langle a, b \mid a^{2p} = b^2 = 1, b^{-1}ab = a^{-1}\rangle$. If $S \cap \langle a\rangle = \emptyset$, then either $S \equiv \{b, ba, ba^p, ba^{p+1}\}$ and $X \cong C_n[2K_1]$, or $p = 7$ and $S \equiv \{b, ba, ba^4, ba^6\}$.*

Finally, we introduce a result [28] regarding the classification of the tetravalent symmetric graphs of order $4p$ where p is a prime. To this end, we introduce several families of tetravalent symmetric graphs of order $4p$. Let p be a prime congruent to 1 modulo 4, and w be an element of order 4 in $\mathbb{Z}_p^*$ with $1 < w < p-1$. Define $\mathcal{CA}_{4p}^0 = \mathrm{Cay}(G, \{a, a^{-1}, a^{w^2}b, a^{-w^2}b\})$ and $\mathcal{CA}_{4p}^1 = \mathrm{Cay}(G, \{a, a^{-1}, a^w b, a^{-w}b\})$, where $G = \langle a\rangle \times \langle b\rangle \cong \mathbb{Z}_{2p} \times \mathbb{Z}_2$.

Let p be an odd prime. The graph $\mathcal{C}(2; p, 2)$ has vertex set $\mathbb{Z}_p \times (\mathbb{Z}_2 \times \mathbb{Z}_2)$ and edge set $\{\{(i,(x,y)),(i+1,(y,z))\} \mid i \in \mathbb{Z}_p, x, y, z \in \mathbb{Z}_2\}$.

Let $G = \mathrm{PGL}(2,7)$ and $H \leqslant G$ such that $H \cong \mathrm{PSL}(2,7)$. By [4, P.285, summary], H has a subgroup T isomorphic to A_4. Let P be a Sylow 3-subgroup of T. Then, $N_G(P) \cong S_3 \times \mathbb{Z}_2$. Take an involution, say a, in the center of $N_G(P)$. Define $\mathcal{G}_{28}$ to have vertex set $\{Tg \mid g \in G\}$, the set of right cosets of T in G, and edge set $\{\{Tg, Tdg\} \mid g \in G, d \in HaH\}$.

Proposition 2.6 *Let p be an odd prime, and X be a connected tetravalent symmetric graph of order $4p$. Then, X is isomorphic to $C_{2p}[2K_1]$, $\mathcal{CA}_{4p}^0$, $\mathcal{CA}_{4p}^1$, $\mathcal{C}(2; p, 2)$ or $\mathcal{G}_{28}$.*

3 Tetravalent non-normal Cayley graphs on Q_{4p}

Let p be an odd prime. In this section, all connected tetravalent non-normal Cayley graphs on $Q_{4p} = \langle a, b \mid a^{2p} = 1, b^2 = a^p, b^{-1}ab = a^{-1}\rangle$ are constructed.

Construction of non-normal Cayley graphs on Q_{4p}: *Set*

$$\Lambda = \{b, b^{-1}, ab, (ab)^{-1}\}. \tag{1}$$

Define $\mathcal{CQ}_{4p} = \mathrm{Cay}(Q_{4p}, \Lambda)$.

Lemma 3.1 *$\mathcal{CQ}_{4p} \cong C_{2p}[2K_1]$. Furthermore, $\mathcal{CQ}_{4p}$ is non-normal.*

Proof. Let $X = \mathcal{CQ}_{4p}$. Since Λ generates Q_{4p}, X is connected. Set $C = \langle R(b^2)\rangle$. Then C is the center of $R(Q_{4p})$. Note that $R(Q_{4p})$ acts on $V(X)$ by right multiplication. The orbit set of C in $V(X)$ is the set of the right cosets of $\langle b^2\rangle$ in Q_{4p}. The orbits adjacent to $\{1, b^2\}$ are $\{1, b^2\}b$ and $\{1, b^2\}ab$. By the normality of C in $R(Q_{4p})$ and the transitivity of $R(Q_{4p})$ on $V(X)$, the quotient graph of X relative to the orbit set of C is a $2p$-cycle and each orbit of C contains no edges. Thus, $X \cong C_{2p}[2K_1]$. Then $\mathrm{Aut}(X) \cong \mathbb{Z}_2^{2p} \rtimes D_{4p}$ and $|\mathrm{Aut}(X)| = 2^{2p+2}p$. Suppose that X is normal. By Proposition 2.3, $\mathrm{Aut}(X) = R(Q_{4p}) \rtimes \mathrm{Aut}(Q_{4p}, \Lambda)$. Since S generates Q_{4p}, $\mathrm{Aut}(Q_{4p}, \Lambda)$ acts faithfully on Λ, implying $\mathrm{Aut}(Q_{4p}, \Lambda) \leqslant S_4$. It follows that $|\mathrm{Aut}(X)| \leqslant 96p < 2^{2p+2}p$, a contradiction. $\square$

Theorem 3.2 *Let p be an odd prime. A connected tetravalent Cayley graph $\mathrm{Cay}(Q_{4p}, S)$ on Q_{4p} is non-normal if and only if $S \equiv \Lambda$.*

Proof. The sufficiency can be obtained by Lemma 3.1, and we only need to prove the necessity. Let $X = \mathrm{Cay}(Q_{4p}, S)$ be a connected tetravalent non-normal Cayley graph. Let $A = \mathrm{Aut}(X)$ and let A_1 be the stabilizer of 1 in A. Then $A = R(Q_{4p})A_1$ and $R(Q_{4p}) \ntrianglelefteq A$. Clearly, $Q_{4p} = \{a^i, a^ib \mid 0 \leqslant i \leqslant 2p-1\}$. It is easily shown that Q_{4p} has automorphism group $\mathrm{Aut}(Q_{4p}) = \{\gamma_{i,j} : a^i \mapsto a, a^jb \mapsto b \mid i \in \mathbb{Z}_{2p}^*, j \in \mathbb{Z}_{2p}\}$. Since S generates Q_{4p}, S contains an element a^ib and its inverse for some $0 \leqslant i \leqslant 2p-1$. Then, $b, b^{-1} \in S^{\gamma_{1,i}}$, and one may let $S = \{b, b^{-1}, a^\ell b, (a^\ell b)^{-1}\}$ or $\{b, b^{-1}, a^\ell, a^{-\ell}\}$ for some $0 \leqslant \ell \leqslant 2p-1$. Again, since S generates Q_{4p}, $(\ell, 2p) = 1$ or 2. If $(\ell, 2p) = 1$ then $S^{\gamma_{\ell,0}} = \{b, b^{-1}, ab, (ab)^{-1}\}$ or $\{b, b^{-1}, a, a^{-1}\}$. Let $(\ell, 2p) = 2$ and $\ell = 2m$. Then, $(\ell + p, 2p) = 1$, $0 < m < p$ and $(m, 2p) = 1$ or 2. If $S = \{b, b^{-1}, a^\ell b, (a^\ell b)^{-1}\}$ then since $(a^\ell b)^{-1} = a^{\ell+p}b$, one has $S^{\gamma_{\ell+p,0}} = \{b, b^{-1}, ab, (ab)^{-1}\}$. If $S = \{b, b^{-1}, a^\ell, a^{-\ell}\}$ then either $S^{\gamma_{m,0}}$ or $S^{\gamma_{p+m,0}}$ is equal to $\{b, b^{-1}, a^2, a^{-2}\}$. Thus, we can assume that $S = \{b, b^{-1}, ab, (ab)^{-1}\}, \{b, b^{-1}, a, a^{-1}\}$ or $\{b, b^{-1}, a^2, a^{-2}\}$.

Let $S = \{b, b^{-1}, a, a^{-1}\}$. It is easy to see that $\gamma_{2p-1,0}, \gamma_{1,p} \in A_1$, implying $|A_1| \geqslant 4$. Consider the number n of 4-cycles in X passing the identity 1 and one of vertices, say v, at distance 2 from 1. Then $n = 0$ when $v = a^2$ or a^{-2} and $n = 1$ otherwise. Note that a^2 and a^{-2} are adjacent to a and a^{-1}, respectively. This implies that A_1/A_1^* has no elements of order 3 or 4, and hence $|A_1/A_1^*| \leqslant 4$, where A_1^* is the kernel of A_1 acting on S. Furthermore, A_1^* fixes each vertex in X at distance 2 from 1. By the connectivity and vertex-transitivity of X, A_1^* fixes all vertices of X, and consequently, $A_1^* = 1$. It follows that $|A_1| = 4$, and hence $A_1 = \langle \gamma_{2p-1,0}, \gamma_{1,p} \rangle = \mathrm{Aut}(Q_{4p}, S)$. By Proposition 2.3, X is normal, a contradiction. Similarly, if $S = \{b, b^{-1}, a^2, a^{-2}\}$, then we also have that X is normal, a contradiction. Thus, $S = \{b, b^{-1}, ab, (ab)^{-1}\} = \Lambda$. $\square$

4 Tetravalent non-normal Cayley graphs on F_{4p} or D_{4p}

Let p be an odd prime. In this section, we shall determine the connected tetravalent non-normal Cayley graphs on F_{4p} or D_{4p}, where

$$\begin{aligned} F_{4p} &= \langle a, b \mid a^p = b^4 = 1, b^{-1}ab = a^\lambda \rangle, \lambda^2 \equiv -1 \pmod p), \\ D_{4p} &= \langle a, b \mid a^{2p} = b^2 = 1, bab = a^{-1} \rangle. \end{aligned}$$

It is easy to show that the automorphism groups of D_{4p} and F_{4p} are as following:

$$\begin{array}{rcl} \mathrm{Aut}(D_{4p}) & = & \{\delta_{m,n} : a^m \mapsto a, ba^n \mapsto b \mid m \in \mathbb{Z}_{2p}^*, n \in \mathbb{Z}_{2p}\}, \\ \mathrm{Aut}(F_{4p}) & = & \{\sigma_{i,j} : a^i \mapsto a, b \mapsto a^j b \mid i \in \mathbb{Z}_p^*, j \in \mathbb{Z}_p\}. \end{array} \tag{2}$$

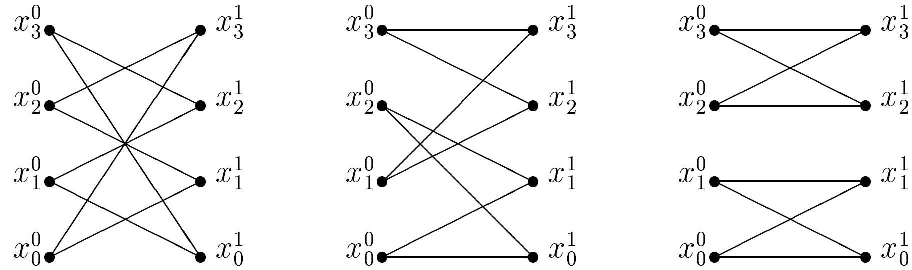
We first prove a lemma.

Lemma 4.1 *Let p be an odd prime, and let $X = \mathrm{Cay}(G, S)$ be a connected tetravalent non-symmetric Cayley graph, where $G = F_{4p}$ or D_{4p}. If the vertex-stabilizer $\mathrm{Aut}(X)_v$ of $v \in V(X)$ is a 2-group, then either $\mathrm{Aut}(X)$ has a normal Sylow p-subgroup or $G = D_{4p}$ and $S \equiv \{b, ba, ba^p, a^p\}$.*

Proof. Set $A = \mathrm{Aut}(X)$. Then $|A| = |R(G)||A_v| = 2^{\ell+2}p$ for some positive integer ℓ. By Proposition 2.2, A is solvable. Let P be a Sylow p-subgroup of A. Then $P \cong \mathbb{Z}_p$. Assume that P is non-normal in A. Take a maximal normal 2-subgroup, say N, of A. By the solvability of A, $PN/N \trianglelefteq A/N$, and hence $PN \trianglelefteq A$. If $P \trianglelefteq PN$, then P is characteristic in PN and hence $P \trianglelefteq A$, a contradiction. Thus, P is non-normal in PN. Consider the quotient graph X_N of X relative to the orbit set of N, and let K be the kernel of A acting on $V(X_N)$. Then $N \leqslant K$ and A/K is vertex-transitive on X_N. Since $|X| = 4p$ and $p > 2$, one has $|X_N| = p$ or $2p$. It follows that $p \mid |A/K|$, and hence K is a 2-group. The maximality of N gives $K = N$. Let Δ be an orbit of N on $V(X)$. Then $|\Delta| = 4$ or 2.

Case 1: $|\Delta| = 4$

In this case, X_N has order p and hence $d(X_N) = 4$ or 2. If $d(X_N) = 4$, then $d(X[\Delta]) = 0$, and $|X_N| = p > 3$. Then the vertex-stabilizer N_v of $v \in \Delta$ fixes each neighbor of v. By the connectivity of X, $N_v = 1$ and hence $|N| = |\Delta||N_v| = 4$. By Sylow Theorem, $P \trianglelefteq PN$, a contradiction. Let $d(X_N) = 2$ and let $V(X_N) = \{\Delta_i \mid i \in \mathbb{Z}_p\}$ with $\Delta_i \sim \Delta_{i+1}$ and $\Delta_0 = \Delta$. Clearly, $A/N \cong \mathbb{Z}_p$ or D_{2p}. This implies that A/N is edge-transitive on X_N. It follows that $X[\Delta_i] \cong C_4$ or $4K_1$ for each $i \in \mathbb{Z}_p$. Furthermore, if $X[\Delta_i] \cong 4K_1$, then $X[\Delta_i \cup \Delta_{i+1}] \cong C_8$ or $2C_4$. Assume first that either $X[\Delta_i] \cong C_4$ or $X[\Delta_i] \cong 4K_1$ and $X[\Delta_i \cup \Delta_{i+1}] \cong C_8$. For the former, each vertex in Δ_i connects exactly one vertex in Δ_{i+1} for each $i \in \mathbb{Z}_p$. By the connectivity of X, N acts faithfully on Δ_i. For the latter, the subgroup N^* of N fixing Δ_i pointwise also fixes Δ_{i+1} pointwise. By the connectivity of X, N^* fixes each vertex of X, and hence $N^* = 1$. Thus, N always acts faithfully on Δ_i, and hence either $N \leqslant \mathrm{Aut}(X[\Delta_i]) \cong D_8$ or $N \leqslant \mathrm{Aut}(X[\Delta_i \cup \Delta_{i+1}]) \cong D_{16}$. Clearly, $|N| \geqslant 4$. Let $|N| = 4$. Since $A/N \leqslant D_{2p}$, one has $G = D_{4p}$ and $R(G) \cap N \cong \mathbb{Z}_2$ is the center of $R(G)$. Clearly, $R(G) \cap N$ normalizes P. Since $p > 2$, by Sylow Theorem, $P \trianglelefteq PN$, a contradiction. If $|N| \geqslant 8$ then $N \cong \mathbb{Z}_8, D_8$ or D_{16}, and hence $\mathrm{Aut}(N)$ is a 2-group. From Proposition 2.1 we obtain that $PN/C_{PN}(N) \leqslant \mathrm{Aut}(N)$. Since $p \geqslant 3$, one has $P \leqslant C_{PN}(N)$, forcing $P \trianglelefteq PN$, a contradiction. Now assume that $X[\Delta_i] \cong 4K_1$ and $X[\Delta_i \cup \Delta_{i+1}] \cong 2C_4$. Set $\Delta_i = \{x_0^i, x_1^i, x_2^i, x_3^i\}$ for each $i \in \mathbb{Z}_p$. Since $X_N = (\Delta_0, \Delta_p, \ldots, \Delta_{p-1})$ is a p-cycle, A has an automorphism, say α, of order p such that $\Delta_i^\alpha = \Delta_{i+1}$ for each $i \in \mathbb{Z}_p$. Without loss of generality, let $(x_j^i)^\alpha = x_j^{i+1}$ for each $j \in \mathbb{Z}_4$ and $i \in \mathbb{Z}_p$. Consider a 4-cycle C in $X[\Delta_0 \cup \Delta_1]$ and let n be the number of edges of C which are in some orbit of α. Then, $n = 0, 1$ or 2 and, consequently, $X[\Delta_0 \cup \Delta_1]$ is one of the three cases:



Case I Case II Case III

It is easy to see that for Case III, $X \cong 2C_p[2K_1]$, contrary to the connectivity of X. For Case I, we have $X \cong C_{2p}[2K_1]$, contrary to the fact that X is non-symmetric. For Case II, we shall show that $X \cong \mathcal{C}(2;p,2)$. Recall that $\mathcal{C}(2;p,2)$ has vertex set $\mathbb{Z}_p \times (\mathbb{Z}_2 \times \mathbb{Z}_2)$ and edge set $\{\{(i,(a,b)),(i+1,(b,c))\} \mid i \in \mathbb{Z}_p, a,b,c \in \mathbb{Z}_2\}$. It is easy to see that the map defined by $x_0^i \mapsto (i,(0,0)), x_1^i \mapsto (i,(0,1)), x_2^i \mapsto (i,(1,0)), x_3^i \mapsto (i,(1,1))$ $(i \in \mathbb{Z}_p)$ is an isomorphism from X to $\mathcal{C}(2;p,2)$. Therefore, $X \cong \mathcal{C}(2;p,2)$. However, by Proposition 2.6, $\mathcal{C}(2;p,2)$ is symmetric, a contradiction.

Case 2: $|\Delta| = 2$

In this case, we have two possibilities: $X[\Delta] \cong 2K_1$ or $X[\Delta] \cong K_2$.

Assume $X[\Delta] \cong 2K_1$. Then $d(X_N) = 4$, 3 or 2. If $d(X_N) = 2$, then $X \cong C_{2p}[2K_1]$ is symmetric, a contradiction. If $d(X_N) = 4$, then it is easy to see that the vertex-stabilizer N_u of $u \in V(X)$ is trivial, and hence $N \cong \mathbb{Z}_2$. This forces that $P \trianglelefteq PN$, a contradiction. Let $d(X_N) = 3$, and let $\Delta_1, \Delta_2, \Delta_3$ be three orbits adjacent to Δ. Since X has valency 4, assume that $X[\Delta \cup \Delta_1] \cong C_4$ and $X[\Delta \cup \Delta_2] \cong X[\Delta \cup \Delta_3] \cong 2K_2$. Set $\Sigma = \{\{\Delta', \Delta''\} \mid X[\Delta' \cup \Delta''] \cong C_4, \Delta', \Delta'' \in V(X_N)\}$. Then Σ is a matching of $V(X_N)$, and A/N is still a vertex-transitive automorphism group of $X_N - \Sigma$. Since X_N is cubic and $|X_N| = 2p$, one has $X_N - \Sigma \cong C_{2p}$ or $2C_p$. Furthermore, the subgraph of X induced by any two orbits of N which are adjacent in $X_N - \Sigma$ is $2K_2$. Let $\Delta = \{u, v\}$. If $X_N - \Sigma \cong C_{2p}$ then N_v fixes all orbits of N pointwise, forcing $N_v = 1$. Let $X_N - \Sigma \cong 2C_p$. Then Δ_1 and Δ are in different p-cycles of $X_N - \Sigma \cong 2C_p$. Since $X[\Delta \cup \Delta_1] \cong C_4$, N_v acts on Δ_1. Let N_v^* be the kernel of N_v on Δ_1. Then N_v^* fixes each orbit of N pointwise and hence $N_v^* = 1$. So, we have $|N_v| \leqslant 2$, and hence $|N| = |\Delta||N_v| \leqslant 4$. Since $P \ntrianglelefteq PN$, by Sylow Theorem, $p = 3$, $|N| = 4$ and $R(G) \cap N = 1$. This implies that $G = D_{12}$. By [18, pp.1111], $|\mathrm{Aut}(X_N)| = 72$ or 12. Since $3^2 \nmid |A/N|$, one has $|A/N| \mid 24$, implying $R(G)N/N \trianglelefteq A/N$. As $\langle R(a^p)\rangle N/N$ is characteristic in $R(G)N/N$, one has $\mathbb{Z}_2 \cong \langle R(a^p)\rangle N/N \trianglelefteq A/N$, and hence $\langle R(a^2)\rangle N \trianglelefteq A$. This is contrary to the fact that N is a maximal normal 2-subgroup of A.

Assume $X[\Delta] \cong K_2$. Then $d(X_N) = 3$ or 2. If $d(X_N) = 3$, then it is easy to see that N is semiregular, that is, $N \cong \mathbb{Z}_2$. Consequently, $P \trianglelefteq NP$, a contradiction. Let $d(X_N) = 2$. Let $V(X_N) = \{\Delta_i \mid i \in \mathbb{Z}_{2p}\}$ with $\Delta_i \sim \Delta_{i+1}$. Then $A/N \leqslant \mathrm{Aut}(X_N) \cong D_{4p}$ and $X[\Delta_0 \cup \Delta_1] \cong C_4$ or K_4. Without loss of generality, assume that $X[\Delta_0 \cup \Delta_1] \cong K_4$. Then $X[\Delta_0 \cup \Delta_{2p-1}] \cong C_4$. This means that A/N is not arc-transitive on X_N, and hence $|A/N| = 2p$. It follows that $R(G)N/N = A/N$, implying that $\mathbb{Z}_2 \cong N \cap R(G) \trianglelefteq R(G)$. Hence, $G = D_{4p}$ and $N \cap R(G) = \langle R(a^p)\rangle$. Let $1 \in \Delta_0$. Recall that $R(G)$ acts on $V(X)$

by right multiplication. Then $\Delta_0 = \{1, a^p\}$, $\Delta_1 = \{x, xa^p\}$ and $\Delta_{2p-1} = \{y, ya^p\}$, where $x, y \in G$. Since $X[\Delta_0 \cup \Delta_1] \cong K_4$, one has $S = \{a^p, x, xa^p, z\}$, where $z \in \Delta_{2p-1}$. It is easy to see that all elements in S are involutions, and $x, z \in \{ba^i \mid i \in \mathbb{Z}_{2p}\}$. Since $\mathrm{Aut}(D_{4p})$ is transitive on $\{ba^i \mid i \in \mathbb{Z}_{2p}\}$, let $x = b$, $z = ba^k$ for some $k \in \mathbb{Z}_{2p}$. As S generates D_{4p}, one has $(k, 2p) = 2$ or 1. If $(k, 2p) = 2$, then $S^{\delta_{k+p,0}\delta_{1,p}} = \{a^p, b, ba, ba^p\}$, and if $(k, 2p) = 1$, then $S^{\delta_{k,0}} = \{a^p, b, ba, ba^p\}$. Thus, $S \equiv \{a^p, b, ba, ba^p\}$. □

Below we shall determine all connected tetravalent non-normal Cayley graphs on F_{4p}.

Construction of non-normal Cayley graphs on F_{4p}: *Set*

$$\begin{array}{ll}\Theta_0 = \{b, b^{-1}, ab^2, a^{-1}b^2\}, & \Theta_1 = \{b, b^{-1}, b^2, ab^2\}, \\ \Theta_2 = \{a, a^{-1}, b, b^{-1}\}, & \Theta_3 = \{b, b^{-1}, ab, (ab)^{-1}\}.\end{array} \tag{3}$$

Define $\mathcal{CF}^i_{4p} = Cay(F_{4p}, \Theta_i)$ *with* $0 \leqslant i \leqslant 3$*, where* $p = 5$ *when* $i = 1$*.*

Theorem 4.2 *Let p be an odd prime. A connected tetravalent Cayley graph* $\mathrm{Cay}(F_{4p}, S)$ *on F_{4p} is non-normal if and only if $S \equiv \Theta_i$ with $0 \leqslant i \leqslant 3$.*

Proof. We first prove the following claim.

Claim: Let $k \in \mathbb{Z}_p^*$ such that $k \neq 1$ in $\mathbb{Z}_p$. Set $T_k = \{b, b^{-1}, ab^2, a^k b^2\}$. Then $\mathrm{Cay}(F_{4p}, T_k)$ is non-normal if and only if $k \equiv -1 \pmod p$ and $\mathrm{Cay}(F_{4p}, T_k) = \mathcal{CF}^0_{4p}$.

We first show the sufficiency of the Claim. Recall that $F_{4p} = \langle a, b \mid a^p = b^4 = 1, b^{-1}ab = a^\lambda\rangle$, where $\lambda^2 \equiv -1 \pmod p$. We also have $F_{4p} = \{a^i, a^ib, a^ib^2, a^ib^{-1} \mid 0 \leqslant i \leqslant p-1\}$. Define a permutation f on F_{4p} as follows:

$$f: \quad a^i \mapsto a^i, a^ib^2 \mapsto a^ib^2, a^ib \mapsto a^{-i}b^{-1}, a^ib^{-1} \mapsto a^{-i}b \ (0 \leqslant i \leqslant p-1). \tag{4}$$

For each $i \in \mathbb{Z}_p$, we have

$$\begin{array}{lll}N_{\mathcal{CF}^0_{4p}}(a^i)^f & = \{a^{-i\lambda}b, a^{i\lambda}b^{-1}, a^{1-i}b^2, a^{-1-i}b^2\} & = N_{\mathcal{CF}^0_{4p}}((a^i)^f), \\ N_{\mathcal{CF}^0_{4p}}(a^ib^2)^f & = \{a^{-i\lambda}b^{-1}, a^{i\lambda}b, a^{1-i}, a^{-1-i}\} & = N_{\mathcal{CF}^0_{4p}}((a^ib^2)f), \\ N_{\mathcal{CF}^0_{4p}}(a^ib)^f & = \{a^{-i\lambda}b^2, a^{i\lambda}, a^{1+i}b, a^{i-1}b\} & = N_{\mathcal{CF}^0_{4p}}((a^ib)^f), \\ N_{\mathcal{CF}^0_{4p}}(a^ib^{-1})^f & = \{a^{i\lambda}b^2, a^{-i\lambda}, a^{i-1}b^{-1}, a^{i+1}b^{-1}\} & = N_{\mathcal{CF}^0_{4p}}((a^ib^{-1})^f).\end{array}$$

It follows that $f \in \mathrm{Aut}(\mathcal{CF}^0_{4p})$. Clearly, f fixes 1. Since f interchanges b and b^{-1}, f is not an automorphism of F_{4p}. By Proposition 2.3, $\mathcal{CF}^0_{4p}$ is non-normal.

We now consider the necessity of the Claim. Let $X = \mathrm{Cay}(F_{4p}, T_k)$ be non-normal. Set $A = \mathrm{Aut}(X)$ and let A_1 be the stabilizer of 1 in A. Then, $R(F_{4p}) \ntrianglelefteq A$, and $|A_1| = 2^s 3^t$ for some integers s and t. If $k \neq \pm\lambda$ in $\mathbb{Z}_p$, then it is easy to see that $(1, b, b^2, b^{-1})$ is the unique 4-cycle in X passing through the identity 1. This implies that $t = 0$ and X is non-symmetric. Let $k = \lambda$ or $-\lambda$. It is easy to see that $\{b, b^{-1}, ab^2, a^{-\lambda}b^2\}^{\sigma_{-\lambda,0}} = \{b, b^{-1}, ab^2, a^\lambda b^2\}$ (see Eq. (2) for the definition of $\sigma_{i,j}$). Hence, one may take $k = \lambda$. In this case, it is easy to see that in X there are two 4-cycles passing through $\{1, b\}$ (or

$\{1, b^{-1}\}$), and there is only one 4-cycle passing through $\{1, ab^2\}$ (or $\{1, a^k b^2\}$). Again, we have that $t = 0$ and X is non-symmetric. By Lemma 4.1, A has a normal Sylow p-subgroup, say P. Then $P = \langle R(a)\rangle$. Since $R(F_{4p})$ acts on $V(X)$ by right multiplication, the four orbits of P are $\Delta_0 = \langle a\rangle, \Delta_1 = \langle a\rangle b, \Delta_2 = \langle a\rangle b^2$ and $\Delta_3 = \langle a\rangle b^3$. Noting that $T_k = \{b, b^{-1}, ab^2, a^k b^2\}$, the quotient graph X_P of X relative to the orbit set of P is K_4. Furthermore, each Δ_i contains no edges, and the induced subgraphs $X[\Delta_0 \cup \Delta_2] \cong C_{2p}$ and $X[\Delta_0 \cup \Delta_1] \cong X[\Delta_0 \cup \Delta_3] \cong pK_2$. Let K be the kernel of A acting on $V(X_P)$. Then, $P \leqslant K$ and $A/K \leqslant \mathrm{Aut}(X_P) \cong S_4$. Since $|A| = |F_{4p}||A_1| = 2^{s+2}p$, one has $A/K \leqslant D_8$. It is easy to see that K acts faithfully on $\Delta_0 \cup \Delta_2$. Therefore, $K \leqslant \mathrm{Aut}(X[\Delta_0 \cup \Delta_2]) \cong D_{4p}$. Since K fixes each Δ_i, one has $|K| \leqslant 2p$. If $|K| < 2p$, then $|A| \leqslant 8p$, forcing $R(F_{4p}) \trianglelefteq A$, a contradiction. Thus, $|K| = 2p$. Let $K_1 = \langle \alpha\rangle$ be the stabilizer of 1 in K. Then, $K_1 \cong \mathbb{Z}_2$ and $KR(F_{4p}) = R(F_{4p}) \rtimes K_1$, implying $K_1 \leqslant \mathrm{Aut}(F_{4p}, T_k)$. By the structure of X, α fixes b and b^{-1} and interchanges ab^2 and $a^k b^2$. Then, $a^\alpha = (ab^2b^2)^\alpha = a^k b^2 b^2 = a^k$. Similarly, $(a^k)^\alpha = a$. It follows that $a^{k^2} = a$ and hence $k^2 \equiv 1 \pmod p$. Since $k \neq 1$ in $\mathbb{Z}_p$, one has $k \equiv -1 \pmod p$ and hence $X = \mathcal{CF}^0_{4p}$. This completes the proof of the Claim.

We now show the sufficiency of Theorem 4.2. By Claim, $\mathcal{CF}^0_{4p}$ is non-normal. With the help of computer software package MAGMA [2], $\mathrm{Aut}(\mathcal{CF}^1_{4p}) \cong S_5$, implying that $\mathcal{CF}^1_{4p}$ is non-normal. Consider $\mathcal{CF}^2_{4p}$. It is easy to check that $f \in \mathrm{Aut}(\mathcal{CF}^2_{4p})$ fixes the identity 1, where f is defined in Eq. (4). Since $f \notin \mathrm{Aut}(F_{4p})$, by Proposition 2.3, $\mathcal{CF}^2_{4p}$ is non-normal. Clearly, $\Theta_3^{\sigma_{\lambda,0}} = \{b, b^{-1}, ba, (ba)^{-1}\}$. By [13, pp.729, Remark], $\mathrm{Cay}(F_{4p}, \{b, b^{-1}, ba, (ba)^{-1}\})$ is non-normal. Thus, $\mathcal{CF}^3_{4p}$ is non-normal.

Finally, we prove the necessity of Theorem 4.2. Let $X = \mathrm{Cay}(F_{4p}, S)$ be a connected tetravalent non-normal Cayley graph. Note that F_{4p} has automorphism group $\mathrm{Aut}(F_{4p}) = \{\sigma_{i,j} : a^i \mapsto a, b \mapsto a^j b \mid 0 < i < p, 0 \leqslant j < p\}$. One may easily obtain that S is equivalent to $\{a, a^{-1}, b, b^{-1}\}$, $\{b, b^{-1}, ab, (ab)^{-1}\}$ or $\{b, b^{-1}, ab^2, a^k b^2\}$ with $k \neq 1 \pmod p$. Without loss of generality, let $S = \{a, a^{-1}, b, b^{-1}\}$, $\{b, b^{-1}, ab, (ab)^{-1}\}$ or $\{b, b^{-1}, ab^2, a^k b^2\}$ with $k \neq 1 \pmod p$. Clearly, $\{a, a^{-1}, b, b^{-1}\} = \Theta_2$ and $\{b, b^{-1}, ab, (ab)^{-1}\} = \Theta_3$. Let $S = \{b, b^{-1}, ab^2, a^k b^2\}$. If $k \neq 0 \pmod p$ then by Claim, X is non-normal if and only if $k \equiv -1 \pmod p$ and $S = \Theta_0$. Let $k \equiv 0 \pmod p$. Then, for each $i \in \mathbb{Z}_p$, the induced subgraph $X[\langle b\rangle a^i] \cong K_4$ is a clique, and $\mathrm{Cay}(F_{4p}, \{b, b^2, b^{-1}\})$ is a union of these p cliques. For any $x \in F_{4p}$, it is easy to check that in X there is a unique clique passing through x which is $X[\langle b\rangle x]$. This implies that $\Omega = \{\langle b\rangle a^i \mid i \in \mathbb{Z}_p\}$ is an A-invariant partition of $V(X)$. Consider the quotient graph X_Ω. Since each clique has order 4, X_Ω has valency at most 4. It is easy to check that $\langle b\rangle$ has 4 neighbors in X_Ω. Then, X_Ω has valency 4 and A acts faithfully on Ω. Since $|\Omega| = p$, by [5, Corollary 3.5B] A is either solvable or 2-transitive on Ω. Furthermore, if A is solvable, then the Sylow p-subgroup $P = \langle R(a)\rangle$ of A is regular on Ω and normal in A, and $C_A(P) = P$. By Proposition 2.1, $A/P \leqslant \mathrm{Aut}(P) \cong \mathbb{Z}_{p-1}$. Then, $R(F_{4p})/P \trianglelefteq A/P$, and hence $R(F_{4p}) \trianglelefteq A$, a contradiction. Thus, A is 2-transitive on Ω. Then, X_Ω is a complete graph. Since X_Ω has valency 4, one has $X_\Omega \cong K_5$. As a result, $p = 5$ and $S = \Theta_1$. □

In the remainder of this section, we consider the connected tetravalent non-normal Cayley graphs on D_{4p}. Recall that $D_{4p} = \langle a, b \mid a^{2p} = b^2 = 1, b^{-1}ab = a^{-1}\rangle$. Clearly, we also have $D_{4p} = \{ba^i, a^j \mid i, j \in \mathbb{Z}_{2p}\}$. Set

$$\digamma = \{ba^i \mid i \in \mathbb{Z}_{2p}\}. \tag{5}$$

It is easy to see that $\mathrm{Aut}(D_{4p})$ is transitive on $\digamma$.

Construction of non-normal Cayley graphs on D_{4p}: *Set*

$$\begin{array}{lll} \Omega_0 = \{b, ba, ba^p, ba^{p+1}\}, & \Omega_1 = \{a, a^{-1}, ba, ba^{-1}\}, & \Omega_2 = \{b, ba^2, ba^6, ba^5\}, \\ \Omega_3 = \{a^2, a^{-2}, b, a^p\}, & \Omega_4 = \{b, ba, ba^2, a^p\}, & \Omega_5 = \{b, ba, ba^p, a^p\}, \\ \Omega_6 = \{b, ba^2, ba^4, a^3\} & \Omega_7 = \{b, ba^2, ba^4, ba\}, & \Omega_8 = \{a, a^{-1}, a^3, b\}, \\ \Omega_9 = \{b, ba^2, ba^6, a^7\}. & & \end{array} \tag{6}$$

Define $\mathcal{CD}^i_{4p} = Cay(D_{4p}, \Omega_i)$ $(0 \leqslant i \leqslant 9)$, where $p = 7$ if $i = 2, 9$, and $p = 3$ if $i = 6, 7, 8$.

Lemma 4.3 *Let* $\mathrm{Cay}(D_{28}, S)$ *be a connected tetravalent Cayley graph on D_{28} such that $3 \mid |\mathrm{Aut}(D_{28}, S)|$. Then $S \equiv \Omega_2$ or Ω_9. Furthermore, $\mathcal{CD}^2_{4p} \cong \mathcal{G}_{28}$ and $\mathcal{CD}^9_{4p} \cong \mathcal{H} \times K_2$, where $\mathcal{G}_{28}$ is given preceding Proposition 2.6 and $\mathcal{H}$ is the Heawood graph.*

Proof. Set $X = \mathrm{Cay}(D_{28}, S)$. Let $\alpha \in \mathrm{Aut}(D_{28}, S)$ have order 3, and let $S = \{s, s^\alpha, s^{\alpha^2}, s'\}$. Since X is connected, S generates G, implying $s \in \digamma$. By the transitivity of $\mathrm{Aut}(D_{28})$ on $\digamma$, one may let $s = b$. Then $a^\alpha = a^k$ and $b^\alpha = ba^\ell$ for some $k \in \mathbb{Z}^*_{2p}$ and $\ell \in \mathbb{Z}_{2p} \setminus \{0\}$. Since α has order 3, $a = a^{\alpha^3} = a^{k^3}$ and $b = b^{\alpha^3} = ba^{(k^2+k+1)\ell}$. It follows that $k^3 \equiv 1 \pmod{14}$ and $(k^2 + k + 1)\ell \equiv 0 \pmod{14}$. From the second equation, we know $(\ell, 14) = 2$. Let $\ell = 2t$. Then $(t, 14) = 1$ or 2. Note that if $(t, 14) = 2$ then $(7 + t, 14) = 1$. Then either $\delta_{t,0}$ or $\delta_{7+t,0}$ maps ba^ℓ to ba^2 and b to b (see Eq. (2) for the definition of $\delta_{m,n}$). Hence, one may let $S = \{b, ba^2, ba^{2(k+1)}, s'\}$. Since $k^3 \equiv 1 \pmod{14}$, $k \equiv 1, -3$ or $9 \pmod{14}$. If $k \equiv 1 \pmod{14}$, then $b = ba^6$, a contradiction. Thus, $k \equiv -3$ or $9 \pmod{14}$ and hence $\alpha = \delta^{-1}_{-3,2}$ or $\delta^{-1}_{9,2}$. Since $\delta_{3,0}\delta^{-1}_{-3,2}\delta^{-1}_{3,0} = \delta_{9,2}$ and $b^{\delta_{3,0}} = b$, one may assume $k \equiv 9 \pmod{14}$ and $\alpha = \delta^{-1}_{9,2}$. Since S generates G, $S = \{b, ba^2, ba^6, a^7\}$ or $\{b, ba^2, ba^6, ba^{2i+1}\}$ for some $i \in \mathbb{Z}_7$. For the latter, one has $ba^{2i+1} = (ba^{2i+1})^\alpha = ba^{18i+11}$. It follows that $8i + 5 \equiv 0 \pmod 7$ and hence $i \equiv 2 \pmod 7$. Then, $S = \{b, ba^2, ba^6, ba^5\}$. Thus, $S \equiv \Omega_2$ or Ω_9.

If $S \equiv \Omega_2$ then by MAGMA [2], $\mathcal{CD}^2_{4p}$ is symmetric, and by Proposition 2.6, $\mathcal{CD}^2_{4p} \cong \mathcal{G}_{28}$. If $S \equiv \Omega_9$ then by MAGMA [2], $\mathcal{CD}^9_{4p} \cong \mathcal{H} \times K_2$. □

Lemma 4.4 *Let p be an odd prime. A connected tetravalent Cayley graph* $\mathrm{Cay}(D_{4p}, S)$ *on D_{4p} is symmetric and non-normal if and only if $S \equiv \Omega_0, \Omega_1$ or Ω_2. Furthermore, $\mathcal{CD}^0_{4p} \cong \mathcal{CD}^1_{4p} \cong C_{2p}[2K_1]$ and $\mathcal{CD}^2_{4p} \cong \mathcal{G}_{28}$.*

Proof. We first show that $\mathcal{CD}^i_{4p}$, $i = 0, 1, 2$, are symmetric and non-normal. For $\mathcal{CD}^0_{4p}$, by Proposition 2.5 $\mathcal{CD}^0_{4p} \cong C_{2p}[2K_1]$ is symmetric and non-normal. For $\mathcal{CD}^1_{4p}$, let $v_{i,j} = b^j a^i$ with $i \in \mathbb{Z}_{2p}$ and $j = 0, 1$. Then, $V(\mathcal{CD}^1_{4p}) = \{v_{i,j} \mid i \in \mathbb{Z}_{2p}, j = 0, 1\}$ and $E(\mathcal{CD}^1_{4p}) = \{\{v_{i,j}, v_{i+1,j}\}, \{v_{i,j}, v_{i+1,j+1}\} \mid i \in \mathbb{Z}_{2p}, j = 0, 1\}$. Clearly, $(v_{0,0}, v_{1,0}, \ldots, v_{2p-1,0})$ is a cycle of length $2p$. This implies that $\mathcal{CD}^1_{4p} \cong C_{2p}[2K_1]$ is symmetric. Then $\mathrm{Aut}(\mathcal{CD}^1_{4p}) \cong (\mathbb{Z}_2^{2p}) \rtimes D_{4p}$ has order $2^{2p+2}p$. If $\mathcal{CD}^1_{4p}$ is normal then by Proposition 2.3, it is easy to show that $|\mathrm{Aut}(\mathcal{CD}^1_{4p})| \leqslant 96p < 2^{2p+2}p$, a contradiction. For $\mathcal{CD}^2_{4p}$, by Lemma 4.3, $\mathcal{CD}^2_{4p} \cong \mathcal{G}_{28}$ is symmetric. By MAGMA [2], $\mathrm{Aut}(\mathcal{CD}^2_{4p})$ has no normal subgroups isomorphic to D_{28}, acting regularly on $V(\mathcal{CD}^2_{4p})$. It follows that $\mathcal{CD}^2_{4p}$ is non-normal.

Now we show the necessity of the first part. Let $X = \mathrm{Cay}(D_{4p}, S)$ be a connected tetravalent symmetric non-normal Cayley graph. By Proposition 2.6, X is isomorphic to $C_{2p}[2K_1], \mathcal{CA}^0_{4p}, \mathcal{CA}^1_{4p}, \mathcal{C}(2;p,2)$ or $\mathcal{G}_{28}$. By [29, Example 3.4], $\mathcal{CA}^i_{4p}$ $(i = 0, 1)$ are 1-regular, and by [23, Theorem 1], every tetravalent 1-regular Cayley graph on D_{4p} is normal. Hence, $X \ncong \mathcal{CA}^0_{4p}, \mathcal{CA}^1_{4p}$. Let $X \cong \mathcal{G}_{28}$. By MAGMA [2], $N_{\mathrm{Aut}(X)}(R(D_{28})) \cong G \rtimes \mathbb{Z}_3$, and by Lemma 4.3, $S \equiv \Omega_2$. Let $X \cong \mathcal{C}(2;p,2)$. If $p > 3$ then by [19, Theorem 3], $\mathcal{C}(2;p,2)$ is a non-Cayley graph, a contradiction. If $p = 3$ then by MAGMA [2], $\mathrm{Aut}(\mathcal{C}(2;3,2))$ has a unique normal subgroup of order 12 which is not isomorphic to D_{12}, a contradiction. Let $X \cong C_{2p}[2K_1]$. Set $V(X) = \{x_i, y_i \mid i \in \mathbb{Z}_{2p}\}$ and $E(X) = \{\{x_i, x_{i+1}\}, \{x_i, y_{i+1}\}, \{y_i, y_{i+1}\} \mid i \in \mathbb{Z}_{2p}\}$. For each $i \in \mathbb{Z}_{2p}$, set $\Delta_i = \{x_i, y_i\}$. Then $\Omega = \{\Delta_i \mid i \in \mathbb{Z}_{2p}\}$ is an A-invariant partition of $V(X)$. Consider the quotient graph X_Ω. Note that $R(D_{4p})$ acts transitively on $V(X)$ by right multiplication. Take $x_0 = 1$. Then, Δ_0 is a subgroup of D_{4p} of order 2, and each Δ_i is a right coset of Δ_0 in D_{4p}. Assume $\Delta_0 \trianglelefteq D_{4p}$. Then $\Delta_0 = \{1, a^p\}$. Let $\{1, a^p\}x$ and $\{1, a^p\}y$ be adjacent to $\{1, a^p\}$ in X_Ω. Then, $S = \{x, y, a^p x, a^p y\}$. Since S generates D_{4p}, one has $x, y \in F$, and consequently, $S \cap \langle a \rangle = \emptyset$. By Proposition 2.5, $S \equiv \Omega_0$. Assume $\Delta_0 \ntrianglelefteq D_{4p}$. Then, $\Delta_0 = \{1, a^\ell b\}$ and for each $i \in \mathbb{Z}_{2p}$, $\Delta_i = \Delta_0 a^j$ for some $j \in \mathbb{Z}_{2p}$. Since $\mathrm{Aut}(D_{4p})$ is transitive on F, let $\Delta_0 = \{1, b\}$. Let $\Delta_0 a^m$ and $\Delta_0 a^n$ be adjacent to Δ_0 in X_Ω. Then $S = \{a^m, ba^m, a^n, ba^n\}$. Since $\langle S \rangle = D_{4p}$, one has $a^m = a^{-n}$ and $(m, 2p) = 1$. Then, $S^{\delta_{m,0}} = \{a, a^{-1}, ba, ba^{-1}\} = \Omega_1$, and hence $S \equiv \Omega_1$. □

Lemma 4.5 *The Cayley graphs* $\mathcal{CD}^i_{4p}$ $(3 \leqslant i \leqslant 9)$ *are non-normal and non-symmetric.*

Proof. It is easy to see that Ω_i $(3 \leqslant i \leqslant 9)$ are generating subsets of D_{4p}. Then, $\mathcal{CD}^i_{4p}$ $(3 \leqslant i \leqslant 9)$ are connected. By MAGMA [2], $\mathrm{Aut}(\mathcal{CD}^9_{4p})$ has no normal subgroups isomorphic to D_{28}, acting regularly on $V(\mathcal{CD}^9_{4p})$. It follows that $\mathcal{CD}^9_{4p}$ is non-normal. Note that if $i = 6, 7$ or 8, then $p = 3$. Similarly, by MAGMA [2], we can obtain that $\mathcal{CD}^i_{4p}$ $(i = 6, 7, 8)$ are non-normal. Consider $\mathcal{CD}^i_{4p}$ $(i = 3, 4, 5)$. Define three permutations on D_{4p} as following:

$$\begin{aligned}
\phi:\ & a^{2i} \mapsto a^{2i}, a^{p+2i} \mapsto ba^{2i}, ba^{2i} \mapsto a^{p+2i}, ba^{2i+p} \mapsto ba^{2i+p}\ (0 \leqslant i \leqslant p-1),\\
\varphi:\ & a^{2i} \mapsto a^{2i}, a^{p+2i} \mapsto ba^{2i+1}, ba^{2i} \mapsto ba^{2i}, ba^{2i+p} \mapsto a^{2i-1}\ (0 \leqslant i \leqslant p-1),\\
\psi:\ & (b\ a^p b)(a^{-1}\ a^{p-1}).
\end{aligned}$$

Clearly, ϕ, φ and ψ fix the identity 1. Since a^{p+2} and ba^2 have different orders, $\phi \notin \mathrm{Aut}(D_{4p})$. Similarly, one has $\varphi, \psi \notin \mathrm{Aut}(D_{4p})$. Recall $\mathcal{CD}^3_{4p} = \mathrm{Cay}(D_{4p}, \Omega_3)$ with $\Omega_3 =$

$\{a^2, a^{-2}, b, a^p\}$. For each $0 \leqslant k \leqslant p-1$, we have

$$\begin{array}{lll}
N_{\mathcal{CD}^3_{4p}}(a^{2k})^\phi & = \{a^{2k+2}, a^{2k-2}, ba^{2k}, a^{p+2k}\} & = N_{\mathcal{CD}^3_{4p}}((a^{2k})^\phi), \\
N_{\mathcal{CD}^3_{4p}}(a^{p+2k})^\phi & = \{ba^{2+2k}, ba^{2k-2}, ba^{p+2k}, a^{2k}\} & = N_{\mathcal{CD}^3_{4p}}((a^{p+2k})^\phi), \\
N_{\mathcal{CD}^3_{4p}}(ba^{2k})^\phi & = \{a^{p+2k+2}, a^{p+2k-2}, ba^{p+2k}, a^{2k+p}\} & = N_{\mathcal{CD}^3_{4p}}((ba^{2k})^\phi), \\
N_{\mathcal{CD}^3_{4p}}(ba^{2k+p})^\phi & = \{ba^{p+2k-2}, ba^{p+2k+2}, ba^{p+2k}, a^{2k}\} & = N_{\mathcal{CD}^3_{4p}}((ba^{p+2k})^\phi).
\end{array}$$

This implies that ϕ is an automorphism of $\mathcal{CD}^3_{4p}$. Similarly, φ and ψ are automorphisms of $\mathcal{CD}^4_{4p}$ and $\mathcal{CD}^5_{4p}$, respectively. Since ϕ, φ and ψ fix 1 and are not in $\mathrm{Aut}(D_{4p})$, by Proposition 2.3, $\mathcal{CD}^i_{4p}$ $(i = 3, 4, 5)$ are non-normal.

By MAGMA [2], $\mathcal{CD}^i_{4p}$ $(i = 6, 7, 8)$ are non-symmetric. Since $a^p \notin \Omega_j$ for $j = 0, 1$ or 2, Ω_j $(j = 0, 1, 2)$ are not equivalent to Ω_i $(i = 3, 4, 5, 9)$. By Lemma 4.4, $\mathcal{CD}^i_{4p}$ $(i = 3, 4, 5, 9)$ are non-symmetric. □

Theorem 4.6 *Let p be an odd prime. A connected tetravalent Cayley graph* $\mathrm{Cay}(D_{4p}, S)$ *on D_{4p} is non-normal if and only if S is equivalent to one of Ω_i $(0 \leqslant i \leqslant 9)$.*

Proof. The sufficiency of Theorem 4.6 has been proved in Lemmas 4.4–4.5. We now consider the necessity. Let $X = \mathrm{Cay}(D_{4p}, S)$ be a connected tetravalent non-normal Cayley graph. If X is symmetric then by Lemma 4.4, $S \equiv \Omega_0, \Omega_1$ or Ω_2. In what follows, assume that X is non-symmetric. Let $A = \mathrm{Aut}(X)$ and $G = D_{4p}$. Then, $R(G) \ntrianglelefteq A$. First we prove a claim.

Claim: Let $\mathcal{B} = \{B_0, B_1\}$ be an A-invariant partition of $V(X)$. If $d(X[B_0]) = 3$, then S is equivalent to one of Ω_i $(6 \leqslant i \leqslant 9)$.

Let A^* be the kernel of A acting on $\mathcal{B}$. Since $d(X[B_0]) = 3$, each vertex in B_0 connects exactly one vertex in B_1. This implies that A^* acts faithfully on B_0, and hence $A^* \leqslant \mathrm{Aut}(X[B_0])$. Since $R(G)$ is transitive on $V(X)$, one has $R(G)/(R(G) \cap A^*) \cong \mathbb{Z}_2$. So, $R(G) \cap A^*$ acts regularly on B_0, and hence $X[B_0]$ is a Cayley graph on $R(G) \cap A^*$. By Proposition 2.4, either $X[B_0]$ is isomorphic to the complete bipartite graph $K_{3,3}$ or the Headwood graph, or $\mathrm{Aut}(X[B_0]) \cong (R(G)) \cap A^*) \rtimes \mathbb{Z}_t$ with $t \leqslant 3$.

Let $X[B_0] \cong K_{3,3}$. Then $G \cong D_{12}$, and $R(G) \cap A^* = \langle R(a) \rangle$ or $\langle R(a^2), R(ba^j) \rangle$ for some $j \in \mathbb{Z}_6$. Without loss of generality, let the identity 1 of G be in B_0. Then $S \cap B_0 = \{a, a^{-1}, a^3\}$ or $\{ba^j, ba^{j+2}, ba^{j+4}\}$. Note that $\{ba^j, ba^{j+2}, ba^{j+4}\}^{\delta_{1,j}} = \{b, ba^2, ba^4\}$. Then S is equivalent either to $\{a, a^{-1}, a^3, ba^i\}$ or to $\Sigma_x = \{b, ba^2, ba^4, x\}$ with $x = a^3, ba, ba^3$ or ba^{-1}. Note that $\delta_{1,i}$ fixes a and maps ba^i to b and that $\langle \delta_{-1,0}, \delta_{-1,2} \rangle$ is transitive on $\{\Sigma_x \mid x = ba, ba^3, ba^{-1}\}$. It follows that $S \equiv \Omega_6, \Omega_7$ or Ω_8.

Let $X[B_0] \ncong K_{3,3}$. Suppose $A^* \cong (R(G) \cap A^*) \rtimes \mathbb{Z}_t$ with $t \leqslant 3$. If $t \leqslant 2$ then $|A^*| \leqslant 4p$ and hence $|A| \leqslant 8p$, forcing $R(G) \trianglelefteq A$, a contradiction. Let $t = 3$. Then $|A^*| = 6p$ and $X[B_0]$ is arc-transitive. Note that a connected cubic arc-transitive graph of order 6 is isomorphic to $K_{3,3}$. Thus, $p > 3$. Take $v \in B_0$. Then $A_v = (A^*)_v$ is a Sylow 3-subgroup of A. Let P be a Sylow p-subgroup of $R(G) \cap A^*$. Then $P = \langle R(a^2) \rangle$ is also a Sylow p-subgroup of A. Since $A^* \cong (R(G)) \cap A^*) \rtimes A_v$, P is characteristic in A^*, and hence normal in A. Let $C = C_A(P)$. Clearly, $\langle R(a)\rangle \leqslant C$, but $R(G) \nleqslant C$. If $A_v \leqslant C$ then $A_v \trianglelefteq A^*$ because $|A^*| = 6p$. By the transitivity of A^* on B_0, A_v fixes all vertices of B_0, forcing $A_v = 1$, a contradiction. Thus, $A_v \nleqslant C$. Since $|A| = 12p$, one has $C = \langle R(a)\rangle$. By Proposition 2.1, $A/\langle R(a)\rangle \leqslant \mathrm{Aut}(P) \cong \mathbb{Z}_{p-1}$. So, $R(G)/\langle R(a)\rangle \trianglelefteq A/\langle R(a)\rangle$, forcing $R(G) \trianglelefteq A$, a contradiction. Thus, $X[B_0]$ is isomorphic to the Heawood graph and $A^* \cong \mathrm{PGL}(2,7)$. By MAGMA [2], $\mathrm{Aut}(\mathrm{PGL}(2,7)) \cong \mathrm{PGL}(2,7)$. Since $A/A^* \cong \mathbb{Z}_2$, by Proposition 2.1, $C_A(A^*) \cong \mathbb{Z}_2$ and hence $A = A^* \times C_A(A^*) \cong \mathrm{PGL}(2,7) \times \mathbb{Z}_2$. By MAGMA [2], all subgroups of A of order 28 are conjugate to $R(G)$, and $|N_A(R(G))| = 84$. Since X is non-symmetric, by Lemmas 4.3 and 4.4, $S \equiv \{b, ba^2, ba^6, a^7\} = \Omega_9$.

In what follows, we let $S = \{s_1, s_2, s_3, s_4\}$. Take $v \in V(X)$. Since X has valency 4, the vertex-stabilizer A_v has order $2^{2+\ell}3^t$ for some non-negative integers ℓ and t. We consider two cases: $t > 0$ and $t = 0$.

Case 1: $t > 0$

Take $v \in V(X)$. Let A_v^* be kernel of A_v acting on the neighborhood $N_X(v)$ of v. Let T be a Sylow 3-subgroup of A_v. If $T \leqslant A_v^*$, the connectivity and vertex-transitivity of X imply $T = 1$, a contradiction. Thus, $T \nleqslant A_v^*$ and hence $\mathbb{Z}_3 \cong TA_v^*/A_v^* \leqslant A_v/A_v^*$. Since X is non-symmetric, one has $A_v/A_v^* \cong \mathbb{Z}_3$ or S_3. It follows that for any $v \in V(X)$, there is a unique vertex $u \in N_X(v)$ such that $A_u = A_v$. Set $F = \{\{u,v\} \in E(X) \mid A_u = A_v\}$ and $\Gamma = X - F$. Then Γ is a cubic graph. For any $g \in A$ and $\{u,v\} \in F$, one has $\{u,v\}^g = \{u^g, v^g\}$. Furthermore, $A_{u^g} = A_u^g = A_v^g = A_{v^g}$. It follows that $\{u,v\}^g = \{u^g, v^g\} \in F$ and hence $F^g = F$. Consequently, A is a vertex-transitive group of automorphisms of Γ. Since $3 \mid |A_v|$, A is also arc-transitive on Γ. By [27, Theorem 2.3], there is no connected cubic symmetric Cayley graph of order $4p$ for each odd prime p. Thus, Γ is disconnected. As Γ is cubic, each component of Γ has order $m = 4$ or $2p$.

Let Γ_i $(0 \leqslant i \leqslant 4p/m)$ be the components of Γ. For each $0 \leqslant i \leqslant 4p/m$, let $B_i = V(\Gamma_i)$ and set $\mathcal{B} = \{B_i \mid 0 \leqslant i \leqslant 4p/m\}$. Then $\mathcal{B}$ is an A-invariant partition of $V(X)$. Suppose $|\Gamma| = 4$. Then $\Gamma \cong K_4$. Consider the quotient graph $X_{\mathcal{B}}$. Clearly, A acts vertex-transitively on $X_{\mathcal{B}}$. It follows that $X_{\mathcal{B}}$ is regular. Since $|\Gamma| = 4$ and $|X_{\mathcal{B}}| = p > 2$, one has $d(X_{\mathcal{B}}) = 4$ or 2. Let $d(X_{\mathcal{B}}) = 4$. Then $p \geqslant 5$ and between any two adjacent vertices in $\mathcal{B}$ there is exactly one edge of X. This implies that A acts faithfully on $\mathcal{B}$. If $p > 5$ then $A \cong \mathbb{Z}_p \rtimes \mathbb{Z}_4$, and hence $R(G) \trianglelefteq A$, a contradiction. If $p = 5$ then $A \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_4, A_5$ or S_5, forcing that A has no subgroups isomorphic to D_{20}, a contradiction. Let $d(X_{\mathcal{B}}) = 2$. Let K be the kernel of A acting on $\mathcal{B}$. Let $B_i = \{x_{2i}, y_{2i}, x_{2i+1}, y_{2i+1}\}$ for each $i \in \mathbb{Z}_p$. Then $V(X) = \{x_j, y_j \mid j \in \mathbb{Z}_{2p}\}$. Clearly, $A/K \cong \mathbb{Z}_p$ or D_{2p}. So, A/K is edge-transitive on $X_{\mathcal{B}}$. Consequently, one may let $E(X) = \{\{x_j, x_{j+1}\}, \{y_j, y_{j+1}\}, \{x_{2j}, y_{2j+1}\}, \{y_{2j}, x_{2j+1}\} \mid j \in \mathbb{Z}_{2p}\}$. Then the stabilizer $K_{x_{2i}}$ of x_{2i} in K also fixes y_{2i} because K fixes each B_i setwise. This implies that $K_{x_{2i}}$ is a 2-group, and hence K is also a 2-group. Since $A/K \cong \mathbb{Z}_p$ or D_{2p}, $|A| = 2^m p$ for some integer m, contrary to $3 \mid |A_v|$. Thus, $|\Gamma| = 2p$. Then $\mathcal{B} = \{B_0, B_1\}$. Since $\Gamma_0 = X[B_0]$ is cubic, by Claim S is equivalent to one of Ω_i $(6 \leqslant i \leqslant 9)$.

Case 2: $t = 0$

In this case, $|A| = 2^{\ell+2}p$. Since $R(G) \ntrianglelefteq A$, one has $\ell \geqslant 2$, and hence $|A| \geqslant 16p$. Let P be a Sylow p-subgroup of A. Since $p > 2$, one has $P \cong \mathbb{Z}_p$. If $P \ntrianglelefteq A$ then by Lemma 4.1,

$S \equiv \Omega_5$. In what follows, assume $P \trianglelefteq A$. Then $P = \langle R(a^2)\rangle$. Consider the quotient graph X_P of X relative to the orbit set of P, and let K be the kernel of A acting on $V(X_P)$. Then, $|X_P| = 4$. Set $V(X_P) = \{\Delta_0, \Delta_1, \Delta_2, \Delta_3\}$, and let $1 \in \Delta_0$. Since $R(G)$ acts on $V(X)$ by right multiplication, one has $\Delta_0 = \langle a^2\rangle$, and Δ_i $(i = 1, 2, 3)$ are right cosets of Δ_0 in G. Clearly, $X_P \cong K_4$ or C_4.

Let $X_P \cong K_4$. Then $A/K \leqslant \mathrm{Aut}(K_4) \cong S_4$. As A/K is a 2-group, one has $A/K \leqslant D_8$. Since $d(X) = 4$, let $X[\Delta_0 \cup \Delta_1] \cong X[\Delta_0 \cup \Delta_2] \cong pK_2$ and $X[\Delta_0 \cup \Delta_3] \cong C_{2p}$. It is easy to see that K acts faithfully on $\Delta_0 \cup \Delta_3$. Then $K \leqslant \mathrm{Aut}(C_{2p}) \cong D_{4p}$. Since K fixes each orbit and $|A| \geqslant 16p$, one has $K \cong D_{2p}$. Assume $s_1 \in \Delta_1, s_2 \in \Delta_2$ and $s_3, s_4 \in \Delta_3$. Let $K_1 = \langle\alpha\rangle$. Then, $K_1 \cong \mathbb{Z}_2$ and α interchanges s_3 and s_4 and fixes s_1 and s_2. Further, $R(G)K = R(G) \rtimes K_1$, implying $K_1 \leqslant \mathrm{Aut}(G, S)$. Clearly, s_3 has order 2 or $2p$. Suppose s_3 has order $2p$. Then, $s_4 = s_3^{-1}$, and $s_1, s_2 \in \mathcal{F}$ (see Eq. (5) for the definition of $\mathcal{F}$). Since $\mathrm{Aut}(G)$ is transitive on $\mathcal{F}$, one may let $s_1 = b$. Since s_3 has order $2p$, there is an automorphism of G mapping s_3 to a and fixing b. Hence, one may let $S = \{b, a, a^{-1}, ba^i\}$ for some $i \in \mathbb{Z}_{2p}$. Since α interchanges a and a^{-1} and fixes b and ba^i, one has $ba^i = (ba^i)^\alpha = ba^{-i}$, implying $a^i = a^p$. It follows that $S = \{a, a^{-1}, b, ba^p\}$. Noting that $S^{\delta_{1,p}} = S$, one has $\delta_{1,p} \in \mathrm{Aut}(G, S)$. Since $|A| = 16p$, $A_1 = \mathrm{Aut}(G, S) = \langle\alpha, \delta_{1,p}\rangle$. By Proposition 2.3, X is normal, a contradiction. Thus, s_3 and s_4 are involutions. Since $X[\Delta_0 \cup \Delta_3] \cong C_{2p}$ and $p > 2$, one has $s_3, s_4 \in \mathcal{F}$ and $\langle s_3, s_4\rangle \cong D_{2p}$. Again since $\mathrm{Aut}(G)$ is transitive on $\mathcal{F}$, one may let $S = \{s_1, s_2, b, ba^{2i}\}$ for some $i \in \mathbb{Z}_p^*$. Clearly, $(i, 2p) = 1$ or 2, and if $(i, 2p) = 2$, then $(i + p, 2p) = 1$. Then, either $\delta_{i,0}$ or $\delta_{i+p,0}$ maps ba^{2i} to ba^2 and fixes b. Hence, one may let $S = \{b, ba^2, s_1, s_2\}$. Without loss of generality, assume further $\Delta_1 = \Delta_0 a^p$ and $\Delta_2 = \Delta_0 ba^p$. Then, $s_1 = a^p$ and $s_2 = ba^{p+2k}$ for some $k \in \mathbb{Z}_p$. Recall that $\alpha \in \mathrm{Aut}(G)$ interchanges b and ba^2 and fixes a^p and ba^{p+2k}. Then, $\alpha = \delta_{-1,2}$. Since $ba^{p+2k} = (ba^{p+2k})^\alpha = ba^{p-2k+2}$, one has $a^{2(2k-1)} = 1$, implying $2k = jp + 1$ for some odd integer j. As a result, $ba^{p+2k} = ba^{(j+1)p+1} = ba$. Thus, $S \equiv \{b, ba, ba^2, a^p\} = \Omega_4$.

Let $X_P \cong C_4$. Then $A/K \leqslant \mathrm{Aut}(C_4) \cong D_8$. Let $\Delta_i \sim \Delta_{i+1}$ with $i \in \mathbb{Z}_4$. We have two possibilities: $d(X[\Delta_0]) = 0$ or $d(X[\Delta_0]) > 0$.

Assume $d(X[\Delta_0]) = 0$. By vertex-transitivity of X, $d(X[\Delta_i]) = 0$ for each $i \in \mathbb{Z}_4$. Then $d(X[\Delta_i \cup \Delta_{i+1}]) = 1, 2$ or 3 for each $i \in \mathbb{Z}_4$. Suppose $d(X[\Delta_i \cup \Delta_{i+1}]) = 2$. Then, $X[\Delta_i \cup \Delta_{i+1}] \cong C_{2p}$. Since $p > 2$, it is easy to see that K acts faithfully on Δ_i. Then $K \leqslant \mathrm{Aut}(C_{2p})$. Since K fixes each orbit, one has $K \leqslant D_{2p}$. As $|A| \geqslant 16p$, one has $K \cong D_{2p}$ and $A/K \cong D_8$. Assume that $s_1, s_2 \in \Delta_1$ and $s_3, s_4 \in \Delta_3$. Let $K_1 = \langle\alpha\rangle$. Then α interchanges s_1 and s_2, and s_3 and s_4. Since $A/K \cong D_8$, there exists $\beta \in A_v$ such that β interchanges $\{s_1, s_2\}$ and $\{s_3, s_4\}$. Hence, $\langle\alpha, \beta\rangle$ is transitive on S, implying that X is symmetric, a contradiction. Thus, $d(X[\Delta_i \cup \Delta_{i+1}]) \neq 2$. So, we can assume $X[\Delta_0 \cup \Delta_1] \cong X[\Delta_2 \cup \Delta_3] \cong pK_2$ and $d(X[\Delta_1 \cup \Delta_2]) = d(X[\Delta_3 \cup \Delta_0]) = 3$. Set $B_0 = \Delta_3 \cup \Delta_0$ and $B_1 = \Delta_1 \cup \Delta_2$. It is easy to see that $\mathcal{B} = \{B_0, B_1\}$ is an A-invariant partition of $V(X)$. By Claim, S is equivalent to one of Ω_i $(6 \leqslant i \leqslant 9)$.

Assume $d(X[\Delta_0]) > 0$. Since $|\Delta_0| = p > 2$, the connectivity and vertex-transitivity of X imply that $X[\Delta_i] \cong C_p$ for each $i \in \mathbb{Z}_4$. So, the set of edges of between Δ_i and Δ_{i+1} is a matching of $\Delta_i \cup \Delta_{i+1}$ for each $i \in \mathbb{Z}_4$. It follows that K acts faithfully on Δ_0. Since $|A| \geqslant 16p$, one has $K \cong \mathrm{Aut}(C_p) \cong D_{2p}$. With no loss of generality, let $s_2, s_4 \in \Delta_0$, $s_1 \in \Delta_1$ and $s_3 \in \Delta_3$. Then s_2 has order p and $s_4 = s_2^{-1}$. Since $G = \langle S \rangle$, one may let $s_1 \in F$, and because of the transitivity of $\mathrm{Aut}(G)$ on F, assume further $s_1 = b$. Let $K_1 = \langle \alpha \rangle$. Then, $|K_1| = 2$ and α interchanges s_2 and s_4 and fixes s_1 and s_3. Furthermore, $KR(G) = R(G) \rtimes K_1$. It follows that $K_1 \leqslant \mathrm{Aut}(G,S)$. Clearly, $s_2 = s_4^{-1} = a^{2j}$ for some $j \in \mathbb{Z}_p^*$. Then, $(j, 2p) = 1$ or 2, and hence either $\delta_{j,0}$ or $\delta_{j+p,0}$ maps a^{2j} to a^2 and fixes b. Thus, one may let $S = \{b, a^2, a^{-2}, s_3\}$. Since $S = S^{-1}$, s_3 is an involution. So, $s_3 = a^p$ or ba^i for some $i \in \mathbb{Z}_{2p}$. If $s_3 = ba^i$ then $ba^i = (ba^i)^\alpha = ba^{-i}$, implying $a^{2i} = 1$. This forces that a^i has order 2, and hence $s_3 = ba^p$. Then, $S = \{a^2, a^{-2}, b, ba^p\}$. Since $S^{\delta_{1,p}} = S$, one has $\delta_{1,p} \in \mathrm{Aut}(G,S)$. As $|A| = 16p$, one has $A_1 = \mathrm{Aut}(G,S) = \langle \alpha, \delta_{1,p} \rangle$. By Proposition 2.3, X is normal, a contradiction. Thus, $s_3 = a^p$, and hence $S = \{a^2, a^{-2}, b, a^p\} = \Omega_3$. □

5 Main result

The main purpose of this paper is to prove the following theorem.

Theorem 5.1 *Let p be a prime and let $X = \mathrm{Cay}(G,S)$ be a connected tetravalent Cayley graph on a group G of order $4p$. Then, either $\mathrm{Aut}(X) = R(G) \rtimes \mathrm{Aut}(G,S)$ or one of the following happens:*

(1) $G = \mathbb{Z}_2^3 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$, $S \equiv \{a, ab, ac, abc\}$, *and* $X = K_{4,4}$.

(2) $G = \mathbb{Z}_4 \times \mathbb{Z}_2 = \langle a \rangle \times \langle b \rangle$, $S \equiv \{a, a^2, a^3, b\}$, *and* $X = K_4 \times K_2$.

(3) $G = \mathbb{Z}_4 \times \mathbb{Z}_2 = \langle a \rangle \times \langle b \rangle$, $S \equiv \{a, a^{-1}, a^2b, b\}$, *and* $X = K_{4,4}$.

(4) $G = \mathbb{Z}_6 \times \mathbb{Z}_2 = \langle a \rangle \times \langle b \rangle$, $S \equiv \{a^3, a, a^{-1}, b\}$ *and* $X = K_{3,3} \times K_2$.

(5) $G = \mathbb{Z}_{2p} \times \mathbb{Z}_2 = \langle a \rangle \times \langle b \rangle$, $S \equiv \{a, ab, a^{-1}, a^{-1}b\}$, *and* $X = C_{2p}[2K_1]$.

(6) $G = \mathbb{Z}_{4p} = \langle a \rangle$, $S \equiv \{a, a^{2p+1}, a^{-1}, a^{2p-1}\}$, *and* $X = C_{2p}[2K_1]$.

(7) $G = Q_8$, $S \equiv \{a, a^{-1}, b, b^{-1}\}$.

(8) $G = D_8$, $S \equiv \{a, a^{-1}, b, a^2\}$, $\{a, a^{-1}, b, ba^2\}$, $\{b, ba, ba^2, ba^{-1}\}$ *or* $\{b, a^2, ba, ba^{-1}\}$.

(9) $G = Q_{4p}$, $S \equiv \Lambda$ (see Eq. (1)).

(10) $G = F_{4p}$, $S \equiv \Theta_i$ $(0 \leqslant i \leqslant 3)$ (see Eq. (3)).

(11) $G = D_{4p}$, $S \equiv \Omega_i$ $(0 \leqslant i \leqslant 8)$ (see Eq. (6)).

Proof. Let $A = \mathrm{Aut}(X)$. If X is normal then $A = R(G) \rtimes \mathrm{Aut}(G,S)$ by Proposition 2.3. To prove the theorem, it suffices to determine the connected tetravalent non-normal Cayley graphs of order $4p$. If G is abelian, then by [1, Theorem 1.2], we have the Cases (1)–(6)

of the theorem. Thus, we may assume that G is non-abelian. From elementary group theory, we know that up to isomorphism there are 4 non-abelian groups of order $4p$.

$$\begin{array}{rcl} Q_{4p} & = & \langle a, b \mid a^{2p} = 1, b^2 = a^p, b^{-1}ab = a^{-1} \rangle; \\ F_{4p} & = & \langle a, b \mid a^p = b^4 = 1, b^{-1}ab = a^{\lambda} \rangle, \lambda^2 \equiv -1 \pmod{p}; \\ D_{4p} & = & \langle a, b \mid a^{2p} = b^2 = 1, b^{-1}ab = a^{-1} \rangle; \\ A_4 & = & \langle a, b \mid a^3 = b^3 = (ab)^2 = 1 \rangle \ (p = 3). \end{array}$$

Let $p > 2$. Suppose $G = A_4$. One may let $a = (1\ 2\ 3)$ and $b = (1\ 2\ 4)$. Since A_4 has only three involutions, S contains a 3-cycle and its inverse. Since all 3-cycles are conjugate in $\mathrm{Aut}(A_4) = S_4$, one may let $S = \{a, a^{-1}, x, y\}$. If x is an involution, then $x, y \in \{(1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$. By MAGMA [2], $|\mathrm{Aut}(X)| = 24$, implying that X is normal, a contradiction. Let x be a 3-cycle. Then, $y = x^{-1}$. Since $\langle (1\ 2\ 3), (1\ 2) \rangle$ is transitive on $\{(1\ 2\ 4), (1\ 4\ 2), (2\ 3\ 4), (2\ 4\ 3), (1\ 3\ 4), (1\ 4\ 3)\}$, one may assume $S = \{a, a^{-1}, b, b^{-1}\}$. By [29, Example 3.7], X is normal, a contradiction. Thus, $G \neq A_4$. By Theorems 3.2,4.2 and 4.6, we have the Cases (9)–(11) of the theorem.

Let $p = 2$. Then $G = Q_8$ or D_8. Assume $G = Q_8$. Since G has an unique involution, one may let $S = \{a, a^{-1}, b, b^{-1}\}$ or $\{a, a^{-1}, ab, (ab)^{-1}\}$. Since a and ab have the same relations as a and b, there is an automorphism of Q_8 mapping a to a and ab to b. Thus, $S \equiv \{a, a^{-1}, b, b^{-1}\}$. It is easy to show that $\mathrm{Cay}(Q_8, \{a, a^{-1}, b, b^{-1}\}) \cong K_{4,4}$ is non-normal. Assume $G = D_8$. Since S generates G, one may let $S = \{a, a^{-1}, b, a^2\}$, or $\{a, a^{-1}, b, x\}$ ($x = ba, ba^2, ba^{-1}$), or $\{b, ba, ba^2, ba^{-1}\}$, or $\{b, a^2, ba, ba^{-1}\}$, or $\{b, a^2, ba^2, ba\}$, or $\{b, a^2, ba^2, ba^{-1}\}$. Note that $\alpha : a \mapsto a^{-1}, b \mapsto b$ and $\beta : a \mapsto a, b \mapsto ba^{-1}$ are automorphisms of G. Then, one may let $S = \{a, a^{-1}, b, a^2\}$, or $\{a, a^{-1}, b, ba\}$, or $\{a, a^{-1}, b, ba^2\}$, or $\{b, ba, ba^2, ba^{-1}\}$, or $\{b, a^2, ba, ba^{-1}\}$. If $S = \{a, a^{-1}, b, ba\}$ then by MAGMA [2], one has $|A| = 16$, implying that X is normal, a contradiction. If $S = \{a, a^{-1}, b, a^2\}$ or $\{b, a^2, ba, ba^{-1}\}$, then by MAGMA [2], one has $|A| = 48$ and since $\mathrm{Aut}(D_8)$ has no elements of order 3, X is non-normal. It is easily checked that $\mathrm{Cay}(G, \{a, a^{-1}, b, ba^2\}) \cong \mathrm{Cay}(G, \{b, ba, ba^2, ba^{-1}\}) \cong K_{4,4}$. This implies that $\mathrm{Cay}(G, \{a, a^{-1}, b, ba^2\})$ and $\mathrm{Cay}(G, \{b, ba, ba^2, ba^{-1}\})$ are non-normal. $\square$

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