A new determinant expression of the zeta function for a hypergraph

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Abstract

Recently, Storm [10] defined the Ihara-Selberg zeta function of a hypergraph, and gave two determinant expressions of it by the Perron-Frobenius operator of a digraph and a deformation of the usual Laplacian of a graph. We present a new determinant expression for the Ihara-Selberg zeta function of a hypergraph, and give a linear algebraic proof of Storm's Theorem. Furthermore, we generalize these results to the Bartholdi zeta function of a hypergraph.

1 Introduction

Graphs and digraphs treated here are finite. Let G be a connected graph and D the symmetric digraph corresponding to G. Set $D(G) = \{(u,v),(v,u) \mid uv \in E(G)\}$. For $e=(u,v) \in D(G)$, set $u=o(e)$ and $v=t(e)$. Furthermore, let $e^{-1}=(v,u)$ be the *inverse* of $e=(u,v)$.

A *path P of length n* in G is a sequence $P=(e_1,\cdots,e_n)$ of n arcs such that $e_i \in D(G)$, $t(e_i)=o(e_{i+1})(1 \leqslant i \leqslant n-1)$. If $e_i=(v_{i-1},v_i)$ for $i=1,\cdots,n$, then we write $P=(v_0,v_1,\cdots,v_{n-1},v_n)$. Set $\mid P \mid = n$, $o(P)=o(e_1)$ and $t(P)=t(e_n)$. Also, P is called an *$(o(P),t(P))$-path*. We say that a path $P=(e_1,\cdots,e_n)$ has a *backtracking* or a *bump* at $t(e_i)$ if $e_{i+1}^{-1}=e_i$ for some $i(1 \leqslant i \leqslant n-1)$. A (v,w)-path is called a *v-cycle* (or *v-closed path*) if $v=w$. The *inverse path* of a path $P=(e_1,\cdots,e_n)$ is the path $P^{-1}=(e_n^{-1},\cdots,e_1^{-1})$.

We introduce an equivalence relation between cycles. Two cycles $C_1=(e_1,\cdots,e_m)$ and $C_2=(f_1,\cdots,f_m)$ are called *equivalent* if $f_j=e_{j+k}$ for all j. The inverse cycle of C is not equivalent to C. Let $[C]$ be the equivalence class which contains a cycle C. Let B^r be the cycle obtained by going r times around a cycle B. Such a cycle is called a *multiple* of B. A cycle C is *reduced* if both C and C^2 have no backtracking. Furthermore, a cycle

C is *prime* if it is not a multiple of a strictly smaller cycle. Note that each equivalence class of prime, reduced cycles of a graph G corresponds to a unique conjugacy class of the fundamental group $\pi_1(G,v)$ of G at a vertex v of G.

The *Ihara-Selberg zeta function* of G is defined by

$$\mathbf{Z}(G,t)=\prod_{[C]}(1-t^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime, reduced cycles of G. Ihara [6] defined zeta functions of graphs, and showed that the reciprocals of zeta functions of regular graphs are explicit polynomials. A zeta function of a regular graph G associated with a unitary representation of the fundamental group of G was developed by Sunada [11,12]. Hashimoto [4] treated multivariable zeta functions of bipartite graphs. Bass [2] generalized Ihara's result on the zeta function of a regular graph to an irregular graph G.

Let G be a connected graph with n vertices and m edges. Then two $2m\times 2m$ matrices $\mathbf{B}=\mathbf{B}(G)=(\mathbf{B}_{e,f})_{e,f\in D(G)}$ and $\mathbf{J}_0=\mathbf{J}_0(G)=(\mathbf{J}_{e,f})_{e,f\in D(G)}$ are defined as follows:

$$\mathbf{B}_{e,f}=\begin{cases}1 & \text{if } t(e)=o(f),\\ 0 & \text{otherwise}\end{cases},\mathbf{J}_{e,f}=\begin{cases}1 & \text{if } f=e^{-1},\\ 0 & \text{otherwise.}\end{cases}$$

Theorem 1 (Bass) *Let G be a connected graph with n vertices and m edges. Then the reciprocal of the Ihara-Selberg zeta function of G is given by*

$$\mathbf{Z}(G,t)^{-1}=\det(\mathbf{I}_{2m}-t(\mathbf{B}-\mathbf{J}_0))=(1-t^2)^{m-n}\det(\mathbf{I}_n-t\mathbf{A}(G)+t^2(\mathbf{D}_G-\mathbf{I}_n)),$$

where $\mathbf{D}_G=(d_{ij})$ is the diagonal matrix with $d_{ii}=\deg_G v_i$ $(V(G)=\{v_1,\cdots,v_n\})$.

The first identity in Theorem 1 was also obtained by Hashimoto [5]. Bass proved the second identity by using a linear algebraic method.

Stark and Terras [9] gave an elementary proof of this formula, and discussed three different zeta functions of any graph. Various proofs of Bass' Theorem were given by Kotani and Sunada [7], and Foata and Zeilberger [3].

Let G be a connected graph. Then the *cyclic bump count* $cbc(\pi)$ of a cycle $\pi=(\pi_1,\cdots,\pi_n)$ is

$$cbc(\pi)=\mid\{i=1,\cdots,n\mid \pi_i=\pi_{i+1}^{-1}\}\mid,$$

where $\pi_{n+1}=\pi_1$.

Bartholdi [1] introduced the Bartholdi zeta function of a graph. The *Bartholdi zeta function* of G is defined by

$$\zeta(G,u,t)=\prod_{[C]}(1-u^{cbc(C)}t^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime cycles of G, and u,t are complex variables with $\mid u\mid$, $\mid t\mid$ sufficiently small.

Bartholdi [1] gave a determinant expression of the Bartholdi zeta function of a graph.

Theorem 2 (Bartholdi) *Let G be a connected graph with n vertices and m unoriented edges. Then the reciprocal of the Bartholdi zeta function of G is given by*

$$\zeta(G,u,t)^{-1} = \det(\mathbf{I}_{2m} - t(\mathbf{B} - (1-u)\mathbf{J}_0))$$
$$= (1-(1-u)^2t^2)^{m-n} \det(\mathbf{I} - t\mathbf{A}(G) + (1-u)(\mathbf{D}_G - (1-u)\mathbf{I})t^2).$$

Storm [10] defined the Ihara-Selberg zeta function of a hypergraph. A hypergraph $H = (V(H), E(H))$ is a pair of a set of *hypervertices* $V(H)$ and a set of *hyperedges* $E(H)$, which the union of all hyperedges is $V(H)$. In general, the union of all hyperedges is a subset of $V(H)$. For example, if a graph (that is, a 2-uniform hypergraph) has an isolated vertex, then the union of all edges is a proper subset of $V(H)$. A hypervertex v is *incident* to a hyperedge e if $v \in e$.

A bipartite graph B_H associated with a hypergraph H is defined as follows: $V(B_H) = V(H) \cup E(H)$ and $v \in V(H)$ and $e \in E(H)$ are *adjacent* in B_H if v is incident to e. Let $V(H) = \{v_1, \ldots, v_n\}$. Then an *adjacency matrix* $\mathbf{A}(H)$ of H is defined as a matrix whose rows and columns are parameterized by $V(H)$, and (i,j)-entry is the number of directed paths in B_H from v_i to v_j of length 2 with no backtracking.

For the bipartite graph B_H associated with a hypergraph H, let $V_1 = V(H)$ and $V_2 = E(H)$. Then, the *halved graph* $B_H^{[i]}$ of B_H is defined to be the graph with vertex set V_i and arc set $\{P :$ reduced path $\mid\ \mid P \mid = 2; o(P), t(P) \in V_i\}$ for $i = 1, 2$.

Let H be a hypergraph. A *path P of length n* in H is a sequence $P = (v_1, e_1, v_2, e_2, \cdots, e_n, v_{n+1})$ of $n+1$ hypervertices and n hyperedges such that $v_i \in V(H)$, $e_j \in E(H)$, $v_1 \in e_1$, $v_{n+1} \in e_n$ and $v_i \in e_i, e_{i-1}$ for $i = 2, \ldots, n-1$. Set $\mid P \mid = n$, $o(P) = v_1$ and $t(P) = v_{n+1}$. Also, P is called an *$(o(P), t(P))$-path*. We say that a path P has a *hyperedge backtracking* if there is a subsequence of P of the form (e, v, e), where $e \in E(H)$, $v \in V(H)$. A (v, w)-path is called a *v-cycle* (or *v-closed path*) if $v = w$.

We introduce an equivalence relation between cycles. Two cycles $C_1 = (v_1, e_1, v_2, \cdots, e_m, v_1)$ and $C_2 = (w_1, f_1, w_2, \cdots, f_m, w_1)$ are called *equivalent* if $w_j = v_{j+k}$ and $f_j = e_{j+k}$ for all j. Let $[C]$ be the equivalence class which contains a cycle C. Let B^r be the cycle obtained by going r times around a cycle B. Such a cycle is called a *multiple* of B. A cycle C is *reduced* if both C and C^2 have no hyperedge backtracking. Furthermore, a cycle C is *prime* if it is not a multiple of a strictly smaller cycle.

The *Ihara-Selberg zeta function* of H is defined by

$$\zeta_H(t) = \prod_{[C]} (1 - t^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime, reduced cycles of H, and t is a complex variable with $\mid t \mid$ sufficiently small(see [10]).

Let H be a hypergraph with $E(H) = \{e_1, \ldots, e_m\}$, and let $\{c_1, \ldots, c_m\}$ be a set of m *colors*, where $c(e_i) = c_i$. Then an *edge-colored graph* GH_c is defined as a graph with vertex set $V(H)$ and edge set $\{vw \mid v, w \in V(H); v \neq w; v, w \in e \in E(H)\}$, where an edge vw is colored c_i if $v, w \in e_i$. Note that GH_c is identified with the "undirected" halved graph $B_H^{[1]}$ with colors.

Let GH^o_c be the symmetric digraph corresponding to the edge-clored graph GH_c. Then the *oriented line graph* $H^o_L = (V_L, E^o_L)$ associated with GH^o_c by

$$V_L = A(GH^o_c), \quad \text{and} \quad E^o_L = \{(e_i, e_j) \in A(GH^o_c) \times A(GH^o_c) \mid c(e_i) \neq c(e_j), t(e_i) = o(e_j)\},$$

where $c(e_i)$ is the same color as the one of the corresponding undirected edge in $D(GH^o_c)$. Also, H^o_L is called the *oriented line graph* of GH_c. The *Perron-Frobenius operator* $\mathbf{T}$: $C(V_L) \longrightarrow C(V_L)$ is given by

$$(\mathbf{T}f)(x) = \sum_{e \in E_o(x)} f(t(e)),$$

where $E_o(x) = \{e \in E^o_L \mid o(e) = x\}$ is the set of all oriented edges with x as their origin vertex, and $C(V_L)$ is the set of functions from V_L to the complex number field $\mathbf{C}$.

Storm [10] gave two nice determinant expressions of the Ihara-Selberg zeta function of a hypergraph by using the results of Kotani and Sunada [7], and Bass [2].

Theorem 3 (Storm) *Let H be a finite, connected hypergraph such that every hypervetex is in at least two hyperedges. Then*

$$\zeta_H(t)^{-1} = \det(\mathbf{I} - t\mathbf{T}) \tag{1}$$

$$= \mathbf{Z}(B_H, \sqrt{t})^{-1} = (1-t)^{m-n} \det(\mathbf{I} - \sqrt{t}\mathbf{A}(B_H) + t\mathbf{Q}_{B_H}), \tag{2}$$

where $n = \mid V(B_H) \mid$, $m = \mid E(B_H) \mid$ and $\mathbf{Q}_{B_H} = \mathbf{D}_{B_H} - \mathbf{I}$.

In Theorem 3, can the equality between the first identity (1) and the second identity (2) be proved by an analogue of Bass' method ?

In Section 2, we present a new determinant expression for the Ihara-Selberg zeta function of a hypergraph. In Section 3, we show that, in Theorem 3, the first identity (1) is obtained from the second identity (2) by using a linear algebraic method. In Section 4, we generalize theses results to the Bartholdi zeta function of a hypergraph.

2 A new determinant expression of the zeta function of a hypergraph

Let $H = (V(H), E(H))$ be a hypergraph, $V(H) = \{v_1, \ldots, v_n\}$ and $E(H) = \{e_1, \ldots, e_m\}$. Let B_H have ν vertices and ϵ edges, where $\nu = n + m$. Then we have

$$D(B_H) = \{(v, e), (e, v) \mid v \in e, \ v \in V(H), e \in E(H)\}.$$

Let $f_1, \ldots, f_\epsilon$ be arcs in B_H such that $o(f_i) \in V(H)$ for each $i = 1, \ldots, \ \epsilon$. Then two $\epsilon \times \epsilon$ matrices $\mathbf{X} = (X_{ij})$ and $\mathbf{Y} = (Y_{ij})$ are defined as follows:

$$X_{ij} = \begin{cases} 1 & \text{if there exists an arc } f_k^{-1} \text{ such that } (f_i, f_k^{-1}, f_j) \text{ is a reduced path,} \\ 0 & \text{otherwise} \end{cases}$$

and

$$Y_{ij} = \begin{cases} 1 & \text{if there exists an arc } f_k \text{ such that } (f_i^{-1}, f_k, f_j^{-1}) \text{ is a reduced path,} \\ 0 & \text{otherwise.} \end{cases}$$

Remark that $\mathbf{Y} = {}^t\mathbf{X}$.

Theorem 4 *Let H be a finite, connected hypergraph such that every hypervetex is in at least two hyperedges. Set $\epsilon = \mid E(B_H) \mid$. Then*

$$\mathbf{Z}(B_H, \sqrt{t})^{-1} = \det(\mathbf{I}_\epsilon - t\mathbf{X}) = \det(\mathbf{I}_\epsilon - t\mathbf{Y}).$$

Proof. Let $H = (V(H), E(H))$ be a hypergraph, $V(H) = \{v_1, \ldots, v_n\}$ and $E(H) = \{e_1, \ldots, e_m\}$. Let B_H have ν vertices and ϵ edges. By Theorem 1, we have

$$\begin{aligned} \mathbf{Z}(B_H, \sqrt{t})^{-1} &= (1-t)^{\epsilon-\nu} \det(\mathbf{I}_\nu - \sqrt{t}\mathbf{A}(B_H) + t(\mathbf{D}_{B_H} - \mathbf{I}_\nu)) \\ &= \det(\mathbf{I}_{2\epsilon} - \sqrt{t}(\mathbf{B}(B_H) - \mathbf{J}_0(B_H))). \end{aligned}$$

Arrange arcs of B_H as follows: $f_1, \ldots, f_\epsilon, f_1^{-1}, \ldots, f_\epsilon^{-1}$. We consider two matrices $\mathbf{B}$ and $\mathbf{J}_0$ under this order. Let

$$\mathbf{B}(B_H) - \mathbf{J}_0(B_H) = \begin{bmatrix} \mathbf{0} & \mathbf{F} \\ \mathbf{G} & \mathbf{0} \end{bmatrix}.$$

It is clear that both $\mathbf{F}$ and $\mathbf{G}$ are symmetric, but $\mathbf{F} \neq {}^t\mathbf{G}$. Furthermore,

$$\mathbf{FG} = \mathbf{X} \textit{ and } \mathbf{GF} = \mathbf{Y}. \tag{3}$$

Thus, we have

$$\begin{aligned} \det(\mathbf{I}_{2\epsilon} - \sqrt{t}(\mathbf{B}(B_H) - \mathbf{J}_0(B_H))) &= \det\left(\begin{bmatrix} \mathbf{I}_\epsilon & -\sqrt{t}\mathbf{F} \\ -\sqrt{t}\mathbf{G} & \mathbf{I}_\epsilon \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} \mathbf{I}_\epsilon - t\mathbf{FG} & -\sqrt{t}\mathbf{F} \\ \mathbf{0} & \mathbf{I}_\epsilon \end{bmatrix}\right) \\ &= \det(\mathbf{I}_\epsilon - t\mathbf{FG}) = \det(\mathbf{I}_\epsilon - t\mathbf{X}) \\ &= \det(\mathbf{I}_\epsilon - t\mathbf{GF}) = \det(\mathbf{I}_\epsilon - t\mathbf{Y}). \end{aligned}$$

Therefore, the result follows. Q.E.D.

3 A linear algebraic proof of Storm Theorem

We show that, in Theorem 3, the identity (1) is obtained from the identity (2) by using a linear algebraic method.

Let $H = (V(H), E(H))$ be a hypergraph, $V(H) = \{v_1, \ldots, v_n\}$ and $E(H) = \{e_1, \ldots, e_m\}$. Let B_H have ν vertices and ϵ edges, and $D(B_H) = \{f_1, \ldots, f_\epsilon, f_1^{-1}, \ldots, f_\epsilon^{-1}\}$ such that $o(f_i) \in V(H)(1 \leqslant i \leqslant \epsilon)$. Furthermore, let $\mathcal{R}$ (or $\mathcal{S}$) be the set of reduced paths P in B_H with length two such that $o(P), t(P) \in V(H)$ (or $o(P), t(P) \in E(H)$). Set $r = \mid \mathcal{R} \mid$ and $s = \mid \mathcal{S} \mid$. For a path $P = (x, y, z)$ of length two in B_H, let

$$oe(P) = (x, y), te(P) = (y, z),$$

where $(x, y, z) = (v, e, w)$ or $(x, y, z) = (e, v, f)$ $(v, w \in V(H); e, f \in E(H))$.

Now, we introduce two $r \times \epsilon$ matrices $\mathbf{K} = (K_{Pf_j^{-1}})_{P \in R; 1 \leqslant j \leqslant \epsilon}$ and $\mathbf{L} = (L_{Pf_j})_{P \in R; 1 \leqslant j \leqslant \epsilon}$ are defined as follows:

$$K_{Pf_j^{-1}} = \begin{cases} 1 & \text{if } te(P) = f_j^{-1}, \\ 0 & \text{otherwise,} \end{cases} \quad L_{Pf_j} = \begin{cases} 1 & \text{if } oe(P) = f_j, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, two $s \times \epsilon$ matrices $\mathbf{M} = (M_{Qf_j^{-1}})_{Q \in S; 1 \leqslant j \leqslant \epsilon}$ and $\mathbf{N} = (N_{Qf_j})_{Q \in S; 1 \leqslant j \leqslant \epsilon}$ are defined as follows:

$$M_{Qf_j^{-1}} = \begin{cases} 1 & \text{if } oe(Q) = f_j^{-1}, \\ 0 & \text{otherwise,} \end{cases} \quad N_{Qf_j} = \begin{cases} 1 & \text{if } te(Q) = f_j, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$ {}^t\mathbf{LK} = \mathbf{F} \quad \textit{and} \quad {}^t\mathbf{MN} = \mathbf{G}. \tag{4}$$

and, $\mathbf{K}\ {}^t\mathbf{M} = (b_{PQ})_{P \in \mathcal{R}; Q \in \mathcal{S}}$ and $\mathbf{N}\ {}^t\mathbf{L} = (c_{QP})_{P \in \mathcal{R}; Q \in \mathcal{S}}$ are given as follows:

$$b_{PQ} = \begin{cases} 1 & \text{if } te(P) = oe(Q), \\ 0 & \text{otherwise,} \end{cases} \quad c_{QP} = \begin{cases} 1 & \text{if } te(Q) = oe(P), \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we have

$$\mathbf{K}\ {}^t\mathbf{MN}\ {}^t\mathbf{L} = \mathbf{T}. \tag{5}$$

Furthermore, by (3) and (4),

$$ {}^t\mathbf{LK}\ {}^t\mathbf{MN} = \mathbf{FG} = \mathbf{X}.$$

Here it is known that, for a $m \times n$ matrix $\mathbf{A}$ and $n \times m$ matrix $\mathbf{B}$,

$$\det(\mathbf{I}_m + \mathbf{AB}) = \det(\mathbf{I}_n + \mathbf{BA}). \tag{6}$$

Therefore, it follows that

$$\det(\mathbf{I}_r - t\mathbf{T}) = \det(\mathbf{I}_\epsilon - t\mathbf{X}).$$

By Theorem 4 and the fact that $\zeta_H(t)^{-1} = \mathbf{Z}(B_H, \sqrt{t})^{-1}$, we have

$$\zeta_H(t)^{-1} = \det(\mathbf{I}_r - t\mathbf{T}).$$

Q.E.D.

4 Bartholdi zeta function of a hypergraph

Let H be a hypergraph. Then a path $P = (v_1, e_1, v_2, e_2, \cdots, e_n, v_{n+1})$ has a *(broad) backtracking* or *(broad) bump* at e or v if there is a subsequence of P of the form (e, v, e) or (v, e, v), where $e \in E(H)$, $v \in V(H)$. Furthermore, the *cyclic bump count* $cbc(C)$ of a cycle $C = (v_1, e_1, v_2, e_2, \cdots, e_n, v_1)$ is

$$cbc(C) = \mid \{i = 1, \cdots, n \mid v_i = v_{i+1}\} \mid + \mid \{i = 1, \cdots, n \mid e_i = e_{i+1}\} \mid,$$

where $v_{n+1} = v_1$ and $e_{n+1} = e_1$.

The *Bartholdi zeta function* of H is defined by

$$\zeta(H, u, t) = \prod_{[C]} (1 - u^{cbc(C)} t^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime cycles of H, and u, t are complex variables with $\mid u \mid$, $\mid t \mid$ sufficiently small.

If $u = 0$, then the Bartholdi zeta function of H is the Ihara-Selberg zeta function of H.

Sato [8] presented a determinant expression of the Bartholdi zeta function of a hypergraph.

Theorem 5 (Sato) *Let H be a finite, connected hypergraph such that every hypervetex is in at least two hyperedges. Then*

$$\begin{aligned} \zeta(H, u, t)^{-1} &= \zeta(B_H, u, \sqrt{t})^{-1} \\ &= (1 - (1-u)^2 t)^{m-n} \det(\mathbf{I} - \sqrt{t}\mathbf{A}(B_H) + (1-u)t(\mathbf{D}_{B_H} - (1-u)\mathbf{I})), \end{aligned}$$

where $n = \mid V(B_H) \mid$ and $m = \mid E(B_H) \mid$.

Let $H = (V(H), E(H))$ be a hypergraph, $V(H) = \{v_1, \ldots, v_n\}$ and $E(H) = \{e_1, \ldots, e_m\}$. Let B_H have ν vertices and ϵ edges, $V_1 = V(H)$ and $V_2 = E(H)$. Then, the *broad halved graph* $B_H^{(i)}$ of B_H is defined to be the graph with vertex set V_i and arc set $\{P :$ path $\mid \mid P \mid = 2; o(P), t(P) \in V_i\}$ for $i = 1, 2$. Furthermore, let $\{c_1, \ldots, c_m\}$ be a set of m colors such that $c(e_i) = c_i$ for $i = 1, \ldots, m$. We color each arc of $B_H^{(1)}$ as follows:

$$c(P) = c(e) \ for \ P = (v, e, w) \in D(B_H^{(1)}).$$

Then the *line digraph* $\vec{L}(B_H^{(1)})$ of $B_H^{(1)}$is defined as follows: $V(\vec{L}(B_H^{(1)})) = D(B_H^{(1)})$, and $(P, Q) \in A(\vec{L}(B_H^{(1)}))$ if and only if $t(P) = o(Q)$ in B_H.

Next, let $\mathcal{R}'$ (or $\mathcal{S}'$) be the set of paths P in B_H with length two such that $o(P), t(P) \in V(H)$ (or $\in E(H)$). Furthermore, let $f_k = (v_{i_k}, e_{j_k})$, $P_k = (v_{i_k}, e_{j_k}, v_{i_k})$ and $Q_k = (e_{j_k}, v_{i_k}, e_{j_k})$ for each $k = 1, \ldots, \epsilon$. Then we have

$$\mathcal{R}' = \mathcal{R} \cup \{P_1, \ldots, P_\epsilon\} \ and \ \mathcal{S}' = \mathcal{S} \cup \{Q_1, \ldots, Q_\epsilon\}.$$

Furthermore, we have $\mid \mathcal{R}' \mid = r + \epsilon$ and $\mid \mathcal{S}' \mid = s + \epsilon$.

Now, we introduce a $(r+\epsilon) \times (r+\epsilon)$ matrix $\mathbf{T}' = (T'_{PP'})_{P,P' \in \mathcal{R}'}$ for the line digraph $\vec{L}(B^{(1)}_H)$ of the halved graph $B^{(1)}_H$ is defined as follows:

$$T'_{PP'} = \begin{cases} u^2 & \text{if } t(P) = o(P'),\ P = P' \in \mathcal{R}' \setminus \mathcal{R}, \\ u^2 & \text{if } t(P) = o(P'),\ P \in \mathcal{R}' \setminus \mathcal{R},\ P' \in \mathcal{R} \text{ and } c(P) = c(P'), \\ u & \text{if } t(P) = o(P'),\ P, P' \in \mathcal{R}' \setminus \mathcal{R} \text{ and } c(P) \neq c(P'), \\ u & \text{if } t(P) = o(P'),\ P \in \mathcal{R}' \setminus \mathcal{R},\ P' \in \mathcal{R} \text{ and } c(P) \neq c(P'), \\ u & \text{if } t(P) = o(P'),\ P \in \mathcal{R}, P' \in \mathcal{R}' \setminus \mathcal{R} \text{ and } c(P) = c(P'), \\ u & \text{if } t(P) = o(P'),\ P, P' \in \mathcal{R} \text{ and } c(P) = c(P'), \\ 1 & \text{if } t(P) = o(P'),\ P \in \mathcal{R}, P' \in \mathcal{R}' \setminus \mathcal{R} \text{ and } c(P) \neq c(P'), \\ 1 & \text{if } t(P) = o(P'),\ P, P' \in \mathcal{R} \text{ and } c(P) \neq c(P'), \\ 0 & \text{otherwise,} \end{cases}$$

We present a new determinant expression for the Bartholdi zeta function of a hypergraph.

Theorem 6 *Let H be a finite, connected hypergraph such that every hypervetex is in at least two hyperedges. Set $\epsilon = \mid E(B_H) \mid$ and $r = \mid \mathcal{R} \mid$. Then*

$$\zeta(H,u,t)^{-1} = \det(\mathbf{I}_{r+\epsilon} - t\mathbf{T}')$$
$$= \det(\mathbf{I}_\epsilon - t(\mathbf{X} + u(\mathbf{F}+\mathbf{G}) + u^2\mathbf{I}_\epsilon)) = \det(\mathbf{I}_\epsilon - t(\mathbf{Y} + u(\mathbf{F}+\mathbf{G}) + u^2\mathbf{I}_\epsilon)).$$

Proof. Let $H = (V(H), E(H))$ be a hypergraph, $V(H) = \{v_1, \ldots, v_n\}$ and $E(H) = \{e_1, \ldots, e_m\}$. Let B_H have ν vertices and ϵ edges. By Theorems 2 and 5, we have

$$\zeta(H,u,t)^{-1} = \det(\mathbf{I}_{2\epsilon} - \sqrt{t}(\mathbf{B}(B_H) - (1-u)\mathbf{J}_0(B_H)))$$
$$= (1-(1-u)^2t)^{\epsilon-\nu} \det(\mathbf{I}_\nu - \sqrt{t}\mathbf{A}(B_H) + (1-u)t(\mathbf{D}_{B_H} - (1-u)\mathbf{I}_\nu)).$$

Arrange arcs of B_H as follows: $f_1, \ldots, f_\epsilon, f_1^{-1}, \ldots, f_\epsilon^{-1}$ such that $o(f_i) \in V(H) (1 \leqslant i \leqslant \epsilon)$. We consider two matrices $\mathbf{B}$ and $\mathbf{J}_0$ under this order. Let

$$\mathbf{B}(B_H) - (1-u)\mathbf{J}_0(B_H) = \begin{bmatrix} \mathbf{0} & \mathbf{F} + u\mathbf{I}_\epsilon \\ \mathbf{G} + u\mathbf{I}_\epsilon & \mathbf{0} \end{bmatrix}.$$

Thus, by (3), we have

$$\begin{aligned} & \det(\mathbf{I}_{2\epsilon} - \sqrt{t}(\mathbf{B}(B_H) - (1-u)\mathbf{J}_0(B_H))) \\ = \ & \det\left(\begin{bmatrix} \mathbf{I}_\epsilon & -\sqrt{t}(\mathbf{F} + u\mathbf{I}_\epsilon) \\ -\sqrt{t}(\mathbf{G} + u\mathbf{I}_\epsilon) & \mathbf{I}_\epsilon \end{bmatrix} \right) \\ = \ & \det\left(\begin{bmatrix} \mathbf{I}_\epsilon - t(\mathbf{F} + u\mathbf{I}_\epsilon)(\mathbf{G} + u\mathbf{I}_\epsilon) & -\sqrt{t}(\mathbf{F} + u\mathbf{I}_\epsilon) \\ \mathbf{0} & \mathbf{I}_\epsilon \end{bmatrix} \right) \\ = \ & \det(\mathbf{I}_\epsilon - t(\mathbf{FG} + u(\mathbf{F}+\mathbf{G}) + u^2\mathbf{I}_\epsilon)) = \det(\mathbf{I}_\epsilon - t(\mathbf{X} + u(\mathbf{F}+\mathbf{G}) + u^2\mathbf{I}_\epsilon)) \\ = \ & \det(\mathbf{I}_\epsilon - t(\mathbf{GF} + u(\mathbf{F}+\mathbf{G}) + u^2\mathbf{I}_\epsilon)) = \det(\mathbf{I}_\epsilon - t(\mathbf{Y} + u(\mathbf{F}+\mathbf{G}) + u^2\mathbf{I}_\epsilon)). \end{aligned}$$

Arrange elements of $\mathcal{R}'$ and $\mathcal{S}'$ as follows:

$$P_1, \ldots, P_\epsilon, \mathcal{R};\ Q_1, \ldots, Q_\epsilon, \mathcal{S},$$

where $P_k = (v_{i_k}, e_{j_k}, v_{i_k})$ and $Q_k = (e_{j_k}, v_{i_k}, e_{j_k})$ if $f_k = (v_{i_k}, e_{j_k})$ for $k = 1, \ldots, \epsilon$. Then we introduce two $(r+\epsilon) \times \epsilon$ matrices $\mathbf{K}' = (K'_{Pf_j^{-1}})_{P \in \mathcal{R}'; 1 \leqslant j \leqslant \epsilon}$ and $\mathbf{L}' = (L'_{Pf_j})_{P \in \mathcal{R}'; 1 \leqslant j \leqslant \epsilon}$ are defined as follows:

$$K'_{Pf_j^{-1}} = \begin{cases} 1 & \text{if } te(P) = f_j^{-1} \text{ and } te(P) \neq oe(P)^{-1}, \\ u & \text{if } te(P) = oe(P)^{-1} = f_j^{-1}, \\ 0 & \text{otherwise,} \end{cases} \qquad L'_{Pf_j} = \begin{cases} 1 & \text{if } oe(P) = f_j, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, two $(s+\epsilon) \times \epsilon$ matrices $\mathbf{M}' = (M'_{Qf_j^{-1}})_{Q \in \mathcal{S}'; 1 \leqslant j \leqslant \epsilon}$ and $\mathbf{N}' = (N'_{Qf_j})_{Q \in \mathcal{S}'; 1 \leqslant j \leqslant \epsilon}$ are defined as follows:

$$M'_{Qf_j^{-1}} = \begin{cases} 1 & \text{if } oe(Q) = f_j^{-1}, \\ 0 & \text{otherwise,} \end{cases} \quad N'_{Qf_j} = \begin{cases} 1 & \text{if } te(Q) = f_j \text{ and } te(Q) \neq oe(Q)^{-1}, \\ u & \text{if } te(Q) = oe(Q)^{-1} = f_j, \\ 0 & \text{otherwise.} \end{cases}$$

Here we have

$$\mathbf{K}' = \begin{bmatrix} u\mathbf{I}_\epsilon \\ \mathbf{K} \end{bmatrix}, \mathbf{L}' = \begin{bmatrix} \mathbf{I}_\epsilon \\ \mathbf{L} \end{bmatrix}, \mathbf{M}' = \begin{bmatrix} \mathbf{I}_\epsilon \\ \mathbf{M} \end{bmatrix} \textit{ and } \mathbf{N}' = \begin{bmatrix} u\mathbf{I}_\epsilon \\ \mathbf{N} \end{bmatrix}.$$

Thus, we have

$$\mathbf{K}'\ {}^t\mathbf{M}'\ \mathbf{N}'\ {}^t\mathbf{L}' = \begin{bmatrix} u^2\mathbf{I}_\epsilon + u\ {}^t\mathbf{M}\mathbf{N} & u^2\ {}^t\mathbf{L} + u\ {}^t\mathbf{M}\mathbf{N}\ {}^t\mathbf{L} \\ u\mathbf{K} + \mathbf{K}\ {}^t\mathbf{M}\mathbf{N} & u\mathbf{K}\ {}^t\mathbf{L} + \mathbf{K}\ {}^t\mathbf{M}\mathbf{N}\ {}^t\mathbf{L} \end{bmatrix}. \tag{7}$$

A nonzero element of $u^2\mathbf{I}_\epsilon$, $u\ {}^t\mathbf{M}\mathbf{N}$, $u^2\ {}^t\mathbf{L}$, $u\ {}^t\mathbf{M}\mathbf{N}\ {}^t\mathbf{L}$, $u\mathbf{K}$, $\mathbf{K}\ {}^t\mathbf{M}\mathbf{N}$, $u\mathbf{K}\ {}^t\mathbf{L}$ and $\mathbf{K}\ {}^t\mathbf{M}\mathbf{N}\ {}^t\mathbf{L}$ corresponds to a sequence of eight paths of length two, respectively:

$$P_i \to Q_i \to P_i;\ P_i \to Q \to P_j(c(P_i) \neq c(P_j));\ P_i \to Q_i \to R(c(P_i) = c(R));$$
$$P_i \to Q \to R(c(P_i) \neq c(R));\ P \to Q_i \to P_i(c(P) = c(P_i));\ P \to Q \to P_i(c(P) \neq c(P_i));$$
$$P \to Q_i \to R(c(P) = c(R));\ P \to Q \to R(c(P) \neq c(R)),$$

where $P, R \in \mathcal{R}$, $Q \in \mathcal{S}$, $i = 1, \ldots, \epsilon$, and the notation $P \to Q$ implies that $te(P) = oe(Q)$ in B_H. Therefore, it follows that

$$\mathbf{K}'\ {}^t\mathbf{M}'\ \mathbf{N}'\ {}^t\mathbf{L}' = \mathbf{T}'. \tag{8}$$

By (3) and (4), we have

$${}^t\mathbf{L}'\mathbf{K}'\ {}^t\mathbf{M}'\mathbf{N}' = u^2\mathbf{I}_\epsilon + u\ {}^t\mathbf{L}\mathbf{K} + u\ {}^t\mathbf{M}\mathbf{N} + {}^t\mathbf{L}\mathbf{K}\ {}^t\mathbf{M}\mathbf{N} = u^2\mathbf{I}_\epsilon + u(\mathbf{F} + \mathbf{G}) + \mathbf{X}. \tag{9}$$

By (6),(8) and (9), it follows that

$$\det(\mathbf{I}_{r+\epsilon} - t\mathbf{T}') = \det(\mathbf{I}_\epsilon - t(\mathbf{X} + u(\mathbf{F} + \mathbf{G}) + u^2\mathbf{I}_\epsilon)).$$

Q.E.D.

If $u = 0$, then Theorem 6 implies (1) of Theorem 3.

Corollary 1 *Let H be a finite, connected hypergraph such that every hypervetex is in at least two hyperedges. Set $r =\mid \mathcal{R} \mid$. Then*

$$\zeta_H(t)^{-1} = \det(\mathbf{I}_r - t\mathbf{T}).$$

Proof. Set $\epsilon =\mid E(B_H) \mid$ and $u = 0$. By Theorem 6 and (5), (7), we have

$$\zeta_H(t)^{-1} = \det(\mathbf{I}_{r+\epsilon} - t\mathbf{T}') = \det\left(\begin{bmatrix} \mathbf{I}_\epsilon & \mathbf{0} \\ -t\mathbf{K}\ {}^t\mathbf{MN} & \mathbf{I}_r - t\mathbf{T} \end{bmatrix}\right) = \det(\mathbf{I}_r - t\mathbf{T}).$$

Q.E.D.

5 Example

Let H be the hypergraph with $V(H) = \{v_1, v_2, v_3\}$ and $E(H) = \{e_1, e_2, e_3\}$, where $e_1 = \{v_1, v_2\}$, $e_2 = \{v_1, v_3\}$ and $e_3 = \{v_1, v_2, v_3\}$. Furthermore, let B_H be the bipartite graph associated with H. Let $f_1 = (v_1, e_1)$,$f_2 = (v_1, e_2)$, $f_3 = (v_1, e_3)$, $f_4 = (v_2, e_1)$, $f_5 = (v_2, e_3)$, $f_6 = (v_3, e_2)$ and $f_7 = (v_3, e_3)$. Then we have $D(B_H) = \{f_1, \ldots, f_7, f_1^{-1}, \ldots, f_7^{-1}\}$. The matrices $\mathbf{X}$ is given as follows:

$$\mathbf{X} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

By Theorem 4, we have

$$\zeta(H, t)^{-1} = \det(\mathbf{I}_7 - t\mathbf{X}) = (1 - t)(1 + t + t^2)(1 - 4t^2 - t^3 + 4t^4).$$

Next, two matrices $\mathbf{F}$ and $\mathbf{G}$ are given as follows:

$$\mathbf{F} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}, \mathbf{G} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Then it is certain that $\mathbf{FG} = \mathbf{X}$.

Furthermore,

$$\mathbf{X} + u\mathbf{F} + u\mathbf{G} + u^2\mathbf{I}_7 = \begin{bmatrix} u^2 & u & u & u & 1 & 0 & 0 \\ u & u^2 & u & 0 & 0 & u & 1 \\ u & u & u^2 & 1 & u & 1 & u \\ u & 1 & 1 & u^2 & u & 0 & 0 \\ 1 & 1 & u & u & u^2 & 1 & u \\ 1 & u & 1 & 0 & 0 & u^2 & u \\ 1 & 1 & u & 1 & u & u & u^2 \end{bmatrix}.$$

By Theorem 6, we have

$$\zeta(H,u,t)^{-1} = \det(\mathbf{I}_7 - t(\mathbf{X} + u\mathbf{F} + u\mathbf{G} + u^2\mathbf{I}_7))$$

$$\begin{aligned} &= (1-(1-u)^2t)(1+(1-2u^2)t+(1-u^2)^2t^2) \\ &\times (1-2u(1+2u)t+(-4-2u-5u^2-6u^3+6u^4)t^2 \\ &- (1-u)^2(1+4u+14u^2+14u^3+4u^4)t^3+(1-u)^4(1+u)^2(2+u)^2t^4). \end{aligned}$$

Now, we consider arcs of $B_H^{(1)}$. Let $R_1 = (v_1, e_1, v_2)$,$R_2 = (v_1, e_2, v_3)$, $R_3 = (v_1, e_3, v_2)$, $R_4 = (v_1, e_3, v_3)$, $R_5 = R_1^{-1}$, $R_6 = R_3^{-1}$, $R_7 = (v_2, e_3, v_3)$, $R_8 = R_2^{-1}$, $R_9 = R_4^{-1}$, $R_{10} = R_7^{-1}$ and $P_i = (f_i, f_i^{-1})(1 \leqslant i \leqslant 7)$. Arrange elements of $\mathcal{R}' = D(B_H^{(1)})$ as follows: $P_1, \cdots, P_7, R_1, \cdots, R_{10}$. We consider the matrix $\mathbf{T}'$ under this order, and then, we have

$$\mathbf{T}' = \begin{bmatrix}
u^2 & u & u & 0 & 0 & 0 & 0 & u^2 & u & u & u & 0 & 0 & 0 & 0 & 0 & 0 \\
u & u^2 & u & 0 & 0 & 0 & 0 & u & u^2 & u & u & 0 & 0 & 0 & 0 & 0 & 0 \\
u & u & u^2 & 0 & 0 & 0 & 0 & u & u & u^2 & u^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & u^2 & u & 0 & 0 & 0 & 0 & 0 & 0 & u^2 & u & u & 0 & 0 & 0 \\
0 & 0 & 0 & u & u^2 & 0 & 0 & 0 & 0 & 0 & 0 & u & u^2 & u^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & u^2 & u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u^2 & u & u \\
0 & 0 & 0 & 0 & 0 & s & u^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u & u^2 & u^2 \\
u & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u & 1 & 1 & 0 & 0 & 0 \\
1 & u & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u & 1 & 1 \\
1 & 1 & u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & u & 0 & 0 & 0 & 0 \\
1 & 1 & u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & u & 0 \\
0 & 0 & 0 & u & 1 & 0 & 0 & u & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & u & 0 & 0 & 1 & 1 & u & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & u \\
0 & 0 & 0 & 0 & 0 & u & 1 & 1 & u & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & u & 1 & 1 & 0 & u & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & u & 0 & 0 & 0 & 0 & 1 & 0 & u & 0 & 0 & 0
\end{bmatrix}.$$

By Theorem 6, we have

$$\det(\mathbf{I}_{17} - t\mathbf{T}') = \zeta(H,u,t)^{-1}.$$

Let $u=0$. By the proof of Corollary 1, the matrix $\mathbf{T}$ in Theorem 3 is the submatrix of $\mathbf{T}'$ consisting of $8,\ldots,17$ rows and $8,\ldots,17$ columns. Thus,

$$\mathbf{T} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

By Theorem 3, we have

$$\det(\mathbf{I}_{10} - t\mathbf{T}) = \zeta(H,t)^{-1}.$$

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