Non-isomorphic graphs with cospectral symmetric powers

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Abstract

The symmetric m-th power of a graph is the graph whose vertices are m-subsets of vertices and in which two m-subsets are adjacent if and only if their symmetric difference is an edge of the original graph. It was conjectured that there exists a fixed m such that any two graphs are isomorphic if and only if their m-th symmetric powers are cospectral. In this paper we show that given a positive integer m there exist infinitely many pairs of non-isomorphic graphs with cospectral m-th symmetric powers. Our construction is based on theory of multidimensional extensions of coherent configurations.

1 Introduction

Let G be a graph with vertex set V.[1] Given a positive integer m the *symmetric m-th power* of G is the graph $G^{\{m\}}$ whose vertices are m-subsets of V and in which two m-subsets are adjacent if and only if their symmetric difference is an edge in G [11]. One of the motivations for studying symmetric powers comes from the graph isomorphism problem which is to recognize in an efficient way whether two given graphs are isomorphic. To be more precise we cite a paragraph from paper [2]:

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[1] All graphs in this paper are undirected, without loops and multiple edges.

> If it were true for some fixed m that any two graphs G and H are isomorphic if and only if their m-th symmetric powers are cospectral, then we would have a polynomial-time algorithm for solving the graph isomorphism problem. For a pessimist this suggests that, for each fixed m, there should be infinitely many pairs of non-isomorphic graphs G and H such that $G^{\{m\}}$ and $H^{\{m\}}$ are cospectral.

In this paper we justify the pessimistic point of view by proving the following theorem.

Theorem 1.1 *Given a positive integer m there exist infinitely many pairs of non-isomorphic graphs G and H such that $G^{\{m\}}$ and $H^{\{m\}}$ are cospectral.*

Let us discuss briefly the main ideas on which our construction is based. It was an old observation of B. Weisfeiler and A. Leman that any isomorphism between two graphs induces the canonical similarity between their schemes [13] (Sections 3 and 2 provide a background on general schemes and schemes of graphs respectively). However, the canonical similarity may exist even for non-isomorphic graphs. In any of these cases the graphs are called equivalent (Definition 3.3). For example any two strongly regular graphs with the same parameters are equivalent. The first crucial observation in the proof of Theorem 1.1 is that any two equivalent graphs are cospectral (Theorem 3.4).

There is an efficient algorithm to test whether or not two graphs are equivalent [13]. Therefore the graph isomorphism problem would be solved if any two equivalent graphs were isomorphic. However, this is not true because the equivalence of two graphs roughly speaking means that there is an isomorphism preserving bijection between the sets of their m-subgraphs only for $m \leqslant 3$. More elaborated technique taking into account the m-subgraphs for larger m was developed in [12]. In scheme theory this method naturally leads to study the m-extension of a scheme which is the canonically defined scheme on the Cartesian m-fold product of the underlying set (see [5] and Section 4). It is almost obvious that the canonical similarity between the schemes of two isomorphic graphs can be extended to the canonical similarity between the m-extensions of that schemes. This enables us to introduce the notion of the m-equivalence of graphs so that the 1-equivalence coincides with the equivalence. The second crucial observation in the proof of Theorem 1.1 is that the m-th symmetric powers of any two m-equivalent graphs are equivalent, and then cospectral (Theorem 4.4).

What we said above shows that to prove Theorem 1.1 it suffices to find an infinite family of pairs of non-isomorphic schemes (associated with appropriate graphs) the m-extensions of which are similar. In Section 5 we modify a construction of such schemes found in [5] so that any involved scheme was the scheme of a suitable graph. The graphs from Theorem 1.1 are exactly those obtained in this way.

After finishing this paper the authors found that Theorem 1.1 was independently proved in the recent article [1]. However, our approach is completely different from the one used in [1]: the technique used there is based on analysis of the m-dimensional Weisfeiler-Lehman algorithm given in [4], whereas we use general theory of schemes in spirit of [8].

2 Preliminaries

In our presentation of the scheme theory we follow recent survey [8].

2.1. Schemes. Let V be a finite set and let $\mathcal{R}$ be a partition of $V \times V$. Denote by $\mathcal{R}^*$ the set of all unions of the elements of $\mathcal{R}$. Obviously, $\mathcal{R}^*$ is closed with respect to taking the complement R^c of R in $V \times V$, unions and intersections. Below for $R \subset V \times V$ we denote by R^T the set of all pairs (u, v) with $(v, u) \in R$ and put $R(u) = \{v \in V : \ (u, v) \in R\}$ for $u \in V$.

Definition 2.1 *A pair $\mathcal{C} = (V, \mathcal{R})$ is called a coherent configuration or a scheme on V if the following conditions are satisfied:*

(C1) *$\mathcal{R}^*$ contains the diagonal $\Delta(V)$ of the Cartesian product $V \times V$,*

(C2) *$\mathcal{R}^*$ contains the relation R^T for all $R \in \mathcal{R}$,*

(C3) *given $R, S, T \in \mathcal{R}$, the number $c_{R,S}(u, v) = |R(u) \cap S^T(v)|$ does not depend on the choice of $(u, v) \in T$.*

The elements of V, $\mathcal{R} = \mathcal{R}(\mathcal{C})$, $\mathcal{R}^* = \mathcal{R}^*(\mathcal{C})$ and the numbers (C3) are called the *points*, the *basis relations*, the *relations* and the *intersection numbers* of $\mathcal{C}$, respectively; the latter are denoted by $c_{R,S}^T$. From the definition it easily follows that

$$R,\, S \in \mathcal{R}^* \ \Rightarrow \ R \cdot S \in \mathcal{R}^*, \tag{1}$$

where $R{\cdot}S$ denotes the relation on V consisting of all pairs (u, w) for which $c_{R,S}(u, w) \neq 0$.

2.2. Fibers. The point set of the scheme $\mathcal{C}$ is the disjoint union of its *fibers* or *homogeneity sets*, i.e. those $X \subset V$ for which $\Delta(X) = \{(x, x) : \ x \in X\}$ is a basis relation. Given $R \in \mathcal{R}$ there exist uniquely determined fibers X and Y such that $R \subset X \times Y$. Moreover, it follows from (C3) that the number

$$|R(u)| = c_{R,R^T}^{\Delta(X)} \tag{2}$$

does not depend on $u \in X$. It is simple but useful fact that sets $X, Y \subset V$ are unions of some fibers if and only if $X \times Y \in \mathcal{R}^*$. The scheme $\mathcal{C}$ is called *homogeneous* (or an *association scheme*, [3]) if the set V is (the unique) fiber of it.

2.3. Isomorphisms and similarities. Two schemes are called *isomorphic* if there exists a bijection between their point sets preserving the basis relations. Any such bijection is called an *isomorphism* of these schemes. Two schemes $\mathcal{C}$ and $\mathcal{C}'$ are called *similar* if

$$c_{R,S}^T = c_{R^\varphi,S^\varphi}^{T^\varphi}, \qquad R, S, T \in \mathcal{R}, \tag{3}$$

for some bijection $\varphi : \mathcal{R} \to \mathcal{R}'$, $R \mapsto R^\varphi$, such bijection is called a *similarity* from $\mathcal{C}$ to $\mathcal{C}'$. Every isomorphism $f : \mathcal{C} \to \mathcal{C}'$ *induces* a similarity φ such that $R^\varphi = R^f$ for all $R \in \mathcal{R}$

where $R^f = \{(u^f, v^f) : (u,v) \in R\}$. The set of all isomorphisms from $\mathcal{C}$ to $\mathcal{C}'$ inducing a similarity φ is denoted by $\mathrm{Iso}(\mathcal{C}, \mathcal{C}', \varphi)$. The set

$$\mathrm{Aut}(\mathcal{C}) = \mathrm{Iso}(\mathcal{C}, \mathcal{C}, \mathrm{id}_{\mathcal{R}})$$

where $\mathrm{id}_{\mathcal{R}}$ is the identity permutation on $\mathcal{R}$, forms a permutation group on V called the *automorphism group* of the scheme $\mathcal{C}$.

Any similarity $\varphi : \mathcal{C} \to \mathcal{C}'$ induces the bijection $X \mapsto X^\varphi$ between the sets of unions of fibers, and the bijection $R \mapsto R^\varphi$ from $\mathcal{R}^*(\mathcal{C})$ onto $\mathcal{R}^*(\mathcal{C}')$. One can prove that $V^\varphi = V'$ and

$$(R^T)^\varphi = (R^\varphi)^T, \quad R \in \mathcal{R}^*(\mathcal{C}). \tag{4}$$

Moreover, E^φ is an equivalence relation of $\mathcal{C}'$ if and only if E is an equivalence relation of $\mathcal{C}$. It should be noted that all the above bijections preserve the inclusion relation, unions and intersections.

2.4. Quotients. Let $X \subset V$ and let $E \subset V \times V$ be an equivalence relation. Then $E \cap (X \times X)$ is also the equivalence relation; the set of its classes is denoted by X/E. For any $R \subset V \times V$ denote by $R_{X/E}$ the relation on the latter set consisting of all pairs (Y, Z) for which $R_{Y,Z} = R \cap (Y \times Z)$ is non-empty.

Suppose that the set X and E are respectively a union of fibers and an equivalence relation of the scheme $\mathcal{C}$. Then the set $\mathcal{R}_{X/E}$ consisting of all nonempty relations $R_{X/E}$, $R \in \mathcal{R}$, forms a partition of $X/E \times X/E$ and

$$\mathcal{C}_{X/E} = (X/E, \mathcal{R}_{X/E})$$

is a scheme. If $E = \Delta(V)$, we identify X/E with X, set $R_X = R_{X,X}$ and treat $\mathcal{C}_X$ as a scheme on X. Any similarity $\varphi : \mathcal{C} \to \mathcal{C}'$ induces a similarity

$$\varphi_{X/E} : \mathcal{C}_{X/E} \to \mathcal{C}'_{X'/E'}, \ R_{X/E} \mapsto R'_{X'/E'}$$

where $X' = X^\varphi$, $R' = R^\varphi$ and $E' = E^\varphi$.

2.5. Tensor product. Let R_i be a relation on a set V_i, $i = 1, 2$. Denote by $R_1 \otimes R_2$ the relation on $V_1 \times V_2$ consisting of all pairs $((u_1, u_2), (v_1, v_2))$ with $(u_1, v_1) \in R_1$ and $(u_2, v_2) \in R_2$.

Let $\mathcal{C}_1 = (V_1, \mathcal{R}_1)$ and $\mathcal{C}_2 = (V_2, \mathcal{R}_2)$ be schemes. Then the set $\mathcal{R}_1 \otimes \mathcal{R}_2$ consisting of all relations $R_1 \otimes R_2$ with $R_1 \in \mathcal{R}_1$ and $R_2 \in \mathcal{R}_2$ is a partition of $V \times V$ where $V = V_1 \times V_2$, and

$$\mathcal{C}_1 \otimes \mathcal{C}_2 = (V_1 \times V_2, \mathcal{R}_1 \otimes \mathcal{R}_2)$$

is a scheme which is called the *tensor product* of $\mathcal{C}_1$ and $\mathcal{C}_2$. Any two similarities $\varphi_1 : \mathcal{C}_1 \to \mathcal{C}'_1$ and $\varphi_2 : \mathcal{C}_2 \to \mathcal{C}'_2$ induce a similarity

$$\varphi : \mathcal{C}_1 \otimes \mathcal{C}_2 \to \mathcal{C}'_1 \otimes \mathcal{C}'_2, \ R_1 \otimes R_2 \mapsto R'_1 \otimes R'_2$$

where $R'_1 = (R_1)^{\varphi_1}$ and $R'_2 = (R_2)^{\varphi_2}$.

2.6. Direct sum. Let $\mathcal{H}_i$ be the fiber set of the scheme $\mathcal{C}_i$, $i = 1, 2$. Denote by V the disjoint union of V_1 and V_2, and by $\mathcal{R}_0$ the set of all relations $X \times Y$ with $X \in \mathcal{H}_i$ and $Y \in \mathcal{H}_j$ where $\{i, j\} = \{1, 2\}$. Then the set $\mathcal{R}_1 \boxplus \mathcal{R}_2 = \cup_{i=0}^{2} \mathcal{R}_i$ is a partition of the set $V \times V$, and

$$\mathcal{C}_1 \boxplus \mathcal{C}_2 = (V, \mathcal{R}_1 \boxplus \mathcal{R}_2)$$

is a scheme called the *direct sum* of the schemes $\mathcal{C}_1$ and $\mathcal{C}_2$. Clearly, $\mathcal{C}_{V_i} = \mathcal{C}_i$, $i = 1, 2$, and $\mathcal{C}$ is the smallest scheme on V having this property. It was proved in [8] that any two similarities $\varphi_1 : \mathcal{C}_1 \to \mathcal{C}_1'$ and $\varphi_2 : \mathcal{C}_2 \to \mathcal{C}_2'$ induce a uniquely determined similarity

$$\varphi : \mathcal{C}_1 \boxplus \mathcal{C}_2 \to \mathcal{C}_1' \boxplus \mathcal{C}_2'$$

such that $\varphi_{V_i} = \varphi_i$, $i = 1, 2$.

2.7. Closure. The set of all schemes on V is partially ordered by inclusion of their sets of relations:

$$\mathcal{C} \leqslant \mathcal{C}' \overset{def}{\Leftrightarrow} \mathcal{R}^* \subset (\mathcal{R}')^*,$$

in this case we say that $\mathcal{C}$ is a *subscheme* of $\mathcal{C}'$. For sets $\mathcal{R}_1, \ldots, \mathcal{R}_s$ of binary relations on V we denote by $[\mathcal{R}_1, \ldots, \mathcal{R}_s]$ the smallest scheme $\mathcal{C} = (V, \mathcal{R})$ such that $\mathcal{R}_i \subset \mathcal{R}^*$ for all i. Usually instead of $\mathcal{R}_i$ in brackets we write R_i (resp. V_i or $\mathcal{C}_i$), if $\mathcal{R}_i = \{R_i\}$ (resp. $\mathcal{R}_i = \{\Delta(V_i)\}$ or $\mathcal{R}_i = \mathcal{R}(\mathcal{C}_i)$).

3 The scheme of a graph

In their seminal paper, B. Weisfeiler and A. Leman (1968) associated with a graph a special matrix algebra containing its adjacency matrix [13]. In modern terms this algebra is nothing else than the adjacency algebra of a scheme defined as follows.

3.1. Let $G = (V, R)$ be a graph with vertex set V and edge set R. Then $[G] := [R]$ is called the *scheme of* G (see Subsection 2.7). Thus it is the smallest scheme on V for which R is a union of its basis relations. For example, it is easily seen that if G is a complete graph with at least 2 vertices, then the scheme $[G]$ has two basis relations: Δ and Δ^c where $\Delta = \Delta(V)$. Below we write $[G, X_1, X_2, \ldots, X_t]$ instead of $[R, X_1, X_2, \ldots, X_t]$ for $X_i \subset V$.

In general, it is quite difficult to find the scheme $[G]$ explicitly. Some information on its structure is given in the following statement. Below given a set $X \subset V$ and an integer d we put

$$X_d = \{v \in V : \ |R(v) \cap X| = d\}. \tag{5}$$

Clearly, $V_d = \{v \in V : \ d_G(v) = d\}$ where $d_G(v)$ is the valency of the vertex v in the graph G.

Lemma 3.1 *Let G be a graph with vertex set V and d be an integer. If $X \subset V$ is a union of fibers of the scheme $[G]$, then so is the set X_d. In particular, the set V_d is a union of fibers of $[G]$.*

Proof. Suppose that X is a union of fibers of $[G]$. Without loss in generality we may assume that $X_d \neq \emptyset$. Then there is a fiber Y such that $Y \cap X_d \neq \emptyset$. So there exists a vertex $y \in Y$ such that $|R(y) \cap X| = d$. Since R is a union of basis relations of $[G]$, equality (2) shows that $d = |R(y) \cap X| = |R(y') \cap X|$ for all $y' \in Y$. Therefore $Y \subset X_d$. Thus X_d is a union of fibers of the scheme $[G]$ and we are done. ∎

Given graphs $G = (V, R)$ and $K = (U, S)$ with disjoint vertex sets, and a set $X \subset V$ one can form a graph

$$G \boxplus_X K = (V \cup U, R \cup S \cup (X \times U) \cup (U \times X)). \tag{6}$$

For $X = \emptyset$ and $X = V$ this graph is known respectively as the disjoint union and the join of the graphs G and K. The scheme of the disjoint union was found in [7]. Below we find the scheme of the graph $G \boxplus_X K$ for special sets X; this result will be used in Section 5.

Theorem 3.2 *Let $G = (V, R)$ and $K = (U, S)$ be graphs with disjoint vertex sets and $X \subset V$. Suppose that $|V| \leqslant |U|$, and (a) $|X| + d_K(x) < |V|$ for all $x \in U$ and (b) no vertex of G is adjacent to all vertices from X. Then*

$$[G \boxplus_X K] = [G, X] \boxplus [K].$$

Proof. Denote by V' the vertex set of the graph $G' = G \boxplus_X K$. Let us prove that

$$d_{G'}(x) \geqslant n > d_{G'}(y), \qquad x \in X,\ y \in V' \setminus X \tag{7}$$

where $n = |V|$. Indeed, from (6) it follows that $X \subset U_m$ where $m = |U|$ and U_m is defined as in (5) with $X = U$, $d = m$ and R being the edge set of G'. Therefore, given $x \in X$ we have

$$d_{G'}(x) \geqslant m \geqslant n$$

which proves the left-hand side inequality in (7). To prove the right-hand side inequality let $y \in V' \setminus X$. If $y \in V$, then obviously $d_{G'}(y) = d_G(y) \leqslant n - 1$ and we are done. Otherwise, $y \in U$. But then $d_{G'}(y) = |X| + d_K(y)$ and the claim follows from condition (a).

From inequalities (7) and condition (b) it follows respectively that

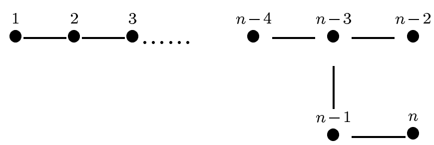
$$X = \bigcup_{d=n}^{n+m} (V')_d, \qquad U = X_k$$

where $k = |X|$. So X, and hence U, is a union of fibers of the scheme $[G']$ by Lemma 3.1. This implies that so is the set $V \setminus X$. However, in this case $R = (R')_V$ and $S = (R')_U$ are relations of $[G']$ where R' is the edge set of the graph G'. Therefore

$$[G \boxplus_X K] \geqslant [R, X, S] \geqslant [G, X] \boxplus [K]$$

(here we used the minimality of the direct sum). Since the converse inclusion is obvious, we are done. ∎

We will apply Theorem 3.2 to the tree $K = T_n$ with $n \geqslant 7$ vertices on the picture below (it has 3, $n-4$ and 1 vertices with valencies 1, 2 and 3 respectively):



A straightforward check shows that the automorphism group $\mathrm{Aut}(T_n)$ of T_n is trivial. On the other hand, from [7, Theorems 4.4,6.3] it follows that given an arbitrary tree T the basis relations of the scheme $[T]$ are the orbits of the group $\mathrm{Aut}(T)$ acting on the pairs of vertices. Thus the scheme $[T_n]$ is *trivial*, i.e. any relation on its point set is the relation of the scheme.

3.2. Let $G = (V, R)$ and $G' = (V, R')$ be graphs with schemes $\mathcal{C}$ and $\mathcal{C}'$ respectively.

Definition 3.3 *The graphs G and G' are called equivalent, $G \sim G'$, if there exists a similarity $\varphi : \mathcal{C} \to \mathcal{C}'$ such that $R^\varphi = R'$.*

It is easy to see that φ is uniquely determined (when it exists); we call it the *canonical similarity* from $\mathcal{C}$ to $\mathcal{C}'$. Not every two equivalent graphs are isomorphic (e.g. take non-isomorphic strongly regular graphs with the same parameters [3]), but if they are, then any isomorphism between them induces the canonical similarity between their schemes (see Subsection 2.3). This simple observation appeared in [12] and the exact sense of it is as follows:

$$\mathrm{Iso}(G, G') = \mathrm{Iso}(\mathcal{C}, \mathcal{C}', \varphi) \tag{8}$$

where the left-hand side is the set of all isomorphisms from G onto G', and the right-hand side is the set of all isomorphisms from $\mathcal{C}$ onto $\mathcal{C}'$ inducing φ (see Subsection 2.3). Thus the graphs G and G' are isomorphic if and only if they are equivalent and the canonical similarity between their schemes is induced by a bijection.

Theorem 3.4 *Any two equivalent graphs are cospectral.*

Proof. Let G be a graph with the adjacency matrix $A = A(G)$ the distinct eigenvalues $\theta_1, \ldots, \theta_s$ of which occur in the spectrum of A with multiplicities $\mu_1, \ldots, \mu_s$. Denote by $\mathcal{A}$ the adjacency algebra of the scheme $[G]$; by definition it is the matrix algebra over the complex number field $\mathbb{C}$ spanned by the set $\{A(R) : R \in \mathcal{R}\}$ where $\mathcal{R}$ is the set of the basis relations of $[G]$. This algebra is closed with respect to the Hadamard (componentwise) product and taking transposes.

Suppose that the graph G is equivalent to a graph G'. Then there exists the canonical similarity $\varphi : [G] \to [G']$ (taking the edge set of G to that of G'). By the linearity it induces the matrix algebra isomorphism (denoted by the same letter)

$$\varphi : \mathcal{A} \to \mathcal{A}', \quad A(R)^\varphi \mapsto A(R^\varphi) \ (R \in \mathcal{R}),$$

preserving the Hadamard product and the transpose, where $\mathcal{A}'$ is the adjacency algebra of the scheme $[G']$, and $A(R)$ and $A(R^\varphi)$ are the adjacency matrices of the relations R and R^φ. By the canonicity φ takes the matrix A to the matrix $A' = A(G')$. Therefore these matrices have the same minimal polynomial and hence the same eigenvalues. Denote by μ_i' the multiplicity of θ_i in A'. Then

$$\sum_{i=1}^{s} (\theta_i)^j \mu_i = \mathrm{tr}(A^j) = \mathrm{tr}((A')^j) = \sum_{i=1}^{s} (\theta_i)^j \mu_i', \qquad 0 \leqslant j \leqslant s-1$$

(see [9, 5.5]). This gives a system of s linear equations with the unknowns $\mu_i - \mu_i'$, $i = 1, \ldots, s$. The determinant of this system being the Vandermonde determinant equal to $\pm \prod_{i \neq j}(\theta_i - \theta_j) \neq 0$. Therefore $\mu_i - \mu_i' = 0$ for all i, and so the matrices A and A' have the same characteristic polynomials. Thus the graphs G and G' are cospectral. ∎

4 The m-equivalence of graphs

4.1. Let m be a positive integer. Following [8] by the *m-extension* of a scheme $\mathcal{C} = (V, \mathcal{R})$ we mean the smallest scheme $\widehat{\mathcal{C}}^{(m)}$ on V^m containing the m-fold tensor power of $\mathcal{C}$ as a subscheme and the reflexive relation corresponding to the diagonal Δ_m of the Cartesian m-fold power of V, or more precisely

$$\widehat{\mathcal{C}}^{(m)} = [\mathcal{C}^m, \Delta_m].$$

Clearly, the 1-extension of $\mathcal{C}$ coincides with $\mathcal{C}$. For $m > 1$ it is difficult to find the basis relations of the m-extension explicitly. However, in any case it contains any *elementary cylindric relation*

$$\mathrm{Cyl}_{i,j}(R) = \{(x, y) \in V^m \times V^m : \ (x_i, y_j) \in R\}$$

where $R \in \mathcal{R}^*$ and $i, j \in \{1, \ldots, m\}$ (see [6, Lemma 6.2]). Since the set of all relations of a scheme is closed with respect to intersections, we obtain the following statement.

Theorem 4.1 *Let $\mathcal{T}$ be a family of relations $R_{i,j} \in \mathcal{R}^*$ where $i, j = 1, \ldots, m$. Then the m-extension of the scheme $\mathcal{C}$ contains any cylindric relation*

$$\mathrm{Cyl}_m(\mathcal{T}) = \bigcap_{i,j=1}^{m} \mathrm{Cyl}_{i,j}(R_{i,j}). \qquad ∎$$

Given a permutation $\sigma \in \mathrm{Sym}(m)$ denote by $\mathcal{T}_\sigma = \mathcal{T}_\sigma(V)$ the family of relations $R_{i,j}$ coinciding with Δ or Δ^c depending on whether or not $j = i^\sigma$ respectively. Since obviously $\Delta, \Delta^c \in \mathcal{R}^*$, from Theorem 4.1 it follows that the m-extension of $\mathcal{C}$ contains the relation

$$C_\sigma = \mathrm{Cyl}_m(\mathcal{T}_\sigma).$$

Given an m-tuple x in the domain of C_σ we have $x_i \neq x_j$ for all $i, j = 1, \ldots, m$ with $j \neq i^\sigma$. This implies that the set $S_x = \{x_1, \ldots, x_m\}$ consists of exactly m elements and hence $|V| \geqslant m$. Under the latter assumption $C_\sigma \neq \emptyset$. If, in addition, the permutation σ is the identity, then it is easy to see that $C_\sigma = \Delta(V_m)$ where V_m is the set of m-tuples of V with pairwise different coordinates,

$$V_m = \{x \in V^m : \ |S_x| = m\}. \tag{9}$$

In particular, V_m is a union of fibers of the m-extension of the scheme $\mathcal{C}$. Denote by $E_m = E_m(V)$ the union of all relations C_σ with $\sigma \in \mathrm{Sym}(V)$. Then obviously

$$E_m = \{(x, y) \in V_m \times V_m : \ S_x = S_y\}. \tag{10}$$

Therefore, E_m is an equivalence relation on V_m. One can see that any of its classes is of the form $\widehat{U} = \{x \in V_m : \ S_x = U\}$ for some set $U \in V^{\{m\}}$. Moreover, the mapping $U \mapsto \widehat{U}$ is a bijection from $V^{\{m\}}$ onto V_m/E_m.

Let $G = (V, R)$ be a graph and $m \leqslant |V|$. Denote by $\mathcal{T}_R = \mathcal{T}_R(V)$ the family of m^2 relations $R_{i,j}$ such that $R_{1,2} = R_{2,1} = \Delta$, $R_{1,1} = R_{2,2} = R$ and $R_{i,j} = \Delta^c$ for the other i, j. Then obviously the relation $R_m = \mathrm{Cyl}_m(\mathcal{T}_R)$ is of the form

$$R_m = \{(x, y) \in V^m \times V^m : \ S_x \,\Delta\, S_y = \{x_1, y_1\} \text{ and } (x_1, y_1) \in R\} \tag{11}$$

where $S_x \,\Delta\, S_y$ is the symmetric difference of the sets S_x and S_y. In particular, $R_1 = R$. Since the latter is a relation of the scheme $\mathcal{C} = [G]$, from Theorem 4.1 it follows that the m-extension of $\mathcal{C}$ contains the relation R_m. The graph with vertex set V_m/E_m and edge set $(R_m)_{V_m/E_m}$ is denoted by G_m. The following statement shows that this graph is isomorphic to the symmetric m-th power $G^{\{m\}}$ of the graph G (see the first paragraph of Section 1).

Theorem 4.2 *Let G be a graph with vertex set V. Then the bijection $f : U \mapsto \widehat{U}$ is an isomorphism of the graph $G^{\{m\}}$ onto the graph G_m. Moreover, the scheme $[G_m]$ is a subscheme of the scheme $(\widehat{\mathcal{C}}^{(m)})_{V_m/E_m}$ where $\mathcal{C} = [G]$.*

Proof. The first statement immediately follows from equality (11). To prove the second statement, it suffices to note that the edge set R of the graph G is a relation of the scheme $\mathcal{C}$, and hence the edge set $(R_m)_{V_m/E_m}$ of the graph G_m is a relation of the scheme $(\widehat{\mathcal{C}}^{(m)})_{V_m/E_m}$. ∎

4.2. Let $\mathcal{C}$ and $\mathcal{C}'$ be similar schemes. A similarity $\varphi : \mathcal{C} \to \mathcal{C}'$ is called the *m-similarity* if there exists a similarity $\widehat{\varphi} = \widehat{\varphi}^{(m)}$ from the m-extension of $\mathcal{C}$ to the m-extension of $\mathcal{C}'$ such that

$$(\Delta_m)^{\widehat{\varphi}} = \Delta'_m \quad \text{and} \quad \widehat{\varphi}|_{\mathcal{C}^m} = \varphi^m,$$

where φ^m is the similarity from $\mathcal{C}^m$ to $\mathcal{C}'^m$ induced by φ (see Subsection 2.5). Clearly, any similarity is 1-similarity. If $m > 1$, then the similarity $\widehat{\varphi}$ does not necessarily exist.

However, if it does, then it is uniquely determined and is called the *m-extension* of φ. It is important to note that any similarity induced by an isomorphism has the obvious m-extension and hence is an m-similarity for all m. Further information on m-similarities can be found in [5, 6].

Let $\varphi : \mathcal{C} \to \mathcal{C}'$ be an m-similarity. It was proved in [6, Lemma 6.2] that for any relation R of the scheme $\mathcal{C}$ the m-extension of φ takes the elementary cylindric relation $\mathrm{Cyl}_{i,j}(R)$ to the elementary cylindric relation $\mathrm{Cyl}_{i,j}(R^\varphi)$. Since the m-extension of φ preserves the intersection of relations of the m-extension of $\mathcal{C}$, we obtain the following statement.

Theorem 4.3 *Let $\varphi : \mathcal{C} \to \mathcal{C}'$ be an m-similarity and $\mathcal{T}$ be a family of relations $R_{i,j}$ of the scheme $\mathcal{C}$, $i,j = 1,\dots,m$. Then*

$$(\mathrm{Cyl}_m(\mathcal{T}))^{\widehat{\varphi}} = \mathrm{Cyl}_m(\mathcal{T}^\varphi)$$

where $\widehat{\varphi}$ is the m-extension of φ and $\mathcal{T}^\varphi$ is the family of relations $R_{i,j}^\varphi$. ∎

Let V and V' be the point sets of the schemes $\mathcal{C}$ and $\mathcal{C}'$ respectively. Let us define the set V'_m and the relation E'_m by formulas (9) and (10) with V replaced by V'. Then $\Delta(V'_m) = C'_\sigma$ with σ being the identity permutation and E'_m the union of all C'_σ with $\sigma \in \mathrm{Sym}(m)$ where $C'_\sigma = \mathrm{Cyl}_m(\mathcal{T}'_\sigma)$ with $\mathcal{T}'_\sigma = \mathcal{T}_\sigma(V')$. However, by Theorem 4.3 the similarity $\widehat{\varphi}$ takes the relation C_σ to the relation C'_σ for all σ. Therefore

$$(V_m)^{\widehat{\varphi}} = V'_m, \qquad (E_m)^{\widehat{\varphi}} = E'_m. \tag{12}$$

Analogously, by Theorem 4.3 for any relation R of the scheme $\mathcal{C}$ the similarity $\widehat{\varphi}$ takes the relation $R_m = \mathrm{Cyl}_m(\mathcal{T}_R)$ to the relation $R'_m = \mathrm{Cyl}_m(\mathcal{T}'_{R'})$ where $R' = R^\varphi$ and $\mathcal{T}'_{R'} = \mathcal{T}_{R'}(V')$. Thus

$$(R_m)^{\widehat{\varphi}} = R'_m. \tag{13}$$

Since V_m and E_m are respectively a union of fibers and a relation of the scheme $\widehat{\mathcal{C}} = \widehat{\mathcal{C}}^{(m)}$, the similarity $\widehat{\varphi}$ induces similarity

$$\widehat{\varphi}_{V_m/E_m} : \widehat{\mathcal{C}}_{V_m/E_m} \to \widehat{\mathcal{C}}'_{V'_m/E'_m} \tag{14}$$

where $\widehat{\mathcal{C}'} = \widehat{\mathcal{C}}'^{(m)}$ (see Subsection 2.4). By equalities (12) and (13) it takes the relation $(R_m)_{V_m/E_m}$ to the relation $(R'_m)_{V'_m/E'_m}$. By the second part of Theorem 4.2 this implies that the similarity (14) induces a similarity from the scheme $[G_m]$ to the scheme $[G'_m]$ preserving their edge sets. Thus the graphs G_m and G'_m are equivalent.

Definition 4.4 *The graphs $G = (V, R)$ and $G' = (V', R')$ are called m-equivalent if there exists an m-similarity $\varphi : [G] \to [G']$ such that $R^\varphi = R'$.*

Clearly, graphs are 1-equivalent if and only if they are equivalent. Moreover, it can be proved that any m-similarity is also a k-similarity for all $k = 1,\dots,m$ (see [6]). So any two m-equivalent graphs are k-equivalent. Now we are ready to prove the main result of this section.

Theorem 4.5 *Let G and G' be m-equivalent graphs. Then the graphs $G^{\{m\}}$ and $(G')^{\{m\}}$ are equivalent. In particular, they are cospectral.*

Proof. The second statement follows from Theorem 3.4. To prove the first one we observe that by Theorem 4.2 the graphs $G^{\{m\}}$ and G_m are isomorphic. By equality (8) this implies that they are equivalent. Similarly, the graphs $(G')^{\{m\}}$ and G'_m are equivalent. Finally, due to the paragraph before Definition 4.4 the graphs G_m and G'_m are also equivalent. Thus $G^{\{m\}} \sim G_m \sim G'_m \sim (G')^{\{m\}}$ and we are done. ∎

5 Construction

5.1. In this subsection we give a brief summary of the results from [5, Section 5]. Below s is a positive integer and $I = \{1, \ldots, s\}$.

Let $\mathcal{C} = (V, \mathcal{R})$ be a scheme on $4s$ points with s fibers $V_1, \ldots, V_s$ each of size 4. Suppose that for each $i \in I$ the scheme $\mathcal{C}_{V_i}$ has 4 basis relations and $\mathrm{Aut}(\mathcal{C}_{V_i})$ is an elementary Abelian group of order 4. Then $\mathcal{C}_{V_i}$ contains exactly three equivalence relations $E_{i,1}$, $E_{i,2}$ and $E_{i,3}$ with two classes of size 2. Moreover, for any distinct $i, j \in I$ the set $\mathcal{R}_{i,j} = \{R \in \mathcal{R} :\ R \subset V_i \times V_j\}$ contains 1, 2 or 4 basis relations. Suppose that

$$|\mathcal{R}_{i,j}| \in \{1, 2\}, \qquad i, j \in I,\ i \neq j.$$

Denote by K the graph with vertex set I in which the vertices i and j are adjacent if and only if $|\mathcal{R}_{i,j}| = 2$. Suppose that K is a cubic graph, i.e. the neighborhood $K(i)$ of any vertex i in K is of cardinality 3.

Definition 5.1 *The scheme $\mathcal{C}$ is called a Klein scheme*[2] *associated with K if for each $i \in I$ there exists a bijection $\alpha : \{1, 2, 3\} \to K(i)$ such that*

$$R \cdot R^T = E_{i,j}, \qquad R \in \mathcal{R}_{i,\alpha(j)},\ j = 1, 2, 3. \tag{15}$$

For any connected cubic graph K on s vertices one can construct a Klein scheme $\mathcal{C} = (V, \mathcal{R})$ on $4s$ points associated with K. Moreover, for each $i \in I$ the mapping $\varphi_i : \mathcal{R} \to \mathcal{R}$ defined by

$$R^{\varphi_i} = \begin{cases} (V_i \times V_j) \setminus R, & \text{if } R \in \mathcal{R}_{i,j} \text{ with } j \in K(i), \\ (V_j \times V_i) \setminus R, & \text{if } R \in \mathcal{R}_{j,i} \text{ with } j \in K(i), \\ R, & \text{otherwise.} \end{cases} \tag{16}$$

is a similarity from $\mathcal{C}$ to itself. Suppose that K has no separators[3] of cardinality greater or equal than $3m$. Then the similarity φ_i is an m-similarity that is not induced by a bijection. Since given a positive integer k there exist infinitely many non-isomorphic cubic graphs with no separators of cardinality k (see e.g. [10]), we obtain the following result.

[2] The adjacency algebra of a Klein scheme belongs to the class $\mathcal{K}^*$ defined and studied in [5, Subsections 5.2-5.4].

[3] A set $X \subset I$ is a *separator* of a graph K with s vertices if any connected component of the subgraph of K induced on $I \setminus X$ has $\leqslant s/2$ vertices.

Theorem 5.2 *Given a positive integer m there exist infinitely many pairwise non-isomorphic Klein schemes $\mathcal{C}$ such that given $i \in I$ the mapping φ_i is an m-similarity from $\mathcal{C}$ to itself that is not induced by a bijection.* ∎

5.2. Let $\mathcal{C} = (V, \mathcal{R})$ be a Klein scheme associated with a graph K. We keep the notation of the previous subsection. A symmetric relation $R \in \mathcal{R}^*$ is called *generic* if

$$j \in K(i) \ \Rightarrow \ R_{i,j} \in \mathcal{R}_{i,j}, \qquad i, j \in I, \tag{17}$$

where $R_{i,j} = R \cap (V_i \times V_j)$. To construct such a relation given $i, j \in I$ with $j \in K(i)$ choose a basis relation $R_{i,j} \in \mathcal{R}_{i,j}$. Then the union of all of them is generic whenever $R_{i,j} = R_{j,i}^T$ for all i, j.

Lemma 5.3 *For any generic relation R we have $[R, V_1, \ldots, V_s] = \mathcal{C}$.*

Proof. Set $\mathcal{C}' = [R, V_1, \ldots, V_s]$. Since R and $\Delta_i = \Delta(V_i)$ $(i \in I)$ are relations of $\mathcal{C}$, the minimality of the scheme $\mathcal{C}'$ implies that $\mathcal{C} \geqslant \mathcal{C}'$. Therefore, it suffices to verify that any relation $S \in \mathcal{R}$ is a relation of $\mathcal{C}'$. However, in this case $S \in \mathcal{R}_{i,j}$ for some $i, j \in I$. Therefore, the required statement holds for $i \neq j$ because in this case we have

$$S = \begin{cases} R_{i,j} = \Delta_i R \Delta_i, & \text{if } |\mathcal{R}_{i,j}| = 1, \\ (V_i \times V_j) \setminus R_{i,j}, & \text{if } |\mathcal{R}_{i,j}| = 2. \end{cases}$$

Suppose that $i = j$. Then $S = \Delta_i$ or $S = E_{i,k} \setminus \Delta_i$ for some $k \in \{1, 2, 3\}$. On the other hand, from (15) it follows that $E_{i,k}$ is a relation of $\mathcal{C}'$ for all k. Thus S is a relation of $\mathcal{C}'$. ∎

Let $\mathcal{C}$ be a Klein scheme on $4s$ points with $s \geqslant 2$ and R be a generic relation of it. Set $G_0 = (V, R)$ and $n_0 = |V|$. By means of operation (6) we successively define the graph

$$G_{i+1} = G_i \boxplus_{V_{i+1}} T_{n_i}, \qquad i = 0, \ldots, s-1,$$

where n_i is the number of vertices of G_i, and $T_{n_i} = (U_i, S_i)$ is the tree defined at the end of Subsection 3.1 (without loss in generality, we may assume that the sets U_i are pairwise disjoint). It immediately follows from the definition that the vertex set and the edge set of the graph G_{i+1} are respectively the union of V with $\cup_{j=0}^{i} U_j$, and the union of R with the symmetric relation

$$R_{i+1} = \bigcup_{j=0}^{i} (S_j \cup (U_j \times V_j) \cup (V_j \times U_j)) \tag{18}$$

where $V_0 = V$. In particular, the graph G_i (as well as the graph T_{n_i}) has $n_i = 2^{i+2}s$ vertices for all i. Moreover,

$$|V_{i+1}| + d_{T_{n_i}}(x) \leqslant 7 < n_i$$

for all vertices x of T_{n_i}. Finally, it is easily seen that any vertex of G_i is adjacent with at most 2 vertices of the set V_{i+1} which is of size 4. Thus by Theorem 3.2 with $G = G_i$, $K = T_{n_i}$ and $X = V_{i+1}$, we conclude that

$$[G_{i+1}] = [G_i, V_{i+1}] \boxplus T_{n_i}, \qquad i = 0, \ldots, s-1.$$

Using induction on i one can see that

$$[G_{i+1}] = [G_0, V_1, \ldots, V_{i+1}] \boxplus ([T_{n_0}] \boxplus \cdots \boxplus [T_{n_i}]). \tag{19}$$

However, since $s \geqslant 2$, the scheme of the graph T_{n_i} is trivial for all i (see the end of Subsection 3.1). Therefore the direct sum in the right-hand side of equality (19) is also trivial. Thus by Lemma 5.3 the scheme of the graph $G(\mathcal{C}, R) = G_s$ can be found as follows:

$$[G(\mathcal{C}, R)] = [R, V_1, \ldots, V_s] \boxplus \mathcal{D} = \mathcal{C} \boxplus \mathcal{D} \tag{20}$$

where $\mathcal{D}$ is a trivial scheme on $n_s - n_0$ points.

5.3. Proof of Theorem 1.1. For a positive integer m denote by $\mathcal{C} = (V, \mathcal{R})$ the Klein scheme from Theorem 5.2. Then given $i \in I$ the mapping φ_i defined by (16) is an m-similarity of the scheme $\mathcal{C}$ to itself that is not induced by a bijection. Let

$$G = G(\mathcal{C}, R), \qquad G' = G(\mathcal{C}, R^{\varphi})$$

where $R \in \mathcal{R}^*$ is a generic relation and φ is the similarity of the scheme in the right-hand side of (20) induced by φ_i. Then by Theorem 4.5 it suffices to prove that G and G' are non-isomorphic m-equivalent graphs.

From (16) and (17) it follows that R is a generic relation of $\mathcal{C}$ if and only if so is the relation $R^{\varphi} = R^{\varphi_i}$. Therefore, formula (20) implies that

$$[G'] = \mathcal{C} \boxplus \mathcal{D} = [G]. \tag{21}$$

On the other hand, from [5, Theorem 7.6] it follows that φ is an m-similarity if and only if both $\varphi_{\mathcal{C}}$ and $\varphi_{\mathcal{D}}$ are m-similarities. However, $\varphi_{\mathcal{C}} = \varphi_i$ is an m-similarity of $\mathcal{C}$ by the choice of φ_i, and $\varphi_{\mathcal{D}}$ is obviously an m-similarity of $\mathcal{D}$. Thus we conclude that φ is an m-similarity of the scheme $\mathcal{C} \boxplus \mathcal{D}$.

From (20) it follows that any basis relation of the scheme $\mathcal{C} \boxplus \mathcal{D}$ other than basis relation of $\mathcal{C}$ is one of the relations $\{u\} \times Y$, $Y \times \{u\}$ where $u \in U_i$ for some i and Y is either V_j or a singleton of U_j for some j. In particular, the similarity φ leaves fixed any such a relation. Therefore,

$$(R \cup R_s)^{\varphi} = R^{\varphi} \cup R_s$$

where the relation R_s is defined by (18) for $i = s-1$. Since $R \cup R_s$ and $R^{\varphi} \cup R_s$ are the edge sets of the graphs G and G' respectively, and φ is an m-similarity of the scheme $[G] = [G']$ to itself, we conclude that the graphs G and G' are m-equivalent. However, from the choice of φ_i it follows that φ is not induced by a bijection. Therefore, due to equality (8) the graphs G and G' are not isomorphic. ∎

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References

[1] A. Alzaga, R. Iglesias, R. Pignol, *Spectra of symmetric powers of graphs and the Weisfeiler-Lehman refinements*, eprint: http://arxiv.org/pdf/0801.2322 (2009), 1–14.

[2] K. Audenaert, C. Godsil, G. Royle, T. Rudolph, *Symmetric squares of graphs*, Journal of Combinatorial Theory, **B97** (2007), 74-90.

[3] A. E. Brouwer, A. M. Cohen, A. Neumaier, *Distance-regular graphs*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, 18. Berlin etc.: Springer-Verlag, 1989.

[4] J. Y. Cai, M. Fürer, N. Immerman, *An optimal lower bound on the number of variables for graph identification*, Combinatorica, **12** (1992), 389–410.

[5] S. Evdokimov, I. Ponomarenko, *On highly closed cellular algebras and highly closed isomorphisms*, Electronic Journal of Combinatorics, **6** (1999), #R18.

[6] S. Evdokimov, I. Ponomarenko, *Separability number and schurity number of coherent configurations*, Electronic Journal of Combinatorics, **7** (2000), #R31.

[7] S. Evdokimov, I. Ponomarenko, G. Tinhofer, *Forestal algebras and algebraic forests (on a new class of weakly compact graphs)*, Discrete Mathematics, **225** (2000), 149–172.

[8] S. Evdokimov, I. Ponomarenko, *Permutation group approach to association schemes*, European Journal of Combinatorics, **30** (2009), 6, 1456–1476.

[9] D. G. Higman, *Coherent configurations. I. Ordinary representation theory*, Geometriae Dedicata, **4** (1975), 1–32.

[10] M. Morgenstern, *Existence and explicit constructions of $q+1$ regular Ramanujan graphs for every prime power q*, Journal of Combinatorial Theory, **B62** (1994), 44-62.

[11] T. Rudolph, *Constructing physically intuitive graph invariants*, eprint: http://arxiv.org/pdf/quant-ph/0206068 (2002), 1–3.

[12] B. Weisfeiler (editor), *On construction and identification of graphs*, Lecture Notes in Mathematics, **558** (1976).

[13] B. Weisfeiler, A. A. Leman, *Reduction of a graph to a canonical form and an algebra arising during this reduction*, Nauchno-Technicheskaya Informatsia, Seriya 2, 1968, no. 9, 12–16. (Russian)