Rainbow matchings in r-partite r-graphs

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Abstract

Given a collection of matchings $\mathcal{M} = (M_1, M_2, \ldots, M_q)$ (repetitions allowed), a matching M contained in $\bigcup \mathcal{M}$ is said to be *s-rainbow* for $\mathcal{M}$ if it contains representatives from s matchings M_i (where each edge is allowed to represent just one M_i). Formally, this means that there is a function $\phi : M \to [q]$ such that $e \in M_{\phi(e)}$ for all $e \in M$, and $|Im(\phi)| \geqslant s$.

Let $f(r, s, t)$ be the maximal k for which there exists a set of k matchings of size t in some r-partite hypergraph, such that there is no s-rainbow matching of size t.

We prove that $f(r, s, t) \geqslant 2^{r-1}(s - 1)$, make the conjecture that equality holds for all values of r, s and t and prove the conjecture when $r = 2$ or $s = t = 2$.

In the case $r = 3$, a stronger conjecture is that in a 3-partite 3-graph if all vertex degrees in one side (say V_1) are strictly larger than all vertex degrees in the other two sides, then there exists a matching of V_1. This conjecture is at the same time also a strengthening of a famous conjecture, described below, of Ryser, Brualdi and Stein. We prove a weaker version, in which the degrees in V_1 are at least twice as large as the degrees in the other sides. We also formulate a related conjecture on edge colorings of 3-partite 3-graphs and prove a similarly weakened version.

1 Preliminaries

An r-graph (namely a hypergraph all of whose edges are of the same size r) is said to be *r-partite* if the vertex set $V(H)$ of H can be partitioned into sets $V_1, V_2, \ldots, V_r$ in such a way that every edge in H meets each V_i at precisely one vertex. Generally, we

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shall use the names $V_1, \ldots, V_r$ for the sides of an r-partite hypergraph, without further explicit mention. (There will be one exception, in which we shall enumerate the sides $V_0, V_1, \ldots, V_{r-1}$.) A *legal k-tuple* (of vertices) is a set of vertices containing at most one vertex from each V_i.

Given a set X of vertices in a hypergraph H we write $E(X)$ (or $E_H(X)$ if explicit mention of the hypergraph H is necessary) for the multiset of partial edges $\{e \setminus X \mid X \subseteq e \in H\}$. For an element x we write $E(x)$ for $E(\{x\})$. We write $deg(X)$ for $|E(X)|$ (repetitions counted). Given a set U of vertices, we write $\Delta(U)$ for $\max\{deg(u) \mid u \in U\}$ and $\delta(U)$ for $\min\{deg(u) \mid u \in U\}$.

A *matching* in a hypergraph H is a subset of $E(H)$ (the edge set of H) consisting of disjoint edges. For the sake of brevity, we shall refer to a matching of size t as a t-matching. The maximal size of a matching in a hypergraph H is denoted by $\nu(H)$.

Let $\mathcal{M} = (M_1, M_2, \ldots, M_q)$ be a collection of (possibly repeating) matchings, and let M be a matching contained in $\bigcup \mathcal{M}$. A function $\phi : M \to [q]$ is called an *earmarking* for M if $\phi(e) \in M_{\phi(e)}$ for all $e \in M$. The pair (M, ϕ) is then said to be an earmarked matching. If $|Im(\phi)| \geqslant s$ then the earmarked matching is said to be *s-rainbow*. If M has an earmarking ϕ such that $|Im(\phi)| \geqslant s$ we say also about M by itself that it is s-rainbow.

In this article we study matchings in r-partite r-graphs, and we are concerned with the following question: what size q of a collection of t-matchings $\mathcal{M} = (M_1, M_2, \ldots, M_q)$ in an r-partite r-graph guarantees the existence of an s-rainbow t-matching? (here t and s are fixed parameters, $s \leqslant t$).

Definition 1.1 *Let r, s, t be numbers such that $s \leqslant t$. We write $f(r, s, t)$ for the maximal size of a family of t-matchings in an r-partite r-uniform hypergraph, possessing no s-rainbow t-matching.*

Conjecture 1.2 *$f(r, s, t) = 2^{r-1}(s-1)$ for all $r > 1$ and for all s and t such that $s \leqslant t$.*

Note that the conjectured value of f is independent of t. One direction of the conjecture can be somewhat strengthened:

Conjecture 1.3 *Let $m = 2^{r-1}(s-1)$. Given matchings $M_1, \ldots, M_{m+1}$ in an r-partite r-graph, where M_i is a t-matching for all $i \leqslant m$ and M_{m+1} consists of one edge, there exists an s-rainbow t-matching.*

2 Motivation

Like others who have studied rainbow matchings (see, e.g., [15, 16]) we are motivated by famous conjectures of Ryser [13], Brualdi [5] and Stein [14]. To formulate them, we need the following definitions.

A matrix is called a *Latin rectangle* if no two symbols in the same row or in the same column are equal (here and below the "symbols" are the elements appearing in the cells of the matrix). A *partial transversal* (or plainly *transversal*) in a Latin $m \times n$ rectangle is a set of entries, each in a different row and in a different column, and each containing a different symbol. The partial transversal is called a *full transversal* if it is of size $\min(m, n)$.

Conjecture 2.1 (Ryser-Brualdi-Stein) *In an $n \times n$ Latin square there exists a partial transversal of size $n-1$. If n is odd, then there exists a transversal of size n.*

(Ryser conjectured the odd case, and Brualdi and Stein independently conjectured the case of general n.) In [9] it was shown that an $n \times n$ Latin square contains a partial transversal of size $n - O(\log^2 n)$.

Forming a 3-partite 3-graph whose sides are the rows, columns and symbols, respectively, and assigning to each entry in the Latin square an edge joining the appropriate row, column and symbol, the conjecture can be restated as:

Conjecture 2.2 *If in an $n \times n \times n$ 3-partite 3-graph H every legal pair of vertices has degree 1 then $\nu(H) \geqslant n-1$.*

Here is a more general conjecture, which possibly better captures the essence of the matter:

Conjecture 2.3 *If in a 3-partite 3-graph $E(x)$ is a matching of size $|V_1|$ for every $x \in V_1$ then $\nu \geqslant |V_1| - 1$.*

And even stronger -

Conjecture 2.4 *If in a 3-partite 3-graph $E(x)$ is a matching of size $|V_1|+1$ for every $x \in V_1$ then V_1 is matchable.*

Note that the condition "V_1 is matchable" can also be formulated as "the matchings $E(x)$, $x \in V_1$, have a $|V_1|$-rainbow $|V_1|$-matching". This is the connection to the topic of the present paper. In this terminology, the conjecture says that any collection $(M_1, M_2, \ldots, M_n)$ of $(n+1)$-matchings in a bipartite graph has an n-rainbow n-matching.

In fact, we believe that something stronger than Conjecture 2.4 is true:

Conjecture 2.5 *If in a 3-partite 3-graph H with sides V_1, V_2, V_3 we have $\delta(V_1) > \Delta(V_2 \cup V_3)$ then $\nu(H) = |V_1|$.*

There is a sharp jump here. If $\delta(V_1) = \Delta(V_2 \cup V_3)$ then it is possible that $\nu(H) = \frac{|V_1|}{2}$, as shown by any disjoint union of copies of the 4-edges hypergraph (a_1, b_1, c_1), (a_1, b_2, c_2), (a_2, b_1, c_2), (a_2, b_2, c_1).

We can prove "half" of this conjecture:

Theorem 2.6 *If $\delta(V_1) \geqslant 2\Delta(V_2 \cup V_3)$ then $\nu(H) = |V_1|$. Moreover, for every edge there exists a matching of V_1 containing e.*

The proof will use the following:

Theorem 2.7 *[2] If for every subset U of V_1 there holds $\nu(E_H(U)) > 2(|U|-1)$ then there exists a matching of V_1.*

Proof (of Theorem 2.6) Let e be an arbitrary edge. Let H' be the hypergraph obtained from H by deleting from $V(H)$ the V_1-vertex of e, and deleting all edges meeting e. We have to show that in H' there exists a matching of the first side, $V_1' := V_1 \setminus e$. We shall show that H' satisfies the conditions of Theorem 2.7. Write $D = \Delta(V_2 \cup V_3)$. Let $U \subseteq V_1'$. Then $|E_H(U)| \geqslant 2D|U|$, and since $e \cap (V_2 \cup V_3)$ meets at most $2(D-1)$ edges apart from e itself, it follows that

$$|E_{H'}(U)| \geqslant 2D|U| - 2(D-1) > 2D(|U|-1) \tag{1}$$

(Edges may be counted with multiplicity). By König's edge coloring theorem, which states that the edge chromatic number of a bipartite graph is equal to the maximal degree of the graph, $E_{H'}(U)$ can be partitioned into D matchings, and by (1) one of these matchings must be of size larger than $2(|U|-1)$, proving the desired condition. □

By a simple trick of duplicating all vertices in $V_2 \cup V_3$ and duplicating the $V_2 \cup V_3$ part of each edge we can deduce another "half" version of the conjecture:

Corollary 2.8 *If $\delta(V_1) \geqslant \Delta(V_2 \cup V_3)$ then $\nu(H) \geqslant \frac{|V_1|}{2}$.*

The same trick would give the following corollary of Conjecture 2.5, if indeed this conjecture is true:

Conjecture 2.9 *For k an integer, if $\delta(V_1) > \frac{1}{k}\Delta(V_2 \cup V_3)$ then $\nu(H) \geqslant \frac{|V_1|}{k}$.*

3 The lower bound in Conjecture 1.2

In this section we prove:

Theorem 3.1 $f(r,s,t) \geqslant 2^{r-1}(s-1)$.

Proof It is convenient in this setting to denote the sides of the r-graph under consideration by $V_0, \ldots, V_{r-1}$. For each function $p : [r-1] \to \{0,1\}$ define a matching $M(p)$ of size t, whose i-th edge ($1 \leqslant i \leqslant t$) is $(u_0^i, u_1^i, \ldots, u_{r-1}^i)$, where $u_j^i = i + \sum_{k \leqslant j} p(k) \bmod t$. Let $\mathcal{M}$ consist of $s-1$ copies of each matching $M(p)$, $p \in \{0,1\}^{[r-1]}$. Let M be a matching of size t contained in the union of the matchings $M(p)$. Clearly, M is perfect, namely it covers all vertices of the hypergraph. We claim that it is equal to some $M(p)$. To prove this, let $e = (1, u_1, \ldots, u_{r-1})$ be the edge in M whose first coordinate is 1, and let $f = (2, v_1, \ldots, v_{r-1})$ be the edge whose first coordinate is 2. Suppose that e belongs to a copy of $M(p)$ and f belongs to a copy of $M(q)$. Assume, for contradiction, that $p \neq q$, and let j be the first index such that $p(j) \neq q(j)$. Then since $u_j \neq v_j$ we have $q(j) \geqslant p(j)$, and thus $q(j) > p(j)$. But then the vertex u_j+1 in the j-th side of the hypergraph cannot belong to any edge in M, contradicting the fact that M is perfect.

Continuing this way we see that all edges in M belong to the same $M(p)$. Since there are only $s-1$ copies of $M(p)$ in $\mathcal{M}$, this means that M is not s-rainbow. $\square$

In this example there are lots of repeated edges in the matchings. With some trepidation we conjecture the following:

Conjecture 3.2 *Any set of $2^{r-2}(s-1)+2$ matchings of size t, no two of which sharing an edge, has an s-colored t-matching contained in its union.*

In the case $r=2$ the conjecture is that a set of $t+1$ disjoint t-matchings has a t-rainbow matching. This is yet another generalization of the Ryser-Brualdi-Stein conjecture.

4 The case $r = 2$

Theorem 4.1 $f(2,s,t) = 2(s-1)$.

Remark 4.2 *Drisko [6] essentially proved $f(2,t,t) = 2(t-1)$, where "essentially" means that he considered only the case in which one side of the bipartite graph is of size t.*

Proof For greater transparency of the proof, we first exhibit the main idea in the special case $s = t$. Namely, we first prove Drisko's result, that $f(2,t,t) = 2(t-1)$. Since by Theorem 3.1 $f(2,t,t) \geqslant 2(t-1)$ we only have to show that $f(2,t,t) \leqslant 2(t-1)$. The proof is shorter than that in [6].

Let $M_1, M_2, \ldots, M_{2t-1}$ be a family of t-matchings in a bipartite graph with sides A and B. Let K be a k-rainbow k-matching of maximal size k. We need to show that $k \geqslant t$. Assume for contradiction that $k < t$, and suppose w.l.o.g that the edges of K are taken from the matchings $M_{2t-k}, M_{2t-k+1}, \ldots, M_{2t-1}$.

Write $X_1 = A \cap supp(K)$ (here and below the support, $supp(M)$ of a matching M is its union), so $|X_1| = |K| = k < t$. Since $|M_1| = t > |X_1|$, there exists some edge $e_1 = \{a_1, b_1\} \in M_1$ disjoint from X_1. If e_1 is disjoint from $supp(K)$, then adding it to K results in a $(k+1)$-rainbow $(k+1)$-matching, contrary to the maximality assumption on k. Thus we may assume that e_1 is incident with an edge $f_1 = \{b_1, c_1\} \in K$. Write $X_2 = (X_1 \cup \{b_1\}) \setminus \{c_1\}$. Then $|X_2| = |X_1| = k$.

Since $|M_2| = t > k$, there exists an edge $e_2 = \{a_2, b_2\} \in M_2$ disjoint from X_2 (possibly with $a_2 = a_1$ or $a_2 = c_1$). If $b_2 \notin supp(K)$, then there exists an alternating path, whose application to K (and earmarking the edges e_i appearing in it by color i) results in a $(k+1)$-rainbow $(k+1)$-matching. Thus we may assume that e_2 is incident with an edge $f_2 = \{b_2, c_2\} \in K$. Write now $X_3 = (X_2 \cup \{b_2\}) \setminus \{c_2\}$.

Continuing this way k steps, all edges of K must appear as f_i, and thus in the $k+1^{st}$ step the edge e_{k+1} does not meet $X_{k+1} = supp(K) \cap B$. This yields an alternating path resulting in a $(k+1)$-rainbow $(k+1)$-matching, contradicting the maximality of k.

The proof of the general case, $s \leqslant t$, is similar, with one main difference: instead of leaving each matching M_i after one edge, we continue choosing edges from it, until all edges in some matching M_j represented in K have appeared as f_ℓ's.

To make this idea precise, let $\hat{K}$ be a k-rainbow t-matching, with maximal possible value of k. Let ϕ be the appropriate earmarking function. Assuming that $k < s$, there are at least s matchings M_i not represented in it, so assume that $M_1, \ldots, M_s \notin Im(\phi)$. Let $K = \hat{K} \setminus \{e\}$, where e is an edge which is not the only one of its color. Now start a process similar to that in the above proof, starting with M_1. But after having chosen $e_1 = \{a_1, b_1\} \in M_1$ disjoint from $X_1 = A \cap supp(K)$, and letting $f_1 = \{b_1, c_1\}$ be the edge in K meeting e_1, we do not necessarily switch to M_2. Unless f_1 is the only one of its color in $(K, \phi \restriction K)$, we continue with M_1. Namely, we choose an edge $e_2 = \{a_2, b_2\} \in M_1$ Disjoint from $X_2 = (X_1 \cup \{b_1\}) \setminus \{c_1\}$. If $b_2 \notin supp(K)$ then applying the alternating path ending at b_2 gives a $(k+1)$-rainbow t-matching, contradicting the maximality of k. Note that we use here the assumption that f_1 is not the only one in its color when claiming that the obtained matching is $(k+1)$-rainbow.

Thus we can assume that e_2 meets at B some edge $f_2 = \{b_2, c_2\} \in K$. We continue this way, until the first time in which the set $F_i = \{f_1, \ldots, f_i\}$ satisfies $F_i \supseteq \phi^{-1}(j_1)$ for some j_1. When this happens, say at an index $i = i_1$, we switch to M_2, namely we find an edge $e_{i_1+1} = \{a_{i_1+1}, b_{i_1+1}\} \in M_2$ disjoint from X_{i_1+1}. Assuming, for contradiction, that $b_{i_1+1} \notin supp(K)$, the matching obtained from K by applying the alternating path ending at b_{i_1+1} is a $(k+1)$-rainbow t-matching. Thus we may assume that e_{i_1+1} meets some edge $f_{i_1+1} \in K$. We now continue with M_2, until for some index $i_2 \neq i_1$ the set $F_{i_2} = \{f_1, \ldots, f_{i_2}\}$ satisfies $F_{i_2} \supseteq \phi^{-1}(j_2)$ for some j_2. We then switch to M_3, and so on.

After k such switches all colors j represented in (K, ϕ) are exhausted, which means that at the $k+1^{st}$ stage the edge e_{i_k+1} does not meet $X_{i_k+1} = B \cap supp(K)$, which results in a $(k+1)$-rainbow t-matching. $\square$

5 The case $s = t = 2$

Theorem 5.1 *$f(r, 2, 2) = 2^{r-1}$ for all r.*

Proof Let $M_i,\ i \leqslant q$ be a set of 2-matchings in an r-partite hypergraph, having no 2-rainbow matching. For each i write $M_i = \{e_i, f_i\}$. Let $A_i = e_i$ for $1 \leqslant i \leqslant q$, $\quad A_i = f_{i-q}$ for $q+1 \leqslant i \leqslant 2q$, and $B_i = f_i$ for $1 \leqslant i \leqslant q$, $\quad B_i = e_{i-q}$ for $q+1 \leqslant i \leqslant 2q$. Then $A_i \cap B_i = \emptyset$, while the assumption that there is no 2-rainbow matching implies that $A_i \cap B_j \neq \emptyset$ for all $i \neq j$. In [4] an upper bound was proved on the size of such a general system (A_i, B_i) satisfying this condition. Alon [3], using a multilinear algebraic proof of Bollobás' theorem discovered by Lovász, proved that if the ground set is partitioned into sets V_m such that $|A_i \cap V_m| = r_m$ and $|B_i \cap V_m| = s_m$ for all i and m, then the number of pairs is at most $\prod_i \binom{r_i+s_i}{r_i}$. In our case, taking the sets V_m to be the sides of the hypergraph, we have $r_m = s_m = 1$, implying that the number of pairs, namely $2q$, does not exceed 2^r. Thus $q \leqslant 2^{r-1}$.

Here is a somewhat shorter proof, due to Roy Meshulam [12]. For each edge $g = (a_1, a_2, \ldots, a_r)$ participating in a matching M_i define a polynomial $P_g = \prod(x_i - z(a_i))$, where $z(a_i)$ are numbers that are chosen to be algebraically independent. Then every edge $g \in \bigcup M_i$ has a substitution $\vec{x}_g$ of values for the variables x_j, such that $P_g(\vec{x}_g) \neq 0$ while $P_h(\vec{x}_g) = 0$ for all edges $h \in \bigcup M_i \setminus \{g\}$. To see this, simply take the other edge, say $(b_1, b_2, \ldots, b_r)$ in the matching M_i containing g, and let $\vec{x}_g = (z(b_1), z(b_2), \ldots, z(b_r))$. Thus the polynomials P_g are all independent, and hence their number does not exceed the dimension of the space of multilinear polynomials in $x_1, x_2, \ldots, x_r$, which is 2^r. Thus, again, $2q \leqslant 2^r$, proving the desired conclusion. □

Again, a slight adaptation of the proof yields also Conjecture 1.3 for $s = t = 2$.

6 Edge colorings in r-partite hypergraphs

As in graphs, the edge chromatic number $\chi_e(H)$ of a hypergraph H is defined to be the minimal number of matchings whose union is the entire edge set of the hypergraph. In [7] the following generalization of König's edge coloring theorem was conjectured:

Conjecture 6.1 *In an r-partite r-graph H with maximal vertex degree Δ there holds: $\chi_e(H) \leqslant (r-1)\Delta$.*

We propose the following stronger:

Conjecture 6.2 *In an r-partite r-graph H with sides $V_1, \ldots, V_r$ there holds: $\chi_e(H) \leqslant \max(\Delta(V_1), \sum_{i=2}^{r} \Delta(V_i))$.*

A special case is:

Conjecture 6.3 *If in a 3-partite hypergraph H it is true that $\delta(V_1) \geqslant 2\Delta(V_2 \cup V_3)$, then $\chi_e(H) = \Delta(H)$.*

This generalizes a conjecture of Hilton [11]:

Conjecture 6.4 *The cells of any $m \times 2m$ Latin rectangle can be decomposed into $2m$ transversals.*

The derivation of Hilton's conjecture is done by the transformation described in Section 2. In [8] an asymptotic version of Conjecture 6.4 was proved, namely that the cells of any $m \times (1+\epsilon)m$ Latin rectangle can be decomposed into $(1+\epsilon)m$ transversals, for m large enough (ϵ being any fixed positive number). Also, "half" of Conjecture 6.4 was proved there: the cells of any $m \times 4m$ Latin rectangle can be decomposed into $4m$ transversals for any m. It is interesting to note that while Hilton's conjecture may be true for $m+1$ replacing $2m$, in Conjecture 6.3 the bound $2\Delta(V_2 \cup V_3)$ on $\Delta(V_1)$ is sharp. The example is obtained from the 4-edges hypergraph $(a_1, b_1, c_1), (a_1, b_2, c_2), (a_2, b_1, c_2), (a_2, b_2, c_1)$ (the example used for the sharpness of Conjecture 2.5), with edges multiplied $\frac{m}{2}$ times, and dangling edges added in V_1, so that the degrees in V_1 are $2m-1$, and $\Delta(V_2 \cup V_3) = m$. Since the line graph of the hypergraph (whose vertices are the edges of the hypergraph, two of them being joined if they intersect) contains a clique of size $2m$, we have $\chi_e(H) \geqslant 2m$, namely the edge chromatic number is larger than the degrees in V_1.

Here we shall prove "half" of Conjecture 6.3:

Theorem 6.5 *If in a 3-partite hypergraph H it is true that $\delta(V_1) \geqslant 4\Delta(V_2 \cup V_3)$, then $\chi_e(H) = \Delta(H)$.*

Proof The proof uses an idea taken from [10]. In fact, we shall use a simplified version, used in [1], for which an appropriate name is the "beating boys" method. Write $k = \Delta(V_2 \cup V_3)$ and $t = \Delta(H)$. let f be a maximum t-coloring of the edges, namely a partial coloring that colors a maximal number of edges. Assuming the negation of the theorem, there exists an edge (x, y, z) not colored by f. For any vertex u denote by $E(u)$ the set of edges containing u. Then there exists a color not appearing among the colors given by f to edges in $E(x)$. Without loss of generality, we may assume that this color is 1. For every $u \in V_1$, if there exists in $E(u)$ an edge e colored 1 by f, remove from $E(u)$ all edges $b = (u, v, w) \in dom(f)$ (where $dom(f)$, the domain of f, is the set of edges colored by f,) for which there exists some edge $h = (p, q, r)$ such that (a) $p \neq u$, (b)$f(h) = f(b)$ and (c) h meets e. (The edge b is a "beating boy" of h, deleted just because it carries the same color as h.) Let E' be the set of edges remaining after all these deletions, and let H' be the hypergraph whose edge set is E'.

Since $|E(u)| \geqslant 4k$ for every $u \in V_1$, and since every edge $e = (u, v, w)$ meets at most $2k$ edges of the form (p, q, r), where $p \neq u$, it follows that $E'(u) \geqslant 2k$ for every $u \in V_1$. By Theorem 2.6 it follows that there exists in H' a matching M of V_1, containing the edge (x, y, z). Color all edges in M by color 1, and for every edge $a = (p, q, r)$ colored 1 by f, if there exists an edge $b = (p, v, w) \in M$ (namely, an edge in M sharing with a its V_1-vertex), re-color a by the color $f(b)$. This produces a coloring f' whose domain is larger than that of f, since (x, y, z) is colored by it. A contradiction (to the assumption that f is not total) will be shown if we prove that f' is a legal coloring. Assuming it is not, there exist two intersecting edges $b = (p, v_1, w_1)$ and $c = (q, v_2, w_2)$ colored by the same color, say i. This could occur only if one of them, say b, was colored 1 by f and it was recolored i because an edge $c \in M \cap E(p)$ was colored i. But this is impossible, because in such a case b would have been removed from E as the "beating boy" of c. $\square$

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