Symmetric functions for the generating matrix of the Yangian of $\mathfrak{gl}_n(\mathbb{C})$

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Abstract

Analogues of classical combinatorial identities for elementary and homogeneous symmetric functions with coefficients in the Yangian are proved. As a corollary, similar relations are deduced for shifted Schur polynomials.

Introduction

In this note we prove some combinatorial relations between the analogues of symmetric functions for the Yangian of the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$. The applications of the results are illustrated by deducing properties of Capelli polynomials and shifted symmetric polynomials. Some of these properties were obtained, for example, in [16] from the definitions of shifted symmetric functions. Here, due to the existence of evaluation homomorphism, they become immediate consequences of similar combinatorial formulas in the Yangian. The elementary symmetric functions in the Yangian of the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ are known to be generators of Bethe subalgebra. Bethe subalgebra finds numerous applications in quantum integrable models of XXX type and Gaudin type ([10], [11], [12]). We describe the inverse of the universal differential operator for higher transfer matrices of XXX model.

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Notations and Preliminary facts

The following notations will be used through the paper. All non-commutative determinants are defined to be row determinants. Namely, if X is a matrix with entries $(x_{ij})_{i,j=1,\ldots n}$ in an associative algebra A, put

$$\det X = \mathrm{rdet} X = \sum_{\sigma\in S_n} (-1)^{\sigma} x_{1\sigma(1)} \ldots x_{n\sigma(n)},$$

where the sum is taken over all permutations of n elements. We also define the following types of powers of the matrix X:

$$X^{[k]} := X_1 \ldots X_k \in \mathrm{End}(\mathbb{C}^n)^{\otimes k} \otimes A,$$

where

$$X_s = \sum_{ij} 1 \otimes \cdots \otimes \underset{s}{E_{ij}} \otimes \cdots \otimes 1 \otimes x_{ij},$$

and

$$X^k := X \ldots X \in \mathrm{End}(\mathbb{C}^n) \otimes A.$$

(This is just regular multiplication of matrices).

Definition of Yangian

Let $P_{l,m}$ be a permutation of l-th and k-th copies of $\mathbb{C}^n$ in $(\mathbb{C}^n)^{\otimes k}$:

$$P_{l,m} = \sum_{ij} 1 \otimes \cdots \otimes 1 \otimes \underset{l}{E_{ij}} \otimes \cdots \otimes \underset{m}{E_{ji}} \otimes \cdots \otimes 1. \tag{1}$$

Let u be an independent variable. Consider the Yang matrix

$$R(u) = 1 - \frac{P_{1,2}}{u} \quad \in \mathrm{End}\,(\mathbb{C}^n)^{\otimes 2}[[u^{-1}]].$$

Definition 1. The Yangian $Y(n)$ of the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ is an associative unital algebra, generated by the elements $\{t_{ij}^{(k)}\}$, $(i,j = 1 \ldots n,\ k = 1, 2, ...)$, satisfying the relation

$$R(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u-v). \tag{2}$$

Here $T(u) = (t_{ij}(u))_{1\leqslant i,j\leqslant k}$ is the generating matrix of $Y(n)$: the entries of $T(u)$ are formal power series with coefficients in $Y(n)$:

$$t_{ij}(u) = \sum_{k=0}^{\infty} \frac{t_{ij}^{(k)}}{u^k}, \quad t_{ij}^{(k)} \in Y(n), \quad t_{ij}^{(0)} = \delta_{i,j}.$$

The definition of $Y(n)$ implies that many formulas involving its generating matrix $T(u)$ contain the shifts of the parameter u. To simplify some of these formulas, it is convenient to introduce a shift-variable τ (we follow [18], [10], [1] in this approach). Any element $f(u)$ of $Y(n)[[u^{-1}]]$ we identify with the operator of multiplication by this formal power series, acting on $Y(n)[[u^{-1}]]$. Let $\tau^{\pm 1} = e^{\frac{\pm d}{du}}$. These operators also act on $Y(n)[[u^{-1}]]$ by shifts of the variable u:

$$\tau^{\pm}(g(u)) = e^{\frac{\pm d}{du}}(g(u)) = g(u \pm 1), \quad g(u) \in Y(n)[[u^{-1}]]. \tag{3}$$

Thus, under this identification of shifts $\tau^{\pm 1}$ and the elements $f(u)$ of $Y(n)[[u^{-1}]]$ with differential operators acting on the algebra $Y(n)[[u^{-1}]]$, we can write the following commutation relation:

$$\tau^{\pm} f(u) = f(u \pm 1)\, \tau^{\pm}. \tag{4}$$

We will use the relation (4) to write the formulas for symmetric functions $e_k(u,\tau)$, $h_k(u,\tau)$, $p_k^{\pm}(u,\tau)$, defined in the next section.

Symmetrizer and antisymmetrizer.

Define the projections to the symmetric and antisymmetric part of $(\mathbb{C}^n)^{\otimes k}$:

$$A_k = \frac{1}{k!} \sum_{\sigma \in \mathcal{S}_k} (-1)^{\sigma} \sigma, \qquad S_k = \frac{1}{k!} \sum_{\sigma \in \mathcal{S}_k} \sigma.$$

These are the elements of the group algebra $\mathbb{C}[\mathcal{S}_k]$ of the permutation group, acting on $(\mathbb{C}^n)^{\otimes k}$ by permuting the tensor components. The operators enjoy the listed below properties.

Proposition 1. *(a)*

$$A_k^2 = A_k \qquad \textit{and} \qquad S_k^2 = S_k.$$

(b) With abbreviated notations $R_{ij} = R_{ij}(v_i - v_j)$, write

$$R(v_1, \dots v_m) = (R_{m-1,m})(R_{m-2,m} R_{m-2,m-1}) \dots (R_{1,m} \dots R_{1,2}).$$

Then $A_k = \frac{1}{k!} R(u, u-1, \dots u-k+1)$, and $S_k = \frac{1}{k!} R(u, u+1, \dots u+k-1)$.

(c)

$$A_k T_1(u) \dots T_m(u-k+1) = T_k(u-k+1) \dots T_1(u) A_k,$$

$$S_k T_1(u) \dots T_k(u+k-1) = T_k(u+k-1) \dots T_1(u) S_k.$$

(d)

$$tr\,(A_n T_1(u) \dots T_n(u-n+1)) = qdet T(u).$$

(the expression $qdet T(u)$ is called the quantum determinant of the matrix $T(u)$ and is defined by $qdet T(u) = \sum_{\sigma \in \mathcal{S}_n} t_{\sigma(1),1}(u) \dots t_{\sigma(n),n}(u-n+1)$, [7], [8].)

(e)

$$A_{k+1} = \frac{1}{k+1} A_k\, R_{k,k+1} \left(\frac{1}{k} \right) A_k,$$

$$S_{k+1} = \frac{1}{k+1} S_k\, R_{k,k+1}\left(-\frac{1}{k}\right)\, S_k.$$

(h) Put

$$B_l^{\mp} := \frac{1}{l!}\, R_{l-1,l}\left(\frac{\pm 1}{l-1}\right) R_{l-2,l-1}\left(\frac{\pm 1}{l-2}\right) \dots R_{1,2}\,(\pm 1)\,.$$

Then

$$S_k = B_2^+ B_3^+ \dots B_k^+, \quad A_k = B_2^- B_3^- \dots B_k^-,$$

Proof. The properties (a) – (d) are contained in Propositions 2.9 – 2.11 in [7]. The property (e) can be shown by induction. The statement of (h) follows from (e). Note that (b) and (h) give different presentations of symmetrizer and antisymmetrizer in terms of R-matrices. For example, by (b), $A_3 = \frac{1}{6} R_{23}(1)\, R_{13}(2)\, R_{12}(1)$, and by property (h), $A_3 = \frac{1}{12} R_{12}(1)\, R_{23}\left(\frac{1}{2}\right)\, R_{12}(1)$. The expressions (h) for the symmetrizer and antisymmetrizer are simple to deduce, but the author is not aware of its appearance in the preceding literature. □

Elementary and homogeneous symmetric functions

Definition 2. The following formal power sums in u^{-1} with coefficients in $Y(n)$ are the analogues of ordinary symmetric functions:

Elementary symmetric functions:

$$e_k(u) = tr\,(A_k T_1(u) \dots T_k(u-k+1)), \quad k = 1, 2, \dots, n.$$

Homogeneous symmetric functions:

$$h_k(u) = tr\,(S_k T_1(u) \dots T_k(u+k-1)), \quad k = 1, 2, \dots.$$

Power sums:

$$p_k^{\pm}(u) = \mathrm{tr}\,(T(u) T(u \pm 1) \dots T(u \pm (k-1)))\,, \quad k = 1, 2, \dots.$$

Bethe subalgebra

Let Z be a matrix of size n by n with complex coefficients. Consider $\mathcal{B}(\mathfrak{gl}_n(\mathbb{C}, Z))$ – the commutative subalgebra of the Yangian $Y(n)$, generated by the coefficients of all the series

$$b_k(u, Z) = \mathrm{tr}\,(A_n T_1(u) \dots T_k(u-k+1) Z_{k+1} \dots Z_n), \quad k = 1, 2 \dots n.$$

It is called *Bethe subalgebra* (see, for example [3], [4], [5], [14]). The introduced above elements $e_k(u)$ are proportional to generators of the (degenerate) Bethe subalgebra – with Z being the identity matrix:

Lemma 1. $e_k(u) = \frac{n!}{k!\,(n-1)^{n-k}}\, b_k(u, Id)$ *for* $k = 1, 2, \dots, n$.

Proof. Let $\mathrm{tr}_{(1\ldots a)}$ denote the trace by the first a components in the tensor product $(\mathrm{End}\,(\mathbb{C}^n))^{\otimes(m+1)}$ for some fixed m, where $m = 0, 1, \ldots, (n-1)$. By Proposition 1 (c), (e), and the cyclic property of the trace, we obtain that

$$\begin{aligned} &\mathrm{tr}_{(1\ldots m+1)}\left(A_{m+1}\, T_1(u)\ldots T_k(u-k+1)\otimes 1^{\otimes^{m+1-k}}\right) \\ &= \frac{(n-1)}{m+1}\,\mathrm{tr}_{(1\ldots m)}\left(A_m\, T_1(u)\ldots T_k(u-k+1)\otimes 1^{\otimes^{m-k}}\right). \end{aligned} \tag{5}$$

From (5) one can show by induction that

$$b_k(u,\mathrm{Id}) = \mathrm{tr}_{(1\ldots n)}(A_n T_1(u)\ldots T_k(u-k+1)\otimes 1^{\otimes^{n-k}}) = \frac{(n-1)^{n-k}k!}{n!}\, e_k(u).$$

□

Remark. In case of Z with simple spectrum, the corresponding Bethe subalgebra is a maximal commutative subalgebra of $Y(n)$. In the case of $Z = \mathrm{Id}$ subalgebra $\mathcal{B}(\mathfrak{gl}_n(\mathbb{C},\mathrm{Id}))$ does not enjoy this property, but the center of $Y(n)$ is contained in the Bethe subalgebra properly. For example, the algebra $\mathcal{B}(\mathfrak{gl}_n(\mathbb{C},\mathrm{Id}))$ contains the coefficients of the series $\mathrm{tr}(T(u)T(u-1)\ldots T(u-k))$, which are not central in general.

Proposition 2. *Let the matrices $B_k^{\pm}$ be defined as in Proposition 1, (h). Then for $k = 1, 2, \ldots, n$*

$$\begin{aligned} e_k(u) &= tr\left(B_k^-\, T_1(u)\ldots T_k(u-k+1)\right), \\ h_k(u) &= tr\left(B_k^+\, T_1(u)\ldots T_k(u+k-1)\right), \\ e_k(u+k-1) &= tr\left(A_k\, T_1(u)\ldots T_k(u+k-1)\right), \\ h_k(u-k+1) &= tr\left(S_k\, T_1(u)\ldots T_k(u-k+1)\right). \end{aligned} \tag{6}$$

Proof. By Proposition 1 part (e),

$$\begin{aligned} e_k(u) &= \frac{1}{k}\mathrm{tr}\left(A_{k-1}\, R_{k-1,k}\left(\frac{1}{k-1}\right) A_{k-1}T_1(u)\ldots T_k(u-k+1)\right), \\ &= \frac{1}{k}\mathrm{tr}\left(R_{k-1,k}\left(\frac{1}{k-1}\right) A_{k-1}T_1(u)\ldots T_k(u-k+1)\, A_{k-1}\right), \\ &= \frac{1}{k}\mathrm{tr}\left(R_{k-1,k}\left(\frac{1}{k-1}\right) A_{k-1}T_1(u)\ldots T_k(u-k+1)\right). \end{aligned} \tag{7}$$

The last equality follows from properties (c) and (a) of the Proposition 1. Applying the same Proposition 1 part (e) to A_{k-1}, and observing, that A_{k-2} commutes with $R_{k-1,k}\left(\frac{1}{k-1}\right)$, we obtain that

$$e_k(u) = \frac{1}{k(k-1)}\mathrm{tr}\left(R_{k-1,k}\left(\frac{1}{k-1}\right) R_{k-2,k-1}\left(\frac{1}{k-2}\right) A_{k-2}T_1(u)\ldots T_k(u-k+1)\right).$$

Proceeding by induction, we obtain the first statement of (6). The second formula is proved similarly, and the last two can be checked directly. □

For $k = 1, 2, \ldots n$, introduce the following notations:

$$\begin{aligned} e_k(u,\tau) &= \mathrm{tr}\left(A_k(T(u)\tau^{-1})^{[k]}\right), \\ h_k(u,\tau) &= \mathrm{tr}\left((S_k T(u)\tau)^{[k]}\right), \\ p_k^{\pm}(u,\tau) &= \mathrm{tr}\left((T(u)\tau^{\pm 1})^k\right). \end{aligned} \tag{8}$$

Observe that

$$e_k(u,\tau) = e_k(u)\tau^{-k}, \quad h_k(u,\tau) = h_k(u)\tau^k, \quad p_k^{\pm}(u,\tau) = p_k^{\pm}(u)\tau^{\pm k}. \tag{9}$$

As it was mentioned, the insertion of the shift τ in the formulas allows to write some relations in the classical form:

Proposition 3. *Let $\lambda = (\lambda_1, \ldots \lambda_m)$ be a composition of number k, $1 \leqslant k \leqslant n$ (the order of parts is important). Let $a_i = \lambda_1 + \cdots + \lambda_i$, $(i = 1, 2, \ldots m)$. Then*

$$e_k(u,\tau) = \sum_{\lambda} \frac{(-1)^{k-m}}{a_1 a_2 \ldots a_m} p^-_{\lambda_1}(u,\tau) \ldots p^-_{\lambda_m}(u,\tau), \tag{10}$$

$$h_k(u,\tau) = \sum_{\lambda} \frac{1}{a_1 a_2 \ldots a_m} p^+_{\lambda_1}(u,\tau) \ldots p^+_{\lambda_m}(u,\tau), \tag{11}$$

where the sums in both equations are taken over all compositions λ of the number k.

Remark. Compare these formulas with (2.14′) in Chapter 1.2 of [6].

Proof. We will prove (10), the arguments for (11) follow the same lines. The matrix B_k^- can be written as a sum of terms of the form

$$(P_{m-1,m} \ldots P_{a_{m-1}-1,a_{m-1}}) \ldots (P_{a_1-1,a_1} \ldots P_{1,2}),$$

with permutation matrices $P_{k,l}$, defined by (1). Each term in this sum corresponds to a decomposition λ of number k, and the coefficients of these terms in the sum are exactly $(-1)^{k-m}(a_1 a_2 \ldots a_m)^{-1}$. Then from (6), the elementary symmetric functions are the sums of the products of terms of the following form:

$$\mathrm{tr}\ \Big(P_{a_i-1,a_i} \ldots P_{a_{i-1}-1,a_{i-1}} T_{a_{i-1}}(u - a_{i-1} + 1) \ldots T_{a_i}(u - a_i + 1)\Big). \tag{12}$$

The following statement can be checked directly.

Lemma 2. *For any k matrices $X(1), \ldots, X(k)$ of the size $n \times n$ with the entries in an associative non-commutative algebra A, one has*

$$tr\ \left(P_{k-1,k} P_{k-2,k-1} \ldots P_{1,2}\, (X(1))_1\, (X(2))_2 \ldots (X(k))_k\right) = tr\ (X(1) X(2) \cdot \cdots \cdot X(k)). \tag{13}$$

From Lemma 2, the expression in (12) is nothing else but $p^-_{\lambda_i}(u - a_{i-1} + 1)$. Thus, $e_k(u)$ is the sum of terms of the form

$$(-1)^{k-m}(a_1 a_2 \ldots a_m)^{-1} p^-_{\lambda_1}(u) p^-_{\lambda_2}(u - a_1) \ldots p^-_{\lambda_m}(u - a_{m-1}),$$

and (10) follows. □

The following Newton identities and some of their corollaries are discussed in [1], using the technics of so-called Manin matrices. Here we give an alternative proof, using the RTT equation for the Yangian. It is inspired by the paper [2] on Newton's identities for RTT algebras with R-matrices that satisfy Hecke type condition.

Proposition 4. *(Newton's formula) For any $m = 1, 2, \ldots, n+1$,*

$$\sum_{k=0}^{m-1} (-1)^{m-k-1} e_k(u,\tau) p^-_{m-k}(u,\tau) = m\, e_m(u,\tau), \tag{14}$$

$$\sum_{k=0}^{m-1} h_k(u,\tau) p^+_{m-k}(u,\tau) = m\, h_m(u,\tau). \tag{15}$$

Proof. By (7),

$$\begin{aligned} m e_m(u) &= \operatorname{tr}\left(R_{m-1,m}\left(\frac{1}{m-1}\right) A_{m-1} T_1(u) \ldots T_m(u-m+1)\right) \\ &= \operatorname{tr}\,(A_{m-1} T_1(u) \ldots T_m(u-m+1)) \\ &\qquad - (m-1) \operatorname{tr}\,(P_{m-1,m}\, A_{m-1} T_1(u) \ldots T_m(u-m+1)) \\ &= e_{m-1}(u) p_1(u-m+1) \\ &\qquad - (m-1) \operatorname{tr}\,(A_{m-1} T_1(u) \ldots T_m(u-m+1) P_{m-1,m}\,). \end{aligned}$$

Applying the cyclic property of the trace, and the Proposition 1, (c) and (e) to the second term in the last expression, we obtain that

$$\begin{aligned} m e_m(u) = e_{m-1}(u)\, p_1(u-m+1) & \\ & - \operatorname{tr}\,(A_{m-2} T_1(u) \ldots T_m(u-m+1) P_{m-1,m}\,) \\ & + (m-2) \operatorname{tr}\,(A_{m-2} T_1(u) \ldots T_m(u-m+1) P_{m-1,m} P_{m-2,m-1}\,), \end{aligned}$$

and by induction,

$$\begin{aligned} m e_m(u) &= e_{m-1}(u) p_1(u-m+1) \\ &\quad - \operatorname{tr}\,(A_{m-2} T_1(u) \ldots T_m(u-m+1) P_{m-1,m}\,) \\ &\quad + \operatorname{tr}\,(A_{m-3} T_1(u) \ldots T_m(u-m+1) P_{m-1,m} P_{m-2,m-1}\,) + \ldots \\ &\quad + (-1)^{m-1} \operatorname{tr}\,(T_1(u) \ldots T_m(u-m+1) P_{m-1,m} \ldots P_{1,2}\,). \end{aligned} \tag{16}$$

Applying Lemma 2 to the terms of the sum, we conclude that each of them has the form

$$(-1)^{m-k-1} e_k(u) p^-_{m-k}(u-k),$$

and the Newton's formula for elementary symmetric functions $e_m(u,\tau)$ follows.

The proof for homogeneous functions is similar. □

Corollary 1. *(a) Coefficients of* $\{p_k^-(u)\}$ *belong to the Bethe subalgebra* $B(n)$*. Therefore, they commute. (See also the Remark after the Proposition 7 in the end of the paper).*

(b)(C.f. Example 8, Chapter 1.2 of [6]). For $m = 1, 2, \ldots, n$*,*

$$m!\, e_m(u) = \det \begin{pmatrix} p_1^-(u) & 1 & 0 & \ldots & 0 \\ p_2^-(u) & p_1^-(u-1) & 2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ p_m^-(u) & p_{m-1}^-(u-1) & p_{m-2}^-(u-2) & \ldots & p_1^-(u-m+1) \end{pmatrix},$$

$$m!\, h_m(u) = \det \begin{pmatrix} p_1^+(u) & -1 & 0 & \ldots & 0 \\ p_2^+(u) & p_1^+(u+1) & -2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ p_m^+(u) & p_{m-1}^+(u+1) & p_{m-2}^+(u+2) & \ldots & p_1^+(u+m-1) \end{pmatrix},$$

$$p_m^-(u) = \det \begin{pmatrix} e_1(u) & 1 & 0 & \ldots & 0 \\ 2\, e_2(u) & e_1(u-1) & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ m\, e_m(u) & e_{m-1}(u-1) & e_{m-2}(u-2) & \ldots & e_1(u-m+1) \end{pmatrix},$$

$$(-1)^{m-1}\, p_m^+(u) = \det \begin{pmatrix} h_1(u) & 1 & 0 & \ldots & 0 \\ 2\, h_2(u) & h_1(u+1) & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ m\, h_m(u) & h_{m-1}(u+1) & h_{m-2}(u+2) & \ldots & h_1(u+m-1) \end{pmatrix}.$$

Inverse of the universal differential operator

Consider the universal differential operator for XXX model: the formal polynomial in variable τ^{-1}, which is the generating function of the elements $e_k(u)$ (see e.g. [10], [18]):

$$E(u, \tau) = \sum_{k=0}^{n} (-1)^k e_k(u, \tau).$$

Using the Newton's identities, it is easy to describe the inverse of this operator.

Namely, for $m = 1, 2, \ldots$ define $h_m^-(u)$ and $h_m^-(u, \tau)$ by the following formulas:

$$h_m^-(u, \tau) := \tau^{-m} h_m^-(u),$$

where

$$m!\, h_m^-(u) = \det \begin{pmatrix} p_1^-(u) & -1 & \ldots & 0 \\ p_2^-(u+1) & p_1^-(u+1) & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ p_{m-1}^-(u+m-2) & p_{m-2}^-(u+m-2) & \ldots & -m+1 \\ p_m^-(u+m-1) & p_{m-1}^-(u+m-1) & \ldots & p_1^-(u+m-1) \end{pmatrix}.$$

Let

$$H^-(u,\tau) = \sum_{l=0}^{\infty} h_l^-(u,\tau),$$

where $h_0^-(u,\tau) = 1$. The following proposition follows directly from Newton's identities.

Proposition 5. *(a) The generating functions $H(u,\tau)$, $E(u,\tau)$ satisfy the following identity:*

$$E(u,\tau)H^-(u+1,\tau) = 1$$

(b) The coefficients of the elements $\{h_k^-(u)\}$ belong to Bethe subalgebra and commute. The relation to elementary symmetric functions is given by

$$e_k(u) = det(h_{j-i+1}^-(u-j+1)).$$

One can go further and introduce combinatorial analogues of Schur functions:

Definition 3. Let $\lambda = (\lambda_1, \dots \lambda_k)$ be a partition of number m with not more than n parts. The *Schur function* $s_\lambda(u)$ is the formal series in u^{-1} with coefficients in $Y(n)$, defined by

$$s_\lambda(u) := det\,[h_{\lambda_i-i+j}^-(u-j+1)]_{1\leqslant i,j\leqslant n}. \tag{17}$$

Proposition 6. *Let λ' be the conjugate partition to λ, and assume that it has not more than n parts. Then*

$$s_\lambda(u) := det\,[e_{\lambda'_i-i+j}(u)]_{1\leqslant i,j\leqslant n}. \tag{18}$$

Proof. The proof is the same as in classical case (see [6], (2.9), (2.9′), (3.4), (3.5)). For any positive number N such that $1 \leqslant N \leqslant n$ consider the matrices

$$H^- = [\,h_{i-j}^-(u-j+1)\,]_{0\leqslant i,j\leqslant N}, \qquad E = [\,(-1)^{i-j}e_{i-j}(u)\,]_{0\leqslant i,j\leqslant N}.$$

Here $h_k^-(u) = e_k(u) = 0$ for any $k < 0$. The Newton's identities show that these matrices are inverses of each other. Therefore, each minor of H^- is equal to the complementary cofactor of the transpose of E, which implies the equality of determinants in (17) and (18) (c.f. [6], formulas (2.9), (2.9′)). ☐

Connection to Capelli polynomials and Shifted Schur polynomials

In this section we show that the proved above identities immediately imply similar relations between Capelli polynomials and shifted Schur polynomials. The theory of higher Capelli polynomials is contained in [13], [15]. The detailed account on shifted symmetric functions and their applications is developed in [16]. Here we briefly remind the main definitions, following these three references.

Let $E = \{e_{ij}\}$ be the matrix of generators of $\mathfrak{gl}_n(\mathbb{C})$. Let $\lambda = (\lambda_1 \dots \lambda_k)$ be a partition of a number m with not more than n parts. Let $\{c_i\}$ be the set of contents of a column tableau of shape λ (see [13] for more details). Consider the Schur projector F_λ in the tensor power $(\mathbb{C}^n)^{\otimes m}$ to the irreducible $\mathfrak{gl}_n(\mathbb{C})$-component V_λ.

Definition 4. The higher Capelli polynomial $c_\lambda(u)$ is a polynomial in variable u and coefficients in the universal enveloping algebra $U(\mathfrak{gl}_n(\mathbb{C}))$, defined by

$$c_\lambda(u) = \mathrm{tr}(F_\lambda \otimes 1\,(u - c_1 + E)_1 \dots (u - c_k + E)_k). \tag{19}$$

The coefficients of Capelli polynomials $c_\lambda(u)$ are in the center of $U(\mathfrak{gl}_n(\mathbb{C}))$. The Capelli element $c_\lambda(u)$ acts in the irreducible representation V_μ with the highest weight μ by multiplication by a scalar, which is the shifted symmetric polynomial $s^*_\lambda(\mu+u)$ in variables $(\mu_1+u, \mu_2+u, \dots, \mu_n+u)$. The constant coefficients $\{c_\lambda(0)\}$ form a linear basis of the center of $U(\mathfrak{gl}_n(\mathbb{C}))$. In particular, we consider the shifted elementary polynomials $e^*_k(u) = s^*_{(1^k)}(\mu+u)$ and shifted homogeneous symmetric polynomials $h^*_k(u) = s^*_{(k)}(\mu+u)$, which take the form

$$e^*_k(u) = \sum_{1\leqslant i_1<i_2<\dots<i_k<\infty} (\mu_{i_1}+u+k-1)(\mu_{i_2}+u+k-2)\dots(\mu_{i_k}+u),$$

$$h^*_k(u) = \sum_{1\leqslant i_1\leqslant i_2\leqslant\dots\leqslant i_k<\infty} (\mu_{i_1}+u-k+1)(\mu_{i_2}+u-k+2)\dots(\mu_{i_k}+u).$$

We identify the corresponding Capelli elements with their shifted Schur polynomials, and use the notations $e^*_k(u)$, $h^*_k(u)$ for $c_{(1^k)}(u)$ and $c_{(k)}(u)$ respectively.

Let $ev : Y(n) \to U(\mathfrak{gl}_n(\mathbb{C}))$ be the evaluation homomorphism:

$$ev : T(u) \mapsto 1 + \frac{E}{u}.$$

Under this map the defined above symmetric functions in $Y(n)$ map to the following Capelli elements:

$$ev\,(e_k(u)) = \frac{e^*_k\,(u-k+1)}{(u\downarrow k)}, \quad ev\,(h_k(u)) = \frac{h^*_k\,(u+k-1)}{(u\uparrow k)},$$

where

$$(u\downarrow k) = u(u-1)\dots(u-k+1) \quad \text{and} \quad (u\uparrow k) = u(u+1)\dots(u+k-1).$$

Moreover, set

$$p_m(u) = \mathrm{tr}\,((E+u)\dots(E+u+m-1)).$$

Then

$$ev\,(\,p^-_m(u+m-1)\,) = ev\,(p^+_m(u)) = \frac{p_m(u)}{(u\uparrow m)},$$

and this implies

$$ev\,(h^-_m(u)) = ev\,(h_m(u)).$$

The eigenvalue of the central polynomial $p_k(u) \in U(\mathfrak{gl}_n(\mathbb{C}))[u]$ in the irreducible representation V_μ can be easily found, using the classical formula for the eigenvalues of Casimir operators from [17]. The eigenvalue of $\mathrm{tr}\,E^k$ is given by the formula

$$\mathrm{tr}\,E^k(\mu) = \sum_{i=1}^n \gamma_i\, m_i^k, \tag{20}$$

where

$$m_i = \mu_i + n - i, \quad \gamma_i = \prod_{j\neq i}\left(1 - \frac{1}{m_i - m_j}\right).$$

Accordingly, the shifted symmetric polynomial $p_k^*(u)$ which gives the eigenvalue of $p_k(u)$ is

$$p_k^*(u) = \sum_{i=1}^{n} \gamma_i\,(m_i+u)(m_i+u+1)\dots(m_i+u+k-1).$$

The combinatorial identities in the Yangian imply immediately the corresponding relations between Capelli polynomials. Some of them are listed below.

Proposition 7.

$$e_k^*(u-k) = \sum_{\lambda} \frac{(-1)^{k-m}}{a_1 a_2 \dots a_m} p^*_{\lambda_1}(u-a_1)\dots p^*_{\lambda_m}(u-a_m), \tag{21}$$

$$h_k^*(u+k-1) = \sum_{\lambda} \frac{1}{a_1 a_2 \dots a_m} p^*_{\lambda_1}(u)\dots p^*_{\lambda_m}(u+a_{m-1}), \tag{22}$$

$$\sum_{k=0}^{m} (-1)^k e_k^*(u-k+1)\, h^*_{m-k}(u-k) = \delta_{m,0}. \tag{23}$$

Remark. 1) The defined here polynomials $p_k^*(u)$ do not coincide with the shifted power sums under the same notation in [16].

2) The identity (23) is similar to the relation (12.18), [16] on the generating functions of the elements e_k^*, h_k^*.

3) The images of $p_k^{\pm}(u)$ under the evaluation homomorphism coincide up to a shift of the variable, but we do not know yet a simple combinatorial relation between those two series, and for this reason we do not state that the coefficients of $p_k^+(u)$ belong to the Bethe subalgebra. We believe that such relation can be obtained from some anti-automorphism of the Yangian (or may be, its double), applied to the universal R-matrix.

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