Some gregarious cycle decompositions of complete equipartite graphs

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Abstract

A k-cycle decomposition of a multipartite graph G is said to be gregarious if each k-cycle in the decomposition intersects k distinct partite sets of G. In this paper we prove necessary and sufficient conditions for the existence of such a decomposition in the case where G is the complete equipartite graph, having n parts of size m, and either $n \equiv 0, 1 \pmod{k}$, or k is odd and $m \equiv 0 \pmod{k}$. As a consequence, we prove necessary and sufficient conditions for decomposing complete equipartite graphs into gregarious cycles of prime length.

1 Introduction and preliminaries

We begin with some relevant definitions and terminology.

Let K_n denote the complete graph on n vertices and $\overline{K}_n$ denote the empty graph on n vertices. For any graph G and any positive integer λ, we denote the multigraph obtained from G by replacing each of its edges with λ parallel edges by λG.

We denote the k-cycle containing edges $v_1v_2, v_2v_3, \ldots, v_{k-1}v_k$ and v_1v_k by $(v_1, v_2, \ldots, v_k)$, or $(v_k, v_{k-1}, \ldots, v_1)$, or by any cyclic shift of these. We denote the k-path containing edges $v_1v_2, v_2v_3, \ldots, v_kv_{k+1}$ by $[v_1, v_2, \ldots, v_{k+1}]$ or $[v_{k+1}, v_k, \ldots, v_1]$ (hence, in our terminology, a k-path contains k edges and $k+1$ vertices). Similarly, we denote the directed k-cycle containing directed edges $(v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k)$ and (v_k, v_1) by $(v_1, v_2, \ldots, v_k)_D$, or by any cyclic shift of this.

The *lexicographic product* $G * H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$, and with an edge joining (g_1, h_1) to (g_2, h_2) if and only if $g_1g_2 \in E(G)$, or $g_1 = g_2$ and $h_1h_2 \in E(H)$. For our purposes we will primarily be concerned with lexicographic products of the form $G * \overline{K}_m$, for some G and m. We note that for all graphs G and positive integers a and ℓ,

$$G * \overline{K}_{a\ell} \cong (G * \overline{K}_a) * \overline{K}_\ell \cong (G * \overline{K}_\ell) * \overline{K}_a.$$

A subgraph of a multipartite graph G is said to be *gregarious in* G (or simply *gregarious* when G is clear) if each of its vertices lies in a different partite set of G. In order to apply this definition, it must be clear what the partite sets of G are. With this in mind we adopt the following convention.

Suppose G is a graph on n vertices, and a and ℓ are positive integers. Throughout this paper, unless otherwise specified, we assume the graph $G * \overline{K}_{a\ell}$ has vertex set $\{v_1, v_2, \ldots, v_{a\ell} \mid v \in V(G)\}$, and an edge joining u_i and v_j in $G * \overline{K}_{a\ell}$ if and only if there is an edge joining u and v in G. The partite sets of $G * \overline{K}_{a\ell}$ are the n sets $\{v_1, v_2, \ldots, v_{a\ell}\}$, $v \in V(G)$. However, we may occasionally choose to express the lexicographic product of G and $\overline{K}_{a\ell}$ as $G * \overline{K}_{a(\ell)}$, or indeed $G * \overline{K}_{\ell(a)}$. In the first case, it is assumed that there is a further partitioning of each of the sets $\{v_1, v_2, \ldots, v_{a\ell}\}$ into a subsets, each of size ℓ. It is these na sets of size ℓ which are considered the partite sets of $G * \overline{K}_{a(\ell)}$. Similarly in the second case, it is assumed that there is a further partitioning of each of the sets $\{v_1, v_2, \ldots, v_{a\ell}\}$ into ℓ subsets, each of size a. It is these $n\ell$ sets of size a which are considered the partite sets of $G * \overline{K}_{\ell(a)}$. Note that this means a given subgraph H may be gregarious in $G * \overline{K}_{a(\ell)}$, but not gregarious in $G * \overline{K}_{a\ell}$.

A *decomposition* of a graph G is a collection of subgraphs of G whose edge-sets partition the edge-set of G. Let $\mathcal{H} = \{\mathcal{H}_\infty, \mathcal{H}_\in, \ldots, \mathcal{H}_\sqcup\}$ be a family of mutually nonisomorphic nontrivial graphs. An $\mathcal{H}$-*decomposition* of G is a decomposition, $\mathcal{D}$ say, of G such that

- for each $D \in \mathcal{D}$ there is some $H_i \in \mathcal{H}$ with $D \cong H_i$; and
- for each $H_i \in \mathcal{H}$ there is some $D \in \mathcal{D}$ with $H_i \cong D$.

If $\mathcal{H} = \{\mathcal{H}\}$ we often refer to such a decomposition as simply an H-*decomposition*. A decomposition of a multipartite graph G is said to be *gregarious* if each of the subgraphs in the decomposition is gregarious in G.

In this paper we will be concerned with gregarious k-cycle decompositions of $K_n * \overline{K}_m$, the complete equipartite graph having n parts of size m. Note that every vertex in $K_n * \overline{K}_m$ has degree $(n-1)m$ and the total number of edges in $K_n * \overline{K}_m$ is $n(n-1)m^2/2$. Hence, obvious necessary conditions for the existence of such a decomposition are that

- $n \geqslant k \geqslant 3$;
- $(n-1)m$ is even; and
- $n(n-1)m^2 \equiv 0 \pmod{2k}$.

The study of gregarious cycle decompositions is relatively new and thus, compared to "nongregarious" cycle decompositions, few results are known. Of course 3-cycle decompositions of complete equipartite graphs are necessarily gregarious and so by Hanani [5], the above necessary conditions are sufficient when $k = 3$. Sufficiency of these conditions has also been proved by Billington and Hoffman [2] when $k = 4$, by Smith [9] when $k = 5$ and by Billington, Hoffman and Smith [4] when $k \in \{6, 8\}$. More generally, in [3] Billington, Hoffman and Rodger prove there exists a *resolvable* gregarious n-cycle decomposition of $K_n * \overline{K}_m$ if and only if $(n, m) \neq (3, 2)$ or $(3, 6)$; that is, the cycles in the decomposition can be partitioned into sets in such a way that the cycles in each set induce a 2-factor of $K_n * \overline{K}_m$. (Note that there exists nonresolvable gregarious 3-cycle decompositions of both $K_3 * \overline{K}_2$ and $K_3 * \overline{K}_6$.) In this paper, using the various new construction techniques presented in Section 2, we prove sufficiency of the above necessary conditions in cases where either $n \equiv 0, 1 \pmod{k}$, or k is odd and $m \equiv 0 \pmod{k}$. More formally, the main result (split into two parts) of this paper is the following.

Theorem 1.1 *Suppose n, m and k are positive integers with $k \geqslant 3$. Necessary conditions for the existence of a gregarious k-cycle decomposition of $K_n * \overline{K}_m$ are that $n \geqslant k$, $(n-1)m$ is even and $n(n-1)m^2 \equiv 0 \pmod{2k}$.*

- **(i)** *These conditions are sufficient whenever $n \equiv 0, 1 \pmod{k}$.*
- **(ii)** *These conditions are sufficient whenever k is odd and $m \equiv 0 \pmod{k}$.*

Part (i) of the above theorem is proved in Section 3, while part (ii) is proved in Section 4. Also, Theorem 1.1 has the following nice corollary.

Corollary 1.2 *Suppose n, m and p are positive integers with p an odd prime. Then there exists a gregarious p-cycle decomposition of $K_n * \overline{K}_m$ if and only if $n \geqslant p$, $(n-1)m$ is even and $n(n-1)m^2 \equiv 0 \pmod{2p}$.*

Proof Since $n(n-1)m^2 \equiv 0 \pmod{2p}$ and p is prime, we must have either $n \equiv 0, 1 \pmod{p}$ or $m \equiv 0 \pmod{p}$. The result then follows by Theorem 1.1. □

Hence, for arbitrary n and m, Corollary 1.2 gives the first known infinite family of values of k for which the obvious necessary conditions for the existence of a gregarious k-cycle decomposition of $K_n * \overline{K}_m$ are also sufficient.

2 Some new decomposition techniques

In this section we introduce some new techniques for obtaining gregarious cycle decompositions of complete equipartite graphs from cycle decompositions of related multigraphs. The techniques used are similar to those introduced in [11].

We begin with the following definition.

Definition 2.1 Suppose $\mathcal{D} = \{H_1, H_2, \ldots, H_t\}$ is a decomposition of λG. A *λ-weight function* on $\mathcal{D}$ is any function ω which assigns an integer label, from the set $\{0, 1, \ldots, \lambda-1\}$, to each edge of the graphs $H_1, H_2, \ldots, H_t$, in such a way that distinct copies of the same edge receive distinct labels. (Hence for each $\ell \in \{0, 1, \ldots, \lambda - 1\}$, the edges labelled ℓ induce a copy of the graph G.) □

Note that the above definition is a specialisation of a more general type of "weight function" first described in [11].

The next lemma shows how a cycle decomposition of the graph $2G$, together with a 2-weight function, can be used to generate a gregarious cycle decomposition of the graph $G * \overline{K}_2$ when certain extra conditions on the original decomposition are satisfied.

Lemma 2.2 *Suppose there exists a k-cycle decomposition of $2G$ which can be partitioned into pairs of cycles in such a way that the cycles in each pair share two adjacent edges. Then there exists a gregarious k-cycle decomposition of $G * \overline{K}_2$.*

Proof Let $\mathcal{D}$ be a k-cycle decomposition of $2G$ which satisfies the conditions of the lemma and let ω be any 2-weight function on $\mathcal{D}$. Recall that $G*\overline{K}_2$ is the graph obtained from G by replacing each vertex v in G with the set of vertices $\{v_1, v_2\}$, and each edge uv in G with the edges u_1v_1, u_1v_2, u_2v_1 and u_2v_2.

For each k-cycle $C \in \mathcal{D}$ we generate a subgraph, denoted by $\hat{C}$, of $G*\overline{K}_2$ by associating each vertex v in C with the set of vertices $\{v_1, v_2\}$, and each edge uv in C having label ℓ (under the function ω) with the pair of edges $u_1v_{1+\ell}$ and $u_2v_{2+\ell}$ (subscripts calculated mod 2). Note that $\hat{C}$ is either a single $2k$-cycle (if C contains an odd number of edges labelled 1), or two vertex disjoint k-cycles which are gregarious in $G*\overline{K}_2$ (if C contains an even number of edges labelled 1). Since ω is a 2-weight function, the graphs $\hat{C}$ together decompose $G * \overline{K}_2$. Hence we need only show that, if C and C' are any "pair" of cycles from the partition of $\mathcal{D}$, then the graph $\hat{C} \cup \hat{C}'$ admits a decomposition into k-cycles which are gregarious in $G * \overline{K}_2$.

Let uv and vw be the two adjacent edges shared by C and C'. Then $\hat{C} \cup \hat{C}'$ contains the edges u_iv_j and v_iw_j for each $i, j \in \{1, 2\}$. Let H be the subgraph of $\hat{C} \cup \hat{C}'$ spanned by these edges, and let H' be the complement of H in $\hat{C} \cup \hat{C}'$. (Hence $\{H, H'\}$ is a decomposition of $\hat{C} \cup \hat{C}'$.) Then for some $a, b \in \{1, 2\}$ the graph H' decomposes into four gregarious $(k-2)$-paths:

$$L_1 = [w_a, \ldots, u_1]; \quad L_2 = [w_{a+1}, \ldots, u_2]; \quad L_3 = [w_b, \ldots, u_1]; \quad L_4 = [w_{b+1}, \ldots, u_2].$$

Now H decomposes into the four gregarious 2-paths:

$$P_1 = [u_1, v_1, w_a]; \quad P_2 = [u_2, v_1, w_{a+1}]; \quad P_3 = [u_1, v_2, w_b]; \quad P_4 = [u_2, v_2, w_{b+1}].$$

The result then follows by adjoining the $(k-2)$-path L_i to the 2-path P_i, for each $i \in \{1, 2, 3, 4\}$. $\square$

Note that the "pairing" condition in Lemma 2.2 means there must be an even number of cycles in the decomposition of $2G$. In fact, it is easy to see that we can relax this condition slightly and obtain the following generalisation.

Lemma 2.3 *Suppose there exists a k-cycle decomposition of $2G$ which can be partitioned into two parts, say $\mathcal{D}_1$ and $\mathcal{D}_2$, so that every cycle in $\mathcal{D}_1$ shares an edge with some cycle in $\mathcal{D}_2$, and $\mathcal{D}_2$ can be partitioned into pairs of cycles in such a way that the cycles in each pair share two adjacent edges. Then there exists a gregarious k-cycle decomposition of $G * \overline{K}_2$.*

Proof Let $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$. We show there exists a 2-weight function ω on $\mathcal{D}$ with the property that each cycle in $\mathcal{D}_1$ contains an even number of edges labelled 1 (under ω). Using the notation defined in the proof of Lemma 2.2, the result then follows since for each $C \in \mathcal{D}_1$, the graph $\hat{C}$ consists of two k-cycles which are gregarious in $G * \overline{K}_2$, and for each "pair" of cycles C and C' in the partition of $\mathcal{D}_2$ the graph $\hat{C} \cup \hat{C}'$ admits a decomposition into k-cycles which are gregarious in $G * \overline{K}_2$ (using the method described in the proof of Lemma 2.2).

Let ω be any 2-weight function on $\mathcal{D}$. If each cycle in $\mathcal{D}_1$ contains an even number of edges labelled 1 we are done. If not, we modify ω as follows. Suppose $C \in \mathcal{D}_1$ and C contains an odd number of edges labelled 1. Now C contains an edge, e say, which is also contained in some cycle, C' say, in $\mathcal{D}_2$. We switch the label on the edge e in C from either 0 to 1, or vice-versa. We then do the same for the copy of the edge e in C'. Hence the resulting edge labelling still induces a 2-weight function on $\mathcal{D}$. Moreover, the cycle C now contains an even number of edges labelled 1 and we have not affected the edge labellings of any other cycles in $\mathcal{D}_1$. We repeat this process for each cycle in $\mathcal{D}_1$ having an odd number of edges labelled 1. The resulting edge labelling then induces a 2-weight function on $\mathcal{D}$ with the required property. $\square$

In order to more fully exploit these newly defined λ-weight functions we also introduce the idea of the "sum-weight" of a cycle. (Again, this concept was originally defined in [11].)

Definition 2.4 Suppose C is a cycle in the graph G. Suppose furthermore that $\vec{G}$ is the graph obtained by orienting the edges of G in some way, and that ω is a function which assigns an integer label $\omega(e)$ to each edge e in the cycle C. We let $\vec{C}$ be a directed cycle formed by orienting the edges of C, and for each edge e in C we denote the corresponding directed edge in $\vec{C}$ by e'. (Note that there are two possible choices for $\vec{C}$, and that $\vec{C}$ need not be a subgraph of $\vec{G}$.) The *sum-weight with respect to $\vec{G}$ under the function ω* of the cycle C, is then defined to be the absolute value of the sum, over all edges e in C, of $f(e')\omega(e)$, where

$$f(e') = \begin{cases} 1, & \text{if } e' \text{ is an edge in } \vec{G}; \\ -1, & \text{otherwise.} \end{cases}$$

Note that taking the "absolute value" ensures that the sum-weight is independent of the choice of $\vec{C}$. Furthermore, when both $\vec{G}$ and ω are clear, we will often refer simply to the *sum-weight* of the cycle C, rather than the sum-weight with respect to $\vec{G}$ under the function ω. $\square$

In [11] we defined the notion of an *unbalanced* λ-weight function on a cycle decomposition of a graph λG. These functions have the property that under some particular orientation of the edges of G, say $\vec{G}$, each cycle in the decomposition of λG has sum-weight coprime to λ. Hence each k-cycle C in the decomposition of λG can be used to generate a λk-cycle in the graph $G * \overline{K}_\lambda$ by associating each vertex v in C with the partite set $A_v = \{v_1, v_2, \ldots, v_\lambda\}$, and each edge uv labelled ℓ in C with the matching between A_u

and A_v consisting of the edges $u_1v_{1+\ell}, u_2v_{2+\ell}, \ldots, u_\lambda v_{\lambda+\ell}$ if (u,v) is an edge in $\vec{G}$, or the edges $v_1u_{1+\ell}, v_2u_{2+\ell}, \ldots, v_\lambda u_{\lambda+\ell}$ if (v,u) is an edge in $\vec{G}$. The fact that the sum-weight of C is coprime to λ ensures that the k matchings form a single λk-cycle in $G * \overline{K}_\lambda$, rather than a collection of disjoint cycles whose lengths sum to λk. Since here we are interested in gregarious cycle decompositions, we will instead be concerned with λ-weight functions under which each cycle in a decomposition of λG has sum-weight a *multiple* of λ. In this case, a k-cycle in the decomposition of λG will generate, using the same method as described above, a subgraph in $G * \overline{K}_\lambda$ consisting of λ pairwise vertex-disjoint gregarious k-cycles. We call such a function a *balanced* λ-weight function. More formally we define this as follows.

Definition 2.5 Suppose $\mathcal{D}$ is a cycle decomposition of λG and ω is a λ-weight function on $\mathcal{D}$. Then ω is said to be *balanced* if, under some orientation of the edges of G, each cycle in $\mathcal{D}$ has sum-weight a multiple of λ. $\square$

The following lemma proves that such functions do indeed generate gregarious cycle decompositions.

Lemma 2.6 *Suppose $\mathcal{D}$ is a k-cycle decomposition of λG and ω is a balanced λ-weight function on $\mathcal{D}$. Then there exists a gregarious k-cycle decomposition of $G * \overline{K}_\lambda$.*

Proof Let $\vec{G}$ be the particular orientation of the edges of G under which ω is balanced. For each k-cycle $C \in \mathcal{D}$ we generate a subgraph in $G * \overline{K}_\lambda$ by associating each vertex v in C with the partite set $A_v = \{v_1, v_2, \ldots, v_\lambda\}$, and each edge uv in C having label ℓ (under the function ω) with the matching between partite sets A_u and A_v consisting of the edges $u_1v_{1+\ell}, u_2v_{2+\ell}, \ldots, u_\lambda v_{\lambda+\ell}$ if (u,v) is an edge in $\vec{G}$, or the edges $v_1u_{1+\ell}, v_2u_{2+\ell}, \ldots, v_\lambda u_{\lambda+\ell}$ if (v,u) is an edge in $\vec{G}$. Hence each of these subgraphs is a 2-regular graph on λk vertices. In fact, since the sum-weight of each cycle in $\mathcal{D}$ is a multiple of λ, each of these subgraphs necessarily consists of λ pairwise vertex-disjoint k-cycles, each of which is gregarious in $G * \overline{K}_\lambda$. Furthermore, since ω is a λ-weight function, the collection of all such k-cycles forms a decomposition of $G * \overline{K}_\lambda$ as required. $\square$

3 The case $n \equiv 0, 1 \pmod{k}$

The aim of this section is to prove Theorem 1.1 (i). We make extensive use of the following three obvious, but surprisingly useful, results.

Lemma 3.1 *Suppose there exists a gregarious $\{H_1, H_2, \ldots, H_t\}$-decomposition of $G * \overline{K}_a$ and, for each $i \in \{1, 2, \ldots, t\}$, there exists a gregarious H-decomposition of $H_i * \overline{K}_\ell$. Then there exists a gregarious H-decomposition of $G * \overline{K}_{a\ell}$.*

Lemma 3.2 *Suppose there exists a $\{H_1, H_2, \ldots, H_t\}$-decomposition of $G * \overline{K}_a$ and, for each $i \in \{1, 2, \ldots, t\}$, there exists a gregarious H-decomposition of $H_i * \overline{K}_\ell$. Then there exists a gregarious H-decomposition of $G * \overline{K}_{a(\ell)}$.*

Lemma 3.3 *Suppose there exists a gregarious* $\{H_1, H_2, \ldots, H_t\}$*-decomposition of* $G * \overline{K}_a$ *and, for each* $i \in \{1, 2, \ldots, t\}$*, there exists an* H*-decomposition of* $H_i * \overline{K}_\ell$*. Then there exists a gregarious* H*-decomposition of* $G * \overline{K}_{\ell(a)}$*.*

The following is a direct application of Lemma 3.2.

Lemma 3.4 *Suppose there exists a* k*-cycle decomposition of* $G * \overline{K}_a$*. Then, for each positive integer* ℓ*, there exists a gregarious* k*-cycle decomposition of* $G * \overline{K}_{a(\ell)}$*.*

Proof By Lemma 3.2 we need only prove there exists a gregarious k-cycle decomposition of $C * \overline{K}_\ell$, where C is a (generic) k-cycle. Suppose $C = (1, 2, \ldots, k)$. If ℓ is even we take the ℓ^2 cycles $(1_i, 2_j, \ldots, (k-1)_i, k_j)$, where $i, j \in \{1, 2, \ldots, \ell\}$. If ℓ is odd we take the ℓ^2 cycles $(1_i, 2_j, \ldots, (k-1)_j, k_{i \circ j})$, where $i, j \in \{1, 2, \ldots, \ell\}$ and $i \circ j$ is the entry in row i and column j of any (fixed) latin square of order ℓ on the set $\{1, 2, \ldots, \ell\}$. It is an easy exercise to check that these cycles decompose $C * \overline{K}_\ell$ as required. $\square$

We can apply Lemma 3.1 in a similar way and obtain the following easy result.

Lemma 3.5 *Suppose there exists a gregarious* k*-cycle decomposition of* $G * \overline{K}_a$*. Then, for each positive integer* ℓ*, there exists a gregarious* k*-cycle decomposition of* $G * \overline{K}_{a\ell}$*.*

We will also make use of some well-known results involving cycle decompositions of complete graphs and complete equipartite graphs. The first such result, due to Alspach, Gavlas [1] and Săjna [8], gives necessary and sufficient conditions for the existence of a k-cycle decomposition of the complete graph K_n.

Theorem 3.6 ([1],[8]) *Suppose* n *and* k *are positive integers with* $n \geqslant 3$ *and* $k \geqslant 3$*. Then there exists a* k*-cycle decomposition of* K_n *if and only if* $n \geqslant k$*,* n *is odd and* $n(n-1) \equiv 0 \pmod{2k}$*.*

We note that if there exists a k-cycle decomposition of K_n then, by Lemma 3.4, for each positive integer m there exists a gregarious k-cycle decomposition of $K_n * \overline{K}_m$. Hence we have the following obvious corollary to Theorem 3.6.

Corollary 3.7 *Suppose* n*,* m *and* k *are positive integers with* $n \geqslant 3$ *and* $k \geqslant 3$*. Then there exists a gregarious* k*-cycle decomposition of* $K_n * \overline{K}_m$ *whenever* $n \geqslant k$*,* n *is odd and* $n(n-1) \equiv 0 \pmod{2k}$*.*

The next theorem follows from a stronger result of Liu [6],[7] involving resolvable cycle decompositions of complete equipartite graphs. Note that we have removed the "resolvability" condition from Liu's original result since we will not be concerned with that property here (this also allows us to easily remove the "exceptions" from Liu's original result).

Theorem 3.8 ([6],[7]) *Suppose* n*,* m *and* k *are positive integers with* $n \geqslant 3$ *and* $k \geqslant 3$*. Then there exists a* k*-cycle decomposition of* $K_n * \overline{K}_m$ *whenever* $(n-1)m$ *is even and* $nm \equiv 0 \pmod{k}$*.*

Combining this result with Lemma 3.4 we have the following easy corollary.

Corollary 3.9 *Suppose n, ℓ, a and k are positive integers with $n \geqslant 3$ and $k \geqslant 3$. Then there exists a gregarious k-cycle decomposition of $K_n * \overline{K}_{ak(\ell)}$ whenever $(n-1)ak$ is even.*

We now state a result of Billington et al. [3] involving gregarious k-cycle decompositions of $K_k * \overline{K}_m$. We note, as mentioned in the introduction, that they actually proved necessary and sufficient conditions for the existence of a resolvable gregarious k-cycle decomposition of $K_k * \overline{K}_m$ however, as already noted, we will not be concerned with this additional property here.

Theorem 3.10 ([3]) *Suppose k and m are positive integers with $k \geqslant 3$. Then there exists a gregarious k-cycle decomposition of $K_k * \overline{K}_m$ if and only if either k is odd or m is even.*

This result has the following simple corollary in the case that k is even.

Corollary 3.11 *Suppose n, m and k are positive integers with m and k even, and $k \geqslant 4$. Then there exists a gregarious k-cycle decomposition of $K_n * \overline{K}_m$ whenever $n \equiv 0 \pmod{k}$.*

Proof If $n = k$ the result follows immediately by Theorem 3.10. Suppose then that $n = qk$, with $q \geqslant 2$. It is easy to see that there exists a $\{K_k, K_q * \overline{K}_k\}$-decomposition of $K_n \cong K_{qk} * \overline{K}_1$. Moreover, there exist gregarious k-cycle decompositions of $K_k * \overline{K}_m$, by Theorem 3.10, and of $K_q * \overline{K}_{k(m)}$, by Corollary 3.9. Hence the result follows by Lemma 3.2. $\square$

Using the techniques developed in Section 1 (in particular Lemma 2.3), we now present a series of three lemmas (each followed immediately by a corollary) in which we obtain some useful gregarious cycle decompositions of complete equipartite graphs having parts of even size. The first of these decompositions deals with cases in which the cycle length is also even.

Lemma 3.12 *Suppose k and m are positive integers with $k \geqslant 4$. Then there exists a gregarious k-cycle decomposition of $K_{k+1} * \overline{K}_m$ whenever k and m are both even.*

Proof If $k = 4$ the result follows from [2]. Suppose then $k \geqslant 6$. Note that we need only consider the case $m = 2$ and the result then follows by Lemma 3.5. Furthermore, in the case $m = 2$ we need only give a k-cycle decomposition of $2K_{k+1}$ which satisfies the conditions of Lemma 2.3. We do this as follows.

Let the vertex set of $2K_{k+1}$ be $\mathbb{Z}_k \cup \{\infty\}$, and let $\rho = (0\,1\,2\,\cdots\,(k-1))(\infty)$ be a permutation of order k on $V(2K_{k+1})$. We define the k-cycles C and D on $2K_{k+1}$ by

$$\begin{aligned} C &= (0, 1, k-1, 2, k-2, \ldots, k/2-1, k/2+1, \infty); \text{ and} \\ D &= (0, 1, 2, \ldots, k-1). \end{aligned}$$

(See for example Figure 1, which shows the cycles C and D in the case $k = 16$.)

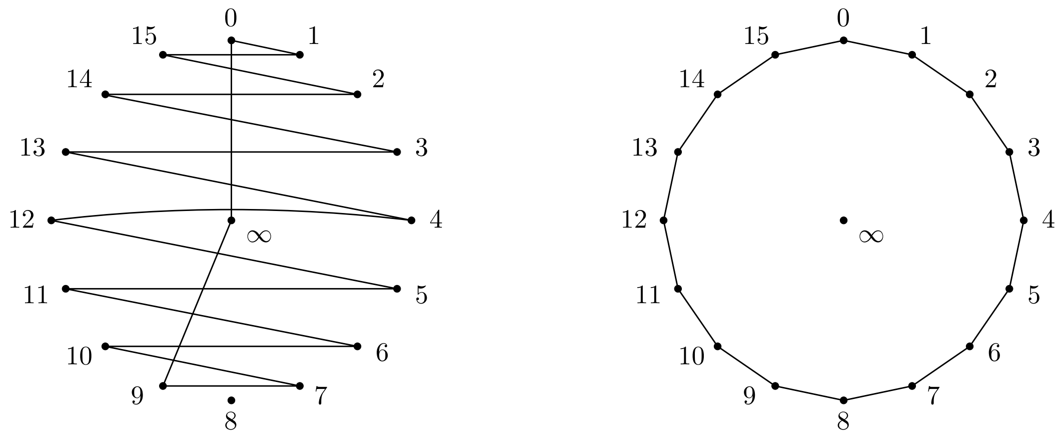

Figure 1: The cycles C and D when $k = 16$

It is a simple exercise to check that $\mathcal{D} = \{D, \rho^0(C), \rho^1(C), \ldots, \rho^{k-1}(C)\}$ is a k-cycle decomposition of $2K_{k+1}$. We note that, for each $i \in \{0, 1, \ldots, k/2 - 1\}$, the cycles $\rho^i(C)$ and $\rho^{i+k/2}(C)$ both contain the 2-path $[i+1, i-1, i+2]$. Hence, setting $\mathcal{D}_1 = \{D\}$ and $\mathcal{D}_2 = \{\rho^0(C), \rho^1(C), \ldots, \rho^{k-1}(C)\}$, it is easy to see that $\mathcal{D}$ satisfies the conditions of Lemma 2.3 and the result follows. □

Corollary 3.13 *Suppose n, m and k are positive integers with m and k even, and $k \geqslant 4$. Then there exists a gregarious k-cycle decomposition of $K_n * \overline{K}_m$ whenever $n \geqslant k$ and $n \equiv 1 \pmod{k}$.*

Proof If $n = k + 1$ the result follows immediately by Lemma 3.12. Suppose then that $n = qk+1$, with $q \geqslant 2$. It is easy to see that there exists a $\{K_{k+1}, K_2 * \overline{K}_k\}$-decomposition of $K_n \cong K_{qk+1} * \overline{K}_1$. Moreover, there exist gregarious k-cycle decompositions of $K_{k+1} * \overline{K}_m$, by Lemma 3.12, and of $K_2 * \overline{K}_{k(m)}$, by Corollary 3.9. Hence the result follows by Lemma 3.2. □

The next result is analogous to that of Lemma 3.12 in the case that k is odd.

Lemma 3.14 *Suppose k and m are positive integers with $k \geqslant 3$. Then there exists a gregarious k-cycle decomposition of $K_{k+1} * \overline{K}_m$ whenever k is odd and m is even.*

Proof As in the proof of Lemma 3.12, we need only give a k-cycle decomposition of $2K_{k+1}$ which satisfies the conditions of Corollary 2.3. The result then follows for $m = 2$ by Lemma 2.3, and subsequently for all even m by Lemma 3.5.

Let the vertex set of $2K_{k+1}$ be $\mathbb{Z}_k \cup \{\infty\}$, and let $\rho = (0\,1\,2 \cdots (k-1))(\infty)$ be a permutation of order k on $V(2K_{k+1})$. We then split the problem according to the congruence of k modulo 4.

Case I. Suppose $k \equiv 1 \pmod 4$. Let $k = 4\ell + 1$ and define the $(4\ell+1)$-cycles C and D on $2K_{4\ell+2}$ by

$$\begin{aligned} C =\ & (0, 1, 4\ell, 2, 4\ell-1, \ldots, \ell, 3\ell+1, \ell+2, 3\ell, \ell+3, \ldots, 2\ell, 2\ell+2, 2\ell+1, \infty); \text{ and} \\ D =\ & (0, 2\ell, 4\ell, 2\ell-1, 4\ell-1, \ldots, 1, 2\ell+1). \end{aligned}$$

(See for example Figure 2, which shows the cycles C and D in the case $k = 17$.)

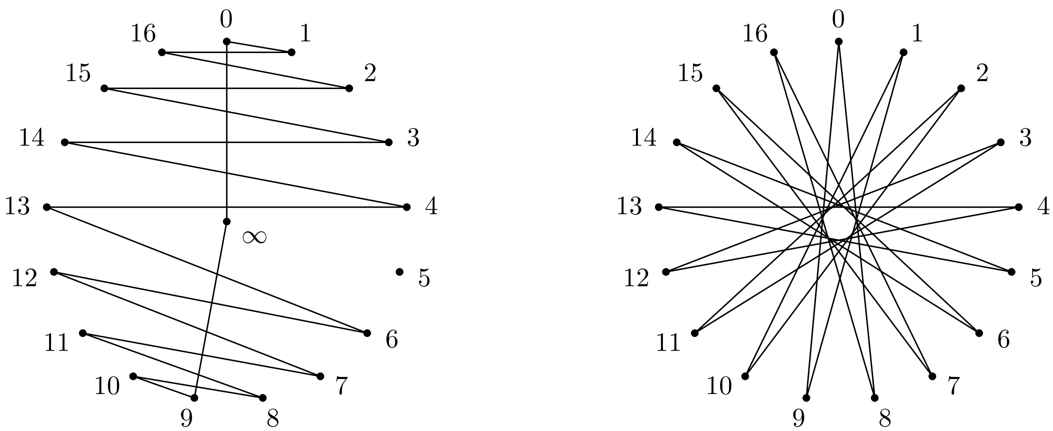

Figure 2: The cycles C and D when $k = 17$

It is a simple exercise to check that $\mathcal{D} = \{D, \rho^0(C), \rho^1(C), \ldots, \rho^{4\ell}(C)\}$ is a k-cycle decomposition of $2K_{4\ell+2}$. We note that, for each $i \in \{0, 1, \ldots, 2\ell-1\}$, the cycles $\rho^i(C)$ and $\rho^{i+2\ell}(C)$ both contain the 2-path $[\infty, i, i+1]$. Hence, setting $\mathcal{D}_1 = \{D, \rho^{4\ell}(C)\}$ and $\mathcal{D}_2 = \{\rho^0(C), \rho^1(C), \ldots, \rho^{4\ell-1}(C)\}$, it is easy to see that $\mathcal{D}$ satisfies the conditions of Lemma 2.3 and the result follows.

Case II. Suppose $k \equiv 3 \pmod 4$. Let $k = 4\ell + 3$ and define the $(4\ell+3)$-cycles C and D on $2K_{4\ell+4}$ by

$$\begin{aligned} C =& (0, 1, 4\ell+2, 2, 4\ell+1, \ldots, 3\ell+3, \ell+1, 3\ell+1, \ell+2, 3\ell, \ldots, 2\ell, 2\ell+2, 2\ell+1, \infty); \text{and} \\ D =& (0, 2\ell+1, 4\ell+2, 2\ell, 4\ell+1, \ldots, 1, 2\ell+2). \end{aligned}$$

It is a simple exercise to check that $\mathcal{D} = \{D, \rho^0(C), \rho^1(C), \ldots, \rho^{4\ell+2}(C)\}$ is a k-cycle decomposition of $2K_{4\ell+4}$. We note that, for each $i \in \{0, 1, 2, \ldots, 2\ell-1\}$, the cycles $\rho^i(C)$ and $\rho^{i+2\ell+2}(C)$ both contain the 2-path $[\infty, i, i+1]$. Hence, setting $\mathcal{D}_1 = \{D, \rho^{2\ell+1}(C)\}$ and $\mathcal{D}_2 = \{\rho^0(C), \rho^1(C), \ldots, \rho^{2\ell}(C), \rho^{2\ell+2}(C), \rho^{2\ell+3}(C), \ldots, \rho^{4\ell+2}(C)\}$, it is easy to see that $\mathcal{D}$ satisfies the conditions of Lemma 2.3 and the result follows. □

Corollary 3.15 *Suppose n, m and k are positive integers with n and m even, k odd and $k \geqslant 3$. Then there exists a gregarious k-cycle decomposition of $K_n * \overline{K}_m$ whenever $n \geqslant k$ and $n \equiv 1 \pmod k$.*

Proof If $k = n-1$, the result follows immediately from Lemma 3.14. Suppose then $n = qk+1$, with $q \geqslant 3$ and q odd (since n is even). It is easy to see that there exists a $\{K_{k+1}, K_q * \overline{K}_k\}$-decomposition of $K_n \cong K_{qk+1} * \overline{K}_1$. Moreover, there exist gregarious k-cycle decompositions of $K_{k+1} * \overline{K}_m$, by Lemma 3.14, and of $K_q * \overline{K}_{k(m)}$, by Corollary 3.9. Hence the result follows by Lemma 3.2. $\square$

The third and final lemma will be used to give the analogous result to Corollary 3.11 in the case that k is odd.

Lemma 3.16 *Suppose k and m are positive integers with $k \geqslant 3$. Then there exists a gregarious k-cycle decomposition of $K_{2k} * \overline{K}_m$ whenever k is odd and m is even.*

Proof The case $k = 3$ (in which all cycles are necessarily gregarious) was settled by Hanani [5]. Assume then $k \geqslant 5$.

Similar to the proof of Lemma 3.14, we need only give a decomposition of $2K_{2k}$ into k-cycles which satisfies the conditions of Lemma 2.2. The result then follows for $m = 2$ by Lemma 2.2, and subsequently for all even m by Lemma 3.5.

Let the vertex set of $2K_{2k}$ be $\mathbb{Z}_{2k-1} \cup \{\infty\}$, and let $\rho = (0\,1\,2\,\cdots\,(2k-2))(\infty)$ be a permutation of order $2k-1$ on $V(2K_{2k})$. We then split the problem according to the congruence of k modulo 4.

Case I. Suppose $k \equiv 1 \pmod 4$. Let $k = 4\ell+1$, then $\ell \geqslant 1$ and $2k-1 = 8\ell+1$. Define the $(4\ell+1)$-cycles C and D on $2K_{8\ell+2}$ by

$$\begin{aligned} C &= (0, 1, 8\ell, 2, 8\ell-1, \ldots, 6\ell+2, 2\ell, 6\ell); \text{ and} \\ D &= (0, 1, 8\ell, 2, 8\ell-1, \ldots, 7\ell+1, \ell+2, 7\ell, \ell+3, 7\ell-1, \ldots, 6\ell+2, 2\ell+1, \infty). \end{aligned}$$

(See for example Figure 3, which shows the cycles C and D in the case $k = 9$.)

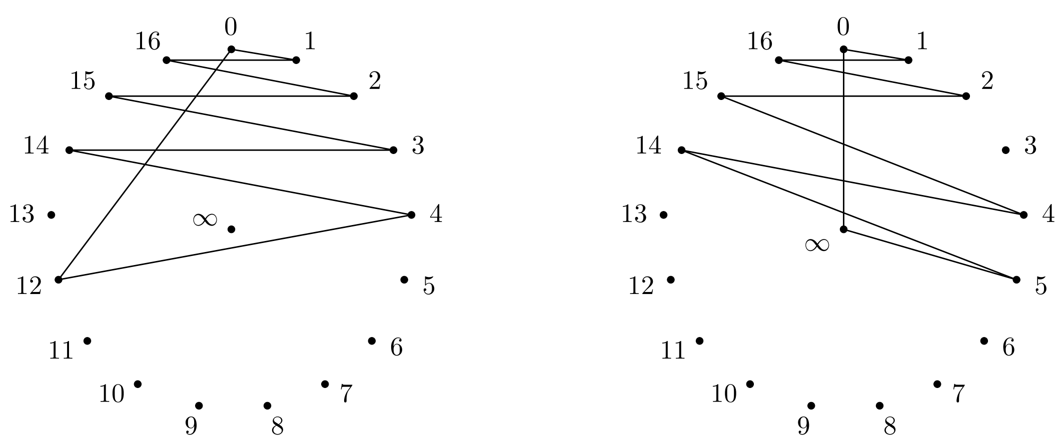

Figure 3: The cycles C and D when $k = 9$

It is a simple exercise to check that $\mathcal{D} = \{\rho^i(C), \rho^i(D) \mid 0 \leqslant i \leqslant 2k-2\}$ is a k-cycle decomposition of $2K_{8\ell+2}$. We note that the cycles C and D both contain the 2-path

$[0, 1, 8\ell]$. Hence it is easy to see that $\mathcal{D}$ satisfies the conditions of Lemma 2.2 and the result follows.

Case II. Suppose $k \equiv 3 \pmod 4$. Let $k = 4\ell + 3$, then $\ell > 1$ and $2k - 1 = 8\ell + 5$. Define the $(4\ell + 3)$-cycles C and D on $2K_{8\ell+6}$ by

$$C = (0, 1, 8\ell + 4, 2, 8\ell + 3, \ldots, 6\ell + 4, 2\ell + 1, 6\ell + 3); \text{ and}$$
$$D = (0, 1, 8\ell + 4, 2, 8\ell + 3, \ldots, \ell, 7\ell + 5, \ell + 2, 7\ell + 4, \ell + 3, \ldots, 6\ell + 4, 2\ell + 2, \infty).$$

It is a simple exercise to check that $\mathcal{D} = \{\rho^i(C), \rho^i(D) \mid 0 \leqslant i \leqslant 2k - 2\}$ is a k-cycle decomposition of $2K_{8\ell+2}$. We note that the cycles C and D both contain the 2-path $[0, 1, 8\ell + 4]$. Hence it is easy to see that $\mathcal{D}$ satisfies the conditions of Lemma 2.2 and the result follows. $\square$

Corollary 3.17 *Suppose n, m and k are positive integers with n and m even, k odd and $k \geqslant 3$. Then there exists a gregarious k-cycle decomposition of $K_n * \overline{K}_m$ whenever $n \geqslant k$ and $n \equiv 0 \pmod k$.*

Proof If $n = 2k$ the result follows immediately from Lemma 3.16. If $n = 4k$ it is easy to see that there exists a $\{K_k, K_4 * \overline{K}_k\}$-decomposition of $K_n \cong K_{4k} * \overline{K}_1$. Moreover, there exists a gregarious k-cycle decomposition of $K_k * \overline{K}_m$, by Theorem 3.10, and there exists a gregarious k-cycle decomposition of $K_4 * \overline{K}_{k(m)}$ as follows. Since m is even, there exists a gregarious K_3-decomposition of $K_4 * \overline{K}_m$ by Hanani [5]. Furthermore, there exists a k-cycle decomposition of $K_3 * \overline{K}_k$ by Theorem 3.8. Hence there exists a gregarious k-cycle decomposition of $K_4 * \overline{K}_{k(m)}$, by Lemma 3.3. The result then follows by Lemma 3.2. Assume then $n = 2qk$ for some $q \geqslant 3$.

It is easy to see that there exists a $\{K_{2k}, K_q * \overline{K}_{2k}\}$-decomposition of $K_n \cong K_{2qk} * \overline{K}_1$. Moreover, there exist gregarious k-cycle decompositions of $K_{2k} * \overline{K}_m$, by Theorem 3.16, and of $K_q * \overline{K}_{2k(m)}$, by Corollary 3.9. Hence the result follows by Lemma 3.2. $\square$

We are now ready to present the proof of Theorem 1.1 (i).

Proof of Theorem 1.1 (i) Let n, m and k satisfy the conditions of the theorem. We note first that if n and m are both odd and $n \equiv 1 \pmod k$, then $n \equiv 1 \pmod{2k}$ since $n(n-1)m^2 \equiv 0 \pmod{2k}$. We then split the problem according to the parity of k and whether $n \equiv 0$ or $1 \pmod k$.

Case I. Suppose k is even and $n \equiv 0 \pmod k$. We note that n is even and hence m is also even. The result then follows by Corollary 3.11.

Case II. Suppose k is even and $n \equiv 1 \pmod k$. If m is even the result follows by Corollary 3.13. If m is odd then n is odd and, as noted above, $n \equiv 1 \pmod{2k}$. The result then follows by Corollary 3.7.

Case III. Suppose k is odd and $n \equiv 0 \pmod k$. If n is odd the result follows by Corollary 3.7. Otherwise, both n and m are even and the result follows by Corollary 3.17.

Case IV. Suppose k is odd and $n \equiv 1 \pmod{k}$. If n is odd the result follows by Corollary 3.7. Otherwise, both n and m are even and the result follows by Corollary 3.15.

This completes the proof. $\square$

4 The case $m \equiv 0 \pmod{k}$

The aim of this section is to prove Theorem 1.1 (ii). The critical ingredient in our proof is the following lemma.

Lemma 4.1 *Suppose n and λ are odd with $n \geqslant \lambda \geqslant 3$. Then:*

(i) there exists a λ-cycle decomposition of λK_n with balanced λ-weight function ω; and

(ii) if $\lambda \geqslant 5$, there exists a gregarious λ-cycle decomposition of $2(K_{n+1} * \overline{K}_\lambda)$ which can be partitioned into pairs in such a way that the cycles in each pair share two adjacent edges.

Proof Note that the following λ-cycle decomposition of λK_n was first given by the author in [10].

Let $n = 2t + 1$, $\lambda = 2\mu + 1$, the vertex set of λK_n be $\mathbb{Z}_{2t+1}$ and $\rho = (0\,1\,\cdots\,2t)$ be a permutation of order $2t + 1$ on $V(\lambda K_{2t+1})$. We define the *difference* of an edge uv in λK_{2t+1} to be the unique value $d \in \{1, 2, \ldots, t\}$, such that u and v differ modulo $2t + 1$ by d. Define values $v_1, v_2, \ldots, v_{2t}$ (calculated modulo $2t + 1$ from the residues $0, 1, \ldots, 2t$) by

$$v_i = -v_{t+i} = \begin{cases} i(-1)^{i+1}, & \text{for } 1 \leqslant i \leqslant \lceil t/2 \rceil; \\ i(-1)^i, & \text{for } \lceil t/2 \rceil + 1 \leqslant i \leqslant t. \end{cases}$$

These values satisfy the following properties.

(P1) For each $v \in \mathbb{Z}_{2t+1} \backslash \{0\}$ there is a unique $i \in \{1, 2, \ldots, 2t\}$ such that $v_i = v$.

(P2) For each $d \in \{1, 2, \ldots, t\}$ there is a unique $i \in \{1, 2, \ldots, t\}$ such that v_i and v_{i+1}, as well as v_{t+i} and v_{t+i+1}, differ modulo $2t + 1$ by d.

(P3) For each $d \in \{1, 2, \ldots, t\}$ there is a unique $i \in \{1, 2, \ldots, t\}$ such that v_i and v_{t+i} differ modulo $2t + 1$ by d.

In particular, property (P1) allows us to think of these v_i as alternative labellings for the non-zero vertices in λK_{2t+1}. Then $\mathcal{D} = \{\rho^\alpha(C_i) \mid 0 \leqslant \alpha \leqslant 2t \text{ and } 1 \leqslant i \leqslant t\}$, where

$$C_i = (v_i, v_{i+1}, \ldots, v_{i+\mu-1}, 0, v_{t+i+\mu-1}, v_{t+i+\mu-2}, \ldots, v_{t+i})$$

for each $i \in \{1, 2, \ldots, t\}$ and subscripts of v are calculated modulo $2t$ from the residues $1, 2, \ldots, 2t$, is a λ-cycle decomposition of λK_{2t+1} (see [10] for more detail). We use this decomposition $\mathcal{D}$ to deal with cases (i) and (ii) separately.

(i) For each $i \in \{1,2,\ldots,t\}$ we label the edges of the cycle C_i so that the edge $v_{t+i}v_i$ has label 0, the edge v_iv_{i+1} has label 1, the edge $v_{i+1}v_{i+2}$ has label 2 and so on, up to the edge $v_{t+i+1}v_{t+i}$ which has label $\lambda-1$. It is clear from properties (P1), (P2) and (P3) above, that for each $d \in \{1,2,\ldots,t\}$ and each $\ell \in \{0,1,\ldots,\lambda-1\}$, there is a unique $i \in \{1,2,\ldots,t\}$ for which the cycle C_i contains an edge of difference d with label ℓ. Taking advantage of this fact, we extend this labelling to the edges of each cycle in $\mathcal{D}$ by assigning, for each $\alpha \in \{1,2,\ldots,2t\}$ and each $i \in \{1,2,\ldots,t\}$, the edge $\rho^\alpha(u)\rho^\alpha(v)$ in cycle $\rho^\alpha(C_i)$ the *same* label as the edge uv in cycle $C_i = \rho^0(C_i)$. The resulting edge labelling thus induces a λ-weight function ω on $\mathcal{D}$. It remains to show that ω is balanced.

Let $\vec{K}_{2t+1}$ be the directed version of K_{2t+1} with directed edge-set

$$E = \{(v, v+d) \mid 0 \leqslant v \leqslant 2t \text{ and } 1 \leqslant d \leqslant t\},$$

where $v+d$ is calculated modulo $2t+1$ from the residues $0,1,\ldots,2t$. Hence $\vec{K}_{2t+1}$ is invariant under the permutation ρ and we need only show that, for each $i \in \{1,2,\ldots,t\}$, the sum-weight (with respect to $\vec{K}_{2t+1}$) of the cycle C_i is a multiple of λ. We do this as follows.

Orient the edges of the cycle C_i to form the directed cycle

$$\vec{C_i} = (v_i, v_{i+1}, \ldots, v_{i+\mu-1}, 0, v_{t+i+\mu-1}, v_{t+i+\mu-2}, \ldots, v_{t+i})_D.$$

Recall that for each $\ell \in \{0,1,\ldots,\lambda-1\}$ there is precisely one edge in C_i labelled ℓ. Since the edge labelled 0 contributes nothing to the sum-weight of C_i, we need only show that for each $\ell \in \{1,2,\ldots,\mu\}$ (recall that $\lambda = 2\mu+1$), if e_1 and e_2 are the directed versions (in $\vec{C_i}$) of the edges labelled ℓ and $\lambda-\ell$ in C_i, then either e_1 and e_2 both belong to E, or neither e_1 nor e_2 belongs to E. (Hence their contribution to the sum-weight of C_i will be either λ or $-\lambda$.) Consider first the directed versions (in $\vec{C_i}$) of the edges labelled μ and $\mu+1$ in C_i; that is, the directed edges $(v_{i+\mu-1}, 0)$ and $(0, v_{t+i+\mu-1})$ respectively. By definition, $v_{t+i+\mu-1} \equiv -v_{i+\mu-1} \pmod{2t+1}$. Hence

$$0 - v_{i+\mu-1} \equiv v_{t+i+\mu-1} - 0 \pmod{2t+1},$$

and the result follows by the definition of E. Similarly, for each $\ell \in \{1,2,\ldots,\mu-1\}$, the directed versions (in $\vec{C_i}$) of the edges labelled ℓ and $\lambda-\ell$ in C_i are $(v_{i+\ell-1}, v_{i+\ell})$ and $(v_{t+i+\ell}, v_{t+i+\ell-1})$ respectively. By definition, $v_{t+i+\ell} \equiv -v_{i+\ell} \pmod{2t+1}$ and $v_{t+i+\ell-1} \equiv -v_{i+\ell-1} \pmod{2t+1}$. Hence

$$v_{i+\ell} - v_{i+\ell-1} \equiv v_{t+i+\ell-1} - v_{t+i+\ell} \pmod{2t+1},$$

and the result follows by the definition of E.

(ii) Note that in this case we assume that $\mu \geqslant 2$. For each $i \in \{1,2,\ldots,t\}$ label the edges of the cycle C_i as in the proof of (i) above, and define D_i to be an exact copy of the edge-labelled cycle C_i. Hence $\{\rho^\alpha(C_i), \rho^\alpha(D_i) \mid 0 \leqslant \alpha \leqslant 2t \text{ and } 1 \leqslant i \leqslant t\}$ is a λ-cycle decomposition of $2\lambda K_{2t+1}$. Moreover, for each $d \in \{1,2,\ldots,t\}$ and each

$\ell \in \{0,1,\ldots,\lambda-1\}$, there is a unique $i \in \{1,2,\ldots,t\}$ for which the cycles C_i and D_i each contain a copy of the *same* edge of difference d with label ℓ.

Now, for each $i \in \{1,2,\ldots,t\}$ the cycles $C_{i-\mu+1}$ and $D_{i-\mu+1}$ (with subscripts calculated modulo t from the residues $1,2,\ldots,t$) each contain the 2-path $[v_i, 0, v_{t+i}]$. Moreover, the edges of this 2-path, namely v_i0 and $0v_{t+i}$, each have difference i, and are labelled μ and $\mu+1$ respectively in both $C_{i-\mu+1}$ and $D_{i-\mu+1}$. For each $i \in \{1,2,\ldots,\mu\}$ we "modify" the cycle $C_{i-\mu+1}$ by removing this 2-path, hence forming the $(\lambda-2)$-path

$$L_{i-\mu+1} = [v_{t+i}, v_{t+i-1}, \ldots, v_{t+i-\mu+1}, v_{i-\mu+1}, v_{i-\mu+2}, \ldots, v_i].$$

Thus in total we remove one edge of each difference $d \in \{1,2,\ldots,\mu\}$ labelled μ, and one edge of each difference $d \in \{1,2,\ldots,\mu\}$ labelled $\mu+1$, from the cycles $C_{i-\mu+1}$, $1 \leqslant i \leqslant \mu$. We do the same, this time for each $i \in \{2,3,\ldots,\mu\}$, to the cycle $D_{i-\mu+1}$, leaving the $(\lambda-2)$-path

$$P_{i-\mu+1} = [v_{t+i}, v_{t+i-1}, \ldots, v_{t+i-\mu+1}, v_{i-\mu+1}, v_{i-\mu+2}, \ldots, v_i]$$

in each case. Thus in total we remove one edge of each difference $d \in \{2,\ldots,\mu\}$ labelled μ, and one edge of each difference $d \in \{2,\ldots,\mu\}$ labelled $\mu+1$, from the cycles $D_{i-\mu+1}$, $2 \leqslant i \leqslant \mu$. (Note that the subscripts for both $L_{i-\mu+1}$ and $P_{i-\mu+1}$ are also calculated modulo t from the residues $1,2,\ldots,t$.) We then define two new $(\lambda-2)$-paths G_0 and H_0 on $2\lambda K_{2t+1}$ as follows (the reason for using this notation will soon become clear).

For each $j \in \{1,2,\ldots,\mu\}$ define

$$x_j = \sum_{i=1}^{j} (-1)^{i+1}(\mu - i + 1),$$

and for each $j \in \{1,2,\ldots,\mu-1\}$ define $y_j = (2t+1) - x_j$. Hence $x_1 = \mu$ and, for each $j \in \{1,2,\ldots,\mu-1\}$, the values x_j and x_{j+1} differ by $\mu - j$. Similarly $y_1 = (2t+1) - \mu$ and, if $\mu > 2$, then for each $j \in \{1,2,\ldots,\mu-2\}$ the values y_j and y_{j+1} differ by $\mu - j$. Then define

$$G_0 = H_0 = [x_\mu, x_{\mu-1}, \ldots, x_1, 0, y_1, y_2, \ldots, y_{\mu-1}].$$

It is easy to see this is indeed a $(\lambda-2)$-path in $2\lambda K_{2t+1}$. We assign the edge $x_\mu x_{\mu-1}$ the label μ in G_0 and $\mu+1$ in H_0. We then assign each of the edges

$$x_{\mu-1}x_{\mu-2}, x_{\mu-2}x_{\mu-3}, \ldots, x_1 0$$

the label μ in both G_0 and in H_0; and we assign the remaining edges

$$0y_1, y_1y_2, \ldots, y_{\mu-2}y_{\mu-1}$$

the label $\mu+1$ in both G_0 and in H_0. Hence G_0 contains one edge of each difference $d \in \{1,2,\ldots,\mu\}$ labelled μ, and one edge of each difference $d \in \{2,3,\ldots,\mu\}$ labelled $\mu+1$. Similarly, the path H_0 contains one edge of each difference $d \in \{2,\ldots,\mu\}$ labelled μ, and one edge of each difference $d \in \{1,2,\ldots,\mu\}$ labelled $\mu+1$. Thus together these two paths use edges of the exact same differences *and* labels as those we removed from the cycles $C_{i-\mu+1}$, $1 \leqslant i \leqslant \mu$, and the cycles $D_{i-\mu+1}$, $2 \leqslant i \leqslant \mu$. Moreover, since $\mu > 1$, the paths G_0 and H_0 share at least two adjacent edges which also have the same labels in both G_0 and H_0.

Then for each $i \in \{1, 2, \ldots, t\}$ we let G_i be either the edge-labelled cycle C_i, if C_i was *not* one of the "modified" cycles from above, or the edge-labelled path L_i otherwise. Hence $t - \mu$ of the graphs G_i are λ-cycles, and the remaining μ (plus the graph G_0) are $(\lambda-2)$-paths. Similarly, we let H_i be either the edge-labelled cycle D_i, or the edge-labelled path P_i, depending on whether D_i was one of the "modified" cycles. Hence $t-\mu+1$ of the graphs H_i are λ-cycles, and the remaining $\mu-1$ (plus the graph H_0) are $(\lambda-2)$-paths. We note also that for each $i \in \{0, 1, \ldots, t\}$ the graphs G_i and H_i share at least two adjacent edges which also have the same labels in both G_i and H_i (even in the one case where G_i is a path and H_i is a cycle).

Hence $\{\rho^\alpha(G_i), \rho^\alpha(H_i) \mid 0 \leqslant \alpha \leqslant 2t \text{ and } 0 \leqslant i \leqslant t\}$ is a decomposition of $2\lambda K_{2t+1}$ into

- $(2t+1)(2t-2\mu+1) = n(n-\lambda+1)$ cycles of length λ; and
- $(2t+1)(2\mu+1) = n\lambda$ paths of length $\lambda - 2$.

Moreover, suppose ω' is the λ-weight function on this decomposition obtained by extending the edge labelling of each of the graphs G_i and H_i under the permutation ρ (as in part (i) above). Then, under ω', each cycle in the decomposition has sum-weight a multiple of λ with respect to $\vec{K}_n$ (also defined above). Hence using the function ω' we generate a decomposition, $\mathcal{D}'$ say, of the graph $2(K_n * \overline{K}_\lambda)$ into $\lambda n(n-\lambda+1)$ gregarious λ-cycles and $n\lambda^2$ gregarious $(\lambda-2)$-paths (by associating vertices and labelled edges of $2\lambda K_n$ with partite sets and matchings of $2(K_n * \overline{K}_\lambda)$ as in the proof of Lemma 2.6).

For each $i \in \{0, 1, \ldots, t\}$ we let $X(G_i)$ (respectively, $X(H_i)$) be the set of λn cycles, or paths, in $\mathcal{D}'$ generated by the graphs $\rho^\alpha(G_i)$ (respectively, $\rho^\alpha(H_i)$), $0 \leqslant \alpha \leqslant 2t$. Then, by the remarks above, we can pair up the graphs in $X(G_i)$ with those in $X(H_i)$ so that the graphs in each pair share two adjacent edges. Finally we note that if G_i (respectively, H_i) is a path, then due to the cyclic nature of the decomposition of $2\lambda K_n$, each vertex in $2(K_n * \overline{K}_\lambda)$ appears as an endpoint of exactly two paths in the set $X(G_i)$ (respectively, $X(H_i)$). Since there are precisely $\mu + 1$ graphs G_i which are paths, say $G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_{\mu+1}}$, and μ graphs H_i which are paths, say $H_{\alpha_{\mu+2}}, H_{\alpha_{\mu+3}}, \ldots, H_{\alpha_\lambda}$, we can thus add an extra partite set of size λ to our graph $2(K_n * \overline{K}_\lambda)$, say $\{\infty_1, \infty_2, \ldots, \infty_\lambda\}$, and attach the ends of each path in the sets $X(G_{\alpha_1}), \ldots, X(G_{\alpha_\mu}), X(H_{\alpha_{\mu+1}}), \ldots, X(H_{\alpha_\lambda})$, to the vertices $\infty_1, \infty_2, \ldots, \infty_\lambda$ respectively. Thus these form gregarious λ-cycles in each case. Note that every vertex in the part $\{\infty_1, \infty_2, \ldots, \infty_\lambda\}$ is attached to every vertex in the graph $2(K_n * \overline{K}_\lambda)$ twice. Hence we have a decomposition of the graph $2(K_{n+1} * \overline{K}_\lambda)$ into gregarious λ-cycles. Moreover, since we haven't affected the underlying graphs from our decomposition of $2(K_n * \overline{K}_\lambda)$, we can partition the cycles in this decomposition into pairs in such a way that the cycles in each pair share two adjacent edges. This completes the proof. $\square$

We are now ready to present the main result of this section.

Proof of Theorem 1.1 (ii) Let n, m and k satisfy the conditions of the theorem. We then split the problem according to whether n is odd or even.

Case I. Suppose n is odd. Since n and k are both odd with $n \geqslant k \geqslant 3$, there exists a gregarious k-cycle decomposition of $K_n * \overline{K}_k$ by Lemma 4.1 (i) and Lemma 2.6. The result then follows by Lemma 3.5.

Case II. Suppose n is even. If $k = 3$ the result follows by Hanani [5]. Suppose then $k \geqslant 5$. Since n is even and $n > k \geqslant 5$, there exists a gregarious k-cycle decomposition of $K_n * \overline{K}_{2k}$ by Lemma 4.1 (ii) and Lemma 2.2. Moreover, since n is even, m is also even and hence $m \equiv 0 \pmod{2k}$. The result then follows by Lemma 3.5. $\square$

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