Automorphism groups of wreath product digraphs

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Abstract

We generalize a classical result of Sabidussi that was improved by Hemminger, to the case of directed color graphs. The original results give a necessary and sufficient condition on two graphs, C and D, for the automorphsim group of the wreath product of the graphs, $\mathrm{Aut}(C \wr D)$ to be the wreath product of the automorphism groups $\mathrm{Aut}(C) \wr \mathrm{Aut}(D)$. Their characterization generalizes directly to the case of color graphs, but we show that there are additional exceptional cases in which either C or D is an infinite directed graph. Also, we determine what $\mathrm{Aut}(C \wr D)$ is if $\mathrm{Aut}(C \wr D) \neq \mathrm{Aut}(C) \wr \mathrm{Aut}(D)$, and in particular, show that in this case there exist vertex-transitive graphs C' and D' such that $C' \wr D' = C \wr D$ and $\mathrm{Aut}(C \wr D) = \mathrm{Aut}(C') \wr \mathrm{Aut}(D')$.

1 Introduction

The main purpose of this paper is to revisit a well-known and important result of Sabidussi [10] giving a necessary and sufficient condition for the wreath product $C \wr D$ (defined below) of two graphs C and D to have automorphism group $\mathrm{Aut}(C) \wr \mathrm{Aut}(D)$, the wreath product of the automorphism group of C and the automorphism group of D (defined

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below). Sabidussi originally considered only almost locally finite graphs X and finite graphs Y. (A graph is almost locally finite if the set of vertices of infinite degree is finite.) The condition that X be almost locally finite is needed for Sabidussi's proof, but is clearly not needed in general. Indeed, note that $(X \wr Y)^c$, the complement of $X \wr Y$, has the same automorphism group as $X \wr Y$, $(X \wr Y)^c = X^c \wr Y^c$, but X^c is not almost locally finite if X is infinite and almost locally finite. Sabidussi improved his own result by weakening the conditions, in a later paper [11]. Then Hemminger [6, 7, 8] fully generalized Sabidussi's result, reaching a complete characterization of wreath product graphs with no "unnatural" automorphisms (to use his terminology).

Since Sabidussi and Hemminger published their papers, the wreath products of digraphs and color digraphs have also been considered in various contexts. It is therefore of interest to explore how their results do or do not generalize to such structures.

We will show that Hemminger's characterization can be directly generalized to any color graphs, and that all of the results in [6, 7, 8, 10] and [11] hold for *finite* digraphs, although their proofs do not suffice to show this. Furthermore, in the case where either C or D is an infinite digraph (with or without colors), we will show that additional conditions are necessary to ensure that there are no unnatural automorphisms. We will explore some of the exceptional digraphs, and prove a theorem that provides conditions under which Hemminger's characterization extends to color digraphs. We will also give several corollaries, covering special cases that are of interest, in which some of the terminology required in the complete characterization can be simplified or omitted.

One such corollary consists of the case where C and D are both finite, vertex-transitive graphs (a common context in which Sabidussi's result is applied). Here we will further show that if C and D are not both complete or both edgeless, then there exist vertex-transitive graphs C' and D' such that $C \wr D = C' \wr D'$ and $\mathrm{Aut}(C \wr D) = \mathrm{Aut}(C') \wr \mathrm{Aut}(D')$.

Finally, the wreath product of Cayley graphs arises naturally in the study of the Cayley Isomorphism problem (definitions are provided in the final section, where this work appears). We show that if C and D are CI-graphs of abelian groups G_1 and G_2, respectively, then $C \wr D$ need not be a CI-graph of $G_1 \times G_2$, and then give a necessary condition on G_1 and G_2 that will ensure that $C \wr D$ is a CI-graph of $G_1 \times G_2$.

In order to simplify awkward terminology, throughout this paper we will refer to substructures of color digraphs simply as "subdigraphs" rather than "color subdigraphs." The reader should understand that the color structure is carried through into the substructure, even though we do not explicitly include the word.

Although most of our definitions and terminology will be included in the next section, we distinguish the definitions of wreath products, both of (color) (di)graphs and of permutation groups.

We begin by defining a more general structure than the wreath product. It was actually the automorphism group of this structure, the C-join of a set of graphs, that Hemminger analyzed in the paper [8], although he confined his analysis to graphs rather than color digraphs.

Definition 1.1 Let C be a color digraph, and let $\{D_c : c \in V(C)\}$ be a collection of color digraphs. The C-join of these color digraphs is the color digraph whose vertex set is the union of all of the vertices in the collection. There is an arc of color k from d_c to $d'_{c'}$, where d_c is a vertex of D_c, $d'_{c'}$ is a vertex of $D_{c'}$ and c, c' are vertices of C, if either of the following holds:

- $c = c'$ and there is an arc of color k from d_c to $d'_{c'}$ in $D_c (= D_{c'})$; or
- there is an arc of color k from c to c' in C.

Another way of describing this structure is that each vertex c of C is replaced by a copy of D_c, and we include all possible arcs of color k from D_c to $D_{c'}$, if and only if there is an arc of color k from c to c' in C.

Now we can define the wreath product.

Definition 1.2 The *wreath product* of two color digraphs C and D is the C-join of $\{D_c : c \in V(C)\}$, where $D_c \cong D$ for every $c \in V(C)$. We denote the wreath product of C and D by $C \wr D$.

It is important to note that we are following the French tradition of denoting the wreath product of both graphs and groups in this paper, consistent with work of Sabidussi, Alspach, and others. There is another school of work, according to whose notation the order of the graphs making up the wreath product is reversed; that is, the graph that we have denoted $C \wr D$, is denoted by $D \wr C$. They also reverse the notation we use for the wreath product of permutation groups (defined below). This is true in work by Praeger, Li and others, and is an unfortunate potential source of confusion.

The name wreath product was chosen because of the close connection (mentioned earlier) to the wreath product of automorphism groups. In Sabidussi's original paper [10], this product is called the "composition" of graphs.

Definition 1.3 Let Γ and Γ' be permutation groups acting on the sets Ω and Ω', respectively. The *wreath product* of Γ with Γ', denoted $\Gamma \wr \Gamma'$, is defined as follows. It is the group of all permutations δ acting on $\Omega \times \Omega'$ for which there exist $\gamma \in \Gamma$ and an element γ'_x of Γ' for each $x \in \Omega$, such that

$$\delta(x, y) = (\gamma(x), \gamma'_x(y)) \text{ for every } (x, y) \in \Omega \times \Omega'.$$

Wreath products can be defined in general for abstract groups, but we will only be considering the special case of permutation groups, so we confine ourselves to this simpler definition.

It is always the case that $\mathrm{Aut}(C) \wr \mathrm{Aut}(D) \leq \mathrm{Aut}(C \wr D)$, for color digraphs C and D. This is mentioned as an observation in [10], for example, in the case of graphs, and color digraphs are equally straightforward.

In fact, it is very often the case that $\mathrm{Aut}(C) \wr \mathrm{Aut}(D) = \mathrm{Aut}(C \wr D)$. Harary claimed that this was always the case in [5], but this was corrected by Sabidussi in [10], who provided a characterization for precisely when $\mathrm{Aut}(C) \wr \mathrm{Aut}(D) = \mathrm{Aut}(C \wr D)$, where C is an almost locally finite graph and D is a finite graph. Hemminger was able to remove all conditions on C and D [6, 7, 8].

Section 2 of this paper will provide background definitions and notation. Section 3 will state Hemminger's result, as well as stating and proving our generalization. Section 4 will provide some useful corollaries and elaborate on one of the conditions required in our result. Section 5 will use results from the third section to consider the question of what the structure of $\mathrm{Aut}(C \wr D)$ can be, if it is not $\mathrm{Aut}(C) \wr \mathrm{Aut}(D)$, and will give the result mentioned previously, on vertex-transitive graphs.

The final section will produce the results that relate to the Cayley Isomorphism problem for graphs that are wreath products of Cayley graphs on abelian groups.

2 Definitions and terminology

Before we can state Hemminger's characterisation of when $\mathrm{Aut}(C \wr D) = \mathrm{Aut}(C) \wr \mathrm{Aut}(D)$, or state and prove our generalization, we need to introduce some additional notation and terminology.

In what follows, everything is stated in terms of color digraphs; color graphs can be modelled as color digraphs by replacing each edge of color k by a digon (arcs in both directions) of color k, for every color k, so all of the definitions and results also hold for color graphs.

Suppose that a color digraph X has arcs of r colors, 1 through r. Whenever we consider color digraphs in this paper, we assume that all non-arcs have been replaced by arcs of a new color, color 0. This serves to simplify our notation and some aspects of the proofs. Thus, for any pair of distinct vertices x and y in a color digraph, there will be an arc of color k from x to y for some $0 \le k \le r$.

Definition 2.1 In any color digraph X, we say that the induced subdigraph on the set S of vertices of X is *externally related in* X, if whenever $x, y \in S$ and $v \in V(X) \setminus S$, the arc from v to x has the same color as the arc from v to y, and the arc from x to v has the same color as the arc from y to v.

That is, an externally related subdigraph in X is an induced subdigraph whose vertices have exactly the same in-neighbours and out-neighbours of every color, within the set $V(X) \setminus S$.

The above definition is given in the context of graphs, in Hemminger's paper [8]. We now define a related concept.

Definition 2.2 In any color digraph X, we say that the vertices x and y are *k-twins*, if $x \neq y$ and the following two conditions hold:

1. there are arcs of color k from x to y, and from y to x; and
2. the subdigraph induced on $\{x, y\}$ is externally related in X.

That is, k-twins are a pair of vertices that are mutually adjacent via two arcs of color k, and that, with the exception of this mutual adjacency, have exactly the same in-neighbours and out-neighbours of every color.

It is straightforward to verify that dropping the requirement that k-twins be distinct yields an equivalence relation that partitions the vertices of any digraph into equivalence classes. We call these equivalence classes *externally related k-classes* of vertices. Notice that the induced digraph on any subset of such a class is itself externally related; we call such subdigraphs *externally related k-cliques.*

Definition 2.3 For any color k, we say that the *k-complement of X is disconnected* if, upon removing all arcs of color k, the underlying graph is disconnected.

That is, the k-complement of X is disconnected if X has a pair of vertices x and y, for which every path between x and y in the underlying graph of X must use an edge of color k.

Notice that saying that the 0-complement of X is disconnected is equivalent to saying that X is disconnected.

Notation 2.4 For any wreath product $C \wr D$ of color digraphs C and D, and any vertex x of C, we use D_x to denote the copy of D in $C \wr D$ that corresponds to the vertex x of C.

To simplify things somewhat, it will be convenient to have a term for automorphisms that fail to behave as we would wish. We therefore adopt the following definition from Hemminger's paper [8].

Definition 2.5 Let $C \wr D$ be a wreath product of color digraphs C and D, and let μ be an automorphism of $C \wr D$. We say that μ is *natural* if for any vertex x of C, there is a vertex y of C (not necessarily distinct) such that $\mu(D_x) = D_y$. If this is not the case, then μ is *unnatural.*

We need some further terminology from Hemminger's work before we can state his result.

Definition 2.6 Let M be a partition of the vertices of C, such that for every $A \in M$, the induced subdigraph on A is externally related in C. We define the color digraph C_M to be the color digraph whose vertices are elements of M, with an arc of color k from A to B if and only if there is an arc of color k in C, from some vertex $x \in A$ to some vertex $y \in B$ (where $A, B \in M$).

Note that due to the subdigraphs on A and B being externally related, we could equivalently have required an arc of color k from every vertex in A to every vertex in B.

Definition 2.7 Let C and C' be color digraphs and σ a map from $V(C)$ to $V(C')$. Then σ is a *smorphism* if whenever $x, y \in V(C)$ with $\sigma(x) \neq \sigma(y)$, there is an arc of color k from $\sigma(x)$ to $\sigma(y)$ in C' if and only if there is an arc of color k from x to y in C.

In other words, σ preserves the colors of arcs between any vertices that have distinct images.

Definition 2.8 Suppose that $X = C \wr D$, where both C and D have more than one vertex, and there is some vertex $x \in V(C)$ for which the induced subdigraph on $V(C) \setminus \{x\}$ is externally related, and D is isomorphic to the C-join of $\{D'_y : y \in V(C)\}$, where $D'_y \cong D$ for every $y \neq x$, and D'_x is arbitrary. Then we call x an *inverting C-point of X*.

Although this definition comes from Hemminger's work, it is a sufficiently odd structure to warrant some further discussion. Clearly, since D'_y is a proper subgraph of D, yet is isomorphic to D, we are considering only infinite structures here. Perhaps the simplest example of a graph that has this structure can be given as follows. Let C be any color digraph that has a vertex x for which the induced subdigraph on $V(C) \setminus \{x\}$ is externally related, and let D be the countably infinite wreath product $C \wr C \wr C \wr \ldots$. Then $X = C \wr D = C \wr C \wr C \wr \ldots \cong D$, and x is an inverting C-point of X. The reason this structure is important (and the reason for its name), is that there is a natural automorphism of X that does something quite unusual. Since $X = C \wr D$, we have copies D_x and $\{D_y : y \in V(C) \setminus \{x\}\}$ of D. Now, D_x is isomorphic to the C-join of $\{D'_y : y \in V(C)\}$, so there is a natural automorphism of X that fixes this D'_x pointwise, while swapping this D'_y with D_y for every vertex y of C with $y \neq x$. The fact that $V(C) \setminus \{x\}$ is externally related guarantees that this is an automorphism. Essentially, this automorphism turns things inside-out (or inverts), by swapping the copies of D that are inside of D_x with those that are outside, leaving only D'_x fixed.

The above definitions and notation are sufficient to allow us to state Hemminger's result. To be able to give our generalization, we need to describe and name a family of digraphs.

Definition 2.9 Let S be any set, with $|S| > 1$ and a total order $<$ defined on its elements. Let the color digraph G be the digraph whose vertices are the elements of S, with an arc of color k from s_i to s_j, and an arc of color k' from s_j to s_i, if and only if $s_i < s_j$, where $k \neq k'$. Then a (k, k') *total order digraph* is any color digraph that can be formed as the G-join of any collection of color digraphs.

We distinguish some special cases of this definition.

Notation 2.10 If $S = \mathbb{Z}$ under the usual total order, we denote the corresponding graph G itself (from Definition 2.9) by Z.

Similarly, if $S = \{1, 2, \ldots, i\}$ under the usual total order, we denote the corresponding graph G by Z_i.

Notice that these are (k, k') total order digraphs, since each is the trivial join of itself with a collection of single vertices.

Definition 2.11 A (k, k') total order digraph G' is *separable* if there is some digraph G, such that $G \wr G'$ has an unnatural automorphism.

The above definition is somewhat unsatisfying, as will be discussed in greater detail later.

As we frequently pass back and forth between referring to a set of vertices in a color digraph, and the subdigraph that they induce, we also need notation for this.

Notation 2.12 If $A \subseteq V(C)$, we let $\bar{A}$ denote the induced subdigraph of C on the vertices of A.

Finally,

Notation 2.13 We denote the color digraph on a single vertex by K_1.

This concludes our background material.

3 Generalizing Sabidussi and Hemminger's Results

We are using some of the generalized terminology that applies to color digraphs to state Hemminger's theorem, but in the situation of graphs (the context in which he proved his result), the colors available are 0 and 1, corresponding to non-edges and edges (respectively).

Theorem 3.1 (Hemminger) *For graphs C and D,* $\mathrm{Aut}(C \wr D) \cong \mathrm{Aut}(C) \wr \mathrm{Aut}(D)$ *if and only if:*

1. *if C has a pair of k-twins, then the k-complement of D is connected, where $k \in \{0,1\}$;*
2. *if*
   - *M is a proper partition of $V(C)$;*
   - *the subgraphs induced by the elements of M are externally related in C; and*
   - *$\sigma : C \to C_M$ is an onto smorphism such that $\bar{A} \wr D \cong \overline{\sigma^{-1}(A)} \wr D$ for all $A \in M$,*

   then all such isomorphisms are natural ones;
3. *if A is an externally related subgraph of C then $A \wr D$ does not have an inverting A-point.*

Our theorem requires some additional conditions on C and D since arcs are permitted.

Theorem 3.2 *Suppose that C and D are color digraphs, with the conditions that*

- i. *if C has an externally related subdigraph that is isomorphic to the Z-join of a collection of color digraphs Y_z, where $Y_i \ncong K_1$ implies $Y_{i-1} \cong Y_{i+1} \cong K_1$ for every integer i, then D is not a separable (k, k') total order digraph; furthermore,*
- ii. *if C has an externally related subdigraph that is isomorphic to the Z_3-join of color digraphs Y_1, Y_2 and Y_3, where $Y_1 \cong Y_3 \cong K_1$ and Y_2 is arbitrary, then D is not a (k, k') total order digraph that is isomorphic to a proper subdigraph of itself.*

Then we have $\mathrm{Aut}(C \wr D) \cong \mathrm{Aut}(C) \wr \mathrm{Aut}(D)$ *if and only if:*

1. *for every $k \in \{0, 1, \ldots, r\}$, if C has a pair of k-twins, then the k-complement of D is connected;*

2. *if*
   - *M is a proper partition of $V(C)$;*
   - *the subdigraphs induced by the elements of M are externally related in C; and*
   - *$\sigma : C \to C_M$ is an onto smorphism such that $\bar{A} \wr D \cong \overline{\sigma^{-1}(A)} \wr D$ for all $A \in M$,*

   then all such isomorphisms are natural ones;
3. *if A is an externally related subdigraph of C then $A \wr D$ does not have an inverting A-point.*

Some additional notation will be useful throughout this section.

Notation 3.3 Consider $C \wr D$ where C and D are color digraphs, and let μ be a predetermined automorphism of $C \wr D$. For each $x \in V(C)$, we use B_x to denote the induced subdigraph of C on $\{y \in V(C) : V(D_y) \cap V(\mu(D_x)) \neq \emptyset\}$. Also, for each $x, y \in V(C)$, define $U_{x,y} = V(\mu(D_x)) \cap V(D_y)$.

Thus $U_{x,y} = \emptyset$ if and only if $y \notin V(B_x)$. If $|V(B_x)| = 1$ for every $x \in V(C)$ holds for every automorphism, then there is no unnatural automorphism. Although it will not be stated explicitly, wherever we assume the existence of an unnatural automorphism, μ, in the results that follow, we will choose μ so that there is some x for which $|V(B_x)| > 1$.

Notation 3.4 With a fixed μ, let $T = \{x \in V(C) : |V(B_x)| > 1\}$.

We begin with a useful lemma.

Lemma 3.5 *Let $X = C \wr D$ be a color digraph with a fixed automorphism, μ. Let v, w, x and y be vertices of C such that:*

- *$v \neq x, y$;*
- *$x, y \in V(B_w)$; and*
- *$U_{w,v} \neq V(D_v)$.*

Then in C, the arc from x to v has the same color as the arc from y to v, and the arc from v to x has the same color as the arc from v to y.

Proof. Since $U_{w,v} \neq V(D_v)$, there is some vertex u of C such that $U_{u,v} \neq \emptyset$. Since X is a wreath product, all arcs from D_w to D_u have the same color (the color of the arc from w to u in C), so all arcs from $\mu(D_w)$ to $\mu(D_u)$ have this same color. In particular, the arcs from vertices in $U_{w,x}$ to vertices in $U_{u,v}$ all have this same color, as do the arcs from vertices in $U_{w,y}$ to vertices in $U_{u,v}$. Again since X is a wreath product, this must be the color of both the arc in C from x to v and the arc in C from y to v, which are therefore the same. Considering the reverse arcs instead of those we have examined, completes the argument. □

The following is an immediate consequence:

Corollary 3.6 *Let $C \wr D$ be a color digraph with a fixed automorphism, μ. Then for any vertex $w \in V(C)$, the subdigraph B_w is externally related in C.*

The next lemma will also be required. As it will be used in a slightly different context in a later section of the paper, we state it in greater generality than is needed for the current context.

Lemma 3.7 *Suppose that C, D, C' and D' are color digraphs, and that $C \wr D = C' \wr D'$. For every vertex v of C, every vertex w of C' for which $V(D'_w) \not\subseteq V(D_v)$, and every vertex $x \neq v$ of C for which $V(D_x) \cap V(D'_w) \neq \emptyset$, we conclude that*

- *there is some color k for which every arc from any vertex of $V(D_v) \setminus V(D'_w)$ to any vertex of $V(D_v) \cap V(D'_w)$ has color k. Furthermore, this color k is the color of the arcs from D_v to D_x.*
- *Similarly, there is some color k' for which every arc to any vertex of $V(D_v) \setminus V(D'_w)$ from any vertex of $V(D_v) \cap V(D'_w)$ has color k'. Furthermore, k' is the color of the arcs from D_x to D_v.*

Proof. First, if $V(D_v) \cap V(D'_w)$ is either $\emptyset$ or $V(D_v)$, the result is vacuously true. So we may assume that there is some vertex y of C' such that $y \neq w$ and $V(D_v) \cap V(D'_w) \neq \emptyset$ and $V(D_v) \cap V(D'_y) \neq \emptyset$.

Since $V(D'_w) \not\subseteq V(D_v)$, we must have some vertex x of C for which $x \neq v$ and $V(D_x) \cap V(D'_w) \neq \emptyset$. Let k be the color of the arcs from D_v to D_x.

Now, since $V(D_v) \cap V(D'_y) \neq \emptyset$ and $V(D_x) \cap V(D'_w) \neq \emptyset$, we must have that all arcs from D'_y to D'_w have color k. But this is true for any y for which $V(D_v) \cap V(D'_y) \neq \emptyset$. So in fact, all arcs from $V(D_v) \setminus V(D'_w)$ to $V(D'_w)$, and therefore in particular, to $V(D_v) \cap V(D'_w)$, have color k.

Reversing the direction of each arc and replacing k with k' in the argument above, completes the proof of the lemma. □

For simplicity of use in this section, we re-write the lemma above in terms of an automorphism, μ. Simply replace each D'_a in the statement and proof of the lemma, by $\mu(D_a)$ to achieve the following result.

Corollary 3.8 *Suppose that C and D are color digraphs, with a fixed automorphism μ of $C \wr D$. For every vertex v of C, every vertex w of C for which $U_{w,v} \neq V(\mu(D_w))$, and every vertex $x \neq v$ of C for which $x \in B_w$, we conclude that*

- *there is some color k for which every arc from any vertex of $V(D_v) \setminus U_{w,v}$ to any vertex of $U_{w,v}$ has color k. Furthermore, this color k is the color of the arcs from D_v to D_x.*
- *Similarly, there is some color k' for which every arc to any vertex of $V(D_v) \setminus U_{w,v}$ from any vertex of $U_{w,v}$ has color k'. Furthermore, k' is the color of the arcs from D_x to D_v.*

Our next lemma explores the circumstances under which the configuration forbidden by condition (i) of Theorem 3.2 can arise.

Lemma 3.9 *Suppose that C and D are color digraphs and conditions (ii) and (1) of Theorem 3.2 hold, but $C \wr D$ has an unnatural automorphism, μ. Given this μ, $T = \{x \in V(C) : |V(B_x)| > 1\}$. Then either*

- *for any $w \in T$, there is at most one $x \in V(B_w)$ such that $U_{w,x} \neq V(D_x)$, or*
- *whenever there is some $w_0 \in T$, with $x_0, x_1 \in V(B_{w_0})$ and $x_0 \neq x_1$, and the arcs between x_0 and x_1 in C have two distinct colors, k and k', we can choose $\{w_i : i \in \mathbb{Z}\}$ so that the induced subdigraph of C on the vertices of $\bigcup_{i\in\mathbb{Z}} V(B_{w_i})$ is an externally related (k, k') total order digraph that is isomorphic to the Z-join of a collection of color digraphs Y_z, where $Y_i \ncong K_1$ implies $Y_{i-1} \cong K_1$ and $Y_{i+1} \cong K_1$.*

Proof. We assume that the first of the conclusions given in the lemma does not hold, and deduce the second. First we show that without loss of generality we can choose w_0, x_0 and x_1 so that $U_{w_0,x_0} \neq V(D_{x_0})$, and $U_{w_0,x_1} \neq V(D_{x_1})$. Because we are assuming that the first conclusion does not hold, the only other possibility is that whenever $w \in T$ with $x, y \in V(B_w)$, $x \neq y$, $U_{w,x} \neq V(D_x)$ and $U_{w,y} \neq V(D_y)$, then the arcs between x and y both have the same color, k (say). By Corollary 3.6, B_w is externally related. Furthermore, two applications of Lemma 3.5 to the vertices of B_w, with first x and then y taking the role of v, yield the conclusion that B_w is an externally related k-clique. In particular, x and y are k-twins. Furthermore, calling on Corollary 3.8, since $x, y \in V(B_w)$ and the arcs between them in both directions have color k, we conclude that the k-complement of D_y (and therefore of D) is disconnected. But this contradicts condition (1) of Theorem 3.2.

There are five significant steps in the remainder of this proof:

1. showing that there is a set $\{w_i : i \in \mathbb{Z}\}$ such that for every i, $V(B_{w_i}) \cap V(B_{w_{i+1}}) \neq \emptyset$;
2. showing that if $i < j$ and $x_i \neq x_j$, then the arc from x_i to x_j has color k and the arc from x_j to x_i has color k', where $x_i \in V(B_{w_{i-1}}) \cap V(B_{w_i})$;
3. showing that if $i \neq j$ then $w_i \neq w_j$ and $x_i \neq x_j$;
4. showing that the induced subdigraph of C on $\bigcup_{i\in\mathbb{Z}} V(B_{w_i})$ is externally related; and
5. showing that the induced subdigraph of C on the vertices of $\bigcup_{i\in\mathbb{Z}} V(B_{w_i})$ is a (k, k') total order digraph that is isomorphic to the Z-join of a collection of color digraphs Y_z, where $Y_i \ncong K_1$ implies $Y_{i-1} \cong K_1$ and $Y_{i+1} \cong K_1$.

As some of these steps require lengthy arguments, separating the proof into these five steps will make it easier to read.

Step 1: showing that there is a set $\{w_i : i \in \mathbb{Z}\}$ such that for every i, $V(B_{w_i}) \cap V(B_{w_{i+1}}) \neq \emptyset$.

We begin to form a chain forwards and backwards from $\mu(D_{w_0})$ for as long as possible, such that for each x_i and x_{i+1} in the chain (where i is an integer), $x_i, x_{i+1} \in V(B_{w_i})$, $x_i \neq x_{i+1}$, and $w_i \neq w_{i+1}$. Notice that the given w_0, x_0 and x_1 satisfy these conditions.

For as long as this chain continues, we will certainly have $V(B_{w_i}) \cap V(B_{w_{i+1}}) \neq \emptyset$, since $x_{i+1} \in V(B_{w_i}) \cap V(B_{w_{i+1}})$.

In order to complete this first section of our proof, we need to show that the chain is infinite (in both directions). Suppose to the contrary that it comes to an end. Going forward, it can only end if $U_{w_i,x_{i+1}} = V(D_{x_{i+1}})$ or $V(\mu(D_{w_i}))$; similarly, going backwards, it can only end if $U_{w_i,x_i} = V(D_{x_i})$ or $V(\mu(D_{w_i}))$. In any of these events, we have that D is isomorphic to a proper subdigraph of itself. We know that B_{w_0} is externally related (by Corollary 3.6). We claim that B_{w_0} is isomorphic to the Z_3-join of the induced subdigraphs of C on x_0, $\{v \in V(B_{w_0}) : v \neq x_0, x_1\}$, x_1 (in that order). To prove this, we need only show that for any vertex $y \in V(B_{w_0})$ with $y \neq x_0, x_1$, we have arcs of color k from x_0 to y and from y to x_1, and arcs of color k' from x_1 to y and from y to x_0. But letting w_0 take the role of w and x_0 take the role of v in Corollary 3.8, and recognising that the colors k and k' are uniquely determined by arcs between U_{w_0,x_0} and $V(D_{x_0}) \setminus U_{w_0,x_0}$ (recall that we have chosen w_0 and x_0 so that both of these sets are nonempty), we see that the lemma tells us that the arcs from D_{x_0} to D_{x_1} have the same color as the arcs from D_{x_0} to D_y, and the arcs from D_{x_1} to D_{x_0} have the same color as the arcs from D_y to D_{x_0}. Similarly, letting x_1 take the role of v in Corollary 3.8 and recalling that U_{w_0,x_1} and $V(D_{x_1}) \setminus U_{w_0,x_1}$ are nonempty, we conclude that the arcs from D_{x_0} to D_{x_1} have the same color as the arcs from D_y to D_{x_1}, and the arcs from D_{x_1} to D_{x_0} have the same color as the arcs from D_{x_1} to D_y. This proves our claim. Finally, notice that either application of Corollary 3.8 allowed us to conclude that D (specifically, D_{x_0} or D_{x_1}) is a (k, k') total order digraph. Now we have a contradiction to condition (ii) of Theorem 3.2.

So we have such a collection.

Step 2: showing that if $i < j$ and $x_i \neq x_j$, then the arc from x_i to x_j has color k and the arc from x_j to x_i has color k', where $x_i \in V(B_{w_{i-1}}) \cap V(B_{w_i})$.

Inductively, assume that all arcs from D_{x_i} to $D_{x_{i+1}}$ have color k (the base case of this, with $i = 0$, has been established in Step 1, and this makes sense, since $x_i \neq x_{i+1}$). Then since $x_i \in V(B_{w_i})$, and $x_{i+1} \in V(B_{w_{i+1}})$, all arcs from $\mu(D_{w_i})$ to $\mu(D_{w_{i+1}})$ must have color k (since $w_i \neq w_{i+1}$). Now, since $x_{i+1} \in V(B_{w_i})$, and $x_{i+2} \in V(B_{w_{i+1}})$, all arcs from $D_{x_{i+1}}$ to $D_{x_{i+2}}$ must have color k. This establishes that all arcs from D_{x_i} to $D_{x_{i+1}}$ have color k for any $i \geq 0$. Similarly, working backwards from our base case of $i = 0$, since $x_i \in V(B_{w_{i-1}})$ and $x_{i+1} \in V(B_{w_i})$, all arcs from $\mu(D_{w_{i-1}})$ to $\mu(D_{w_i})$ must have color k, and since $x_{i-1} \in V(B_{w_{i-1}})$ and $x_i \in V(B_{w_i})$, all arcs from $D_{x_{i-1}}$ to D_{x_i} must have color k. This establishes that all arcs from D_{x_i} to $D_{x_{i+1}}$ have color k for any integer i, which will form the base case for our next induction.

Fix i, let $j \geq i$, and inductively suppose that either all arcs from D_{x_i} to D_{x_j} have color k, or $x_j = x_i$. If $x_j = x_i$ then since all arcs from D_{x_j} to $D_{x_{j+1}}$ have color k (by our last inductive argument), so do all arcs from D_{x_i} to $D_{x_{j+1}}$, completing the induction. Otherwise, since $x_i \in V(B_{w_i})$, and $x_j \in V(B_{w_j})$, all arcs from $\mu(D_{w_i})$ to $\mu(D_{w_j})$ must have color k if $w_i \neq w_j$. And since $x_i \in V(B_{w_i})$ and $x_{j+1} \in V(B_{w_j})$, all arcs from D_{x_i} to $D_{x_{j+1}}$ must have color k. On the other hand, if $w_i = w_j$, then $w_i \neq w_{j+1}$, but $x_j \in V(B_{w_i}) = V(B_{w_j})$ and $x_{j+1} \in V(B_{w_{j+1}})$, so since all arcs from D_{x_j} to $D_{x_{j+1}}$ have color k (by our last inductive argument), so must all arcs from $\mu(D_{w_i})$ to $\mu(D_{w_{j+1}})$. Since $x_i \in V(B_{w_i})$ also, all arcs from D_{x_i} to $D_{x_{j+1}}$ must also have color k. This completes the proof that all arcs from D_{x_i} to D_{x_j} have color k whenever $j > i$, if $x_j \neq x_i$. Reversing the direction of the arcs and replacing k by k' throughout the two inductive arguments that we have just concluded, will prove that all arcs from D_{x_j} to D_{x_i} have color k' whenever $j > i$, if $x_j \neq x_i$.

Step 3: showing that if $i \neq j$, then $w_i \neq w_j$ and $x_i \neq x_j$.

Suppose that there is some x_i such that $x_i = x_j$ for some $j \neq i$. Without loss of generality, assume $j > i$. Then since $x_i \neq x_{i+1}$, we must have $j \geq i+2 > i+1$. Now by Step 2 of this proof, since $i+1 > i$ and $x_i \neq x_{i+1}$, all arcs from D_{x_i} to $D_{x_{i+1}}$ must have color k. However, since $j > i+1$ and $x_i = x_j \neq x_{i+1}$, all arcs from D_{x_j} to $D_{x_{i+1}}$ must have color k'. But $x_i = x_j$, so we can only conclude that $k' = k$, contradicting an assumption.

Now, if $w_i = w_j$ but $j > i$, then we could have chosen $x_{j+1} = x_{i+1}$ (before the choice of w_{j+1} is made), but we have just seen that this is not possible.

Step 4: showing that the induced subdigraph of C on the vertices of $\bigcup_{i\in\mathbb{Z}} V(B_{w_i})$ is externally related.

Let y be any vertex of C for which $y \notin V(B_{w_i})$ for any integer i. For some fixed i, suppose the arc from y to x_i has color ℓ, and let $v \in V(C)$ be such that $y \in V(B_v)$. Since $x_i \in V(B_{w_i}) \cap V(B_{w_{i-1}})$, we must have all arcs from $\mu(D_v)$ to both $\mu(D_{w_i})$ and $\mu(D_{w_{i-1}})$ have color ℓ. Hence the arcs from y to both x_{i-1} and x_{i+1} have color ℓ. Proceeding inductively, we see that all arcs from any vertex of $\mu(D_v)$ to any vertex of $\mu(D_{w_j})$ has color ℓ, for any j, and therefore all arcs from y to any vertex of B_{w_i} has color ℓ, for any i. Reversing the arcs in this argument, we can also prove that the arcs to y from any vertex of $\bigcup_{i\in\mathbb{Z}} V(B_{w_i})$ all have the same color, completing this section of our proof.

Step 5: showing that the induced subdigraph of C on the vertices of $\bigcup_{i\in\mathbb{Z}} V(B_{w_i})$ is a (k,k') total order digraph that is isomorphic to the Z-join of a collection of color digraphs Y_z, where $Y_i \neq K_1$ implies $Y_{i-1} = K_1$ and $Y_{i+1} = K_1$.

All of the vertices x_i (where i is an integer) are distinct, and $k' \neq k$. We claim that if $x \in V(B_{w_i})$ for some i, then the arcs from x_i to x and from x to x_{i+1} in C have color k, and the reverse arcs have color k'. This is a direct consequence of applying Corollary 3.8, recognising that the colors k and k' are uniquely determined by the arcs between U_{w_i,x_i} and $V(D_{x_i}) \setminus U_{w_i,x_i}$. Let G be the induced subdigraph of C on the vertices of $\bigcup_{i\in\mathbb{Z}} V(B_{w_i})$. It is now straightforward to see that G can be formed as the Z-join of a collection of color digraphs Y_z, where each Y_z is either a single vertex (corresponding to each x_i), or the induced subdigraph of C on the vertices of $V(B_{w_i}) \setminus \{x_i, x_{i+1}\}$. This produces the additional structure that if $Y_z \ncong K_1$, then Y_{z-1} and Y_{z+1} must come from some x_i and x_{i+1}, and are therefore single vertices. □

The main focus in our proof of Theorem 3.2, will be to come up with a contradiction to condition (2) of that theorem, if the other conditions are assumed to hold. The partition of $V(C)$ that we construct will be based on the sets $V(B_w)$ where $w \in V(C)$. It will therefore be critical to know that if $v, w \in T$, there is no $x \in V(C)$ for which $x \in V(B_v)$ and $x \in V(B_w)$. The following lemma together with its corollary provide this assurance.

Lemma 3.10 *Suppose that C and D are color digraphs and conditions (i), (ii), and (1) of Theorem 3.2 hold, but $C \wr D$ has an unnatural automorphism, μ. Given this μ, $T = \{x \in V(C) : |V(B_x)| > 1\}$. Then for any $w \in T$, there is at most one $x \in V(B_w)$ such that $U_{w,x} \neq V(D_x)$.*

Proof. Let $w \in T$. Towards a contradiction, suppose that $v, x \in V(B_w)$ with $v \neq x$, $U_{w,v} \neq V(D_v)$, and $U_{w,x} \neq V(D_x)$. We know that all arcs from D_v to D_x must have the same color, k (say), and likewise, all arcs from D_x to D_v have the same color, k' (say).

Corollary 3.6 tells us that B_w is externally related. Suppose that $k' = k$. Then for any vertex y with $y \neq v, x$ and $y \in V(B_w)$, two applications of Lemma 3.5, first using the same labels as in that lemma, and then reversing the roles of x and v, yield the conclusion that B_w is an externally related k-clique, so in particular, v and x are k-twins. Furthermore, calling on Corollary 3.8, since $x, v \in V(B_w)$ and the arcs between them in both directions have color k, we conclude that the k-complement of D_v (and therefore of D) is disconnected. But this contradicts condition (1) of Theorem 3.2.

We must therefore have $k' \neq k$. Again calling on Corollary 3.8, we see that D_v (and therefore D) must be a (k, k') total order digraph. Furthermore, since both $U_{w,v}, U_{w,x} \neq \emptyset$, μ is unnatural, so D must be a separable (k, k') total order digraph. But now Lemma 3.9 either provides a contradiction to condition (i) of Theorem 3.2, or yields the desired conclusion. □

Corollary 3.11 *Suppose that C and D are color digraphs and conditions (i), (ii), and (1) of Theorem 3.2 hold, but $C \wr D$ has an unnatural automorphism, μ. Let $x \in B_w$ such that $U_{w,x} \neq V(\mu(D_w))$. If $y \in V(C)$ with $y \neq w$ and $x \in V(B_y)$, then $U_{y,x} = V(\mu(D_y))$.*

Proof. This is just Lemma 3.10 relative to μ^{-1} instead of to μ. □

With these results in hand, we are ready to prove our main theorem.
Proof.
Necessity.

1. Suppose that x and y are a pair of k-twins in C, and the k-complement of D is disconnected, so that D' is a component of the k-complement of D. Then define μ by $\mu(D'_x) = D'_y$, $\mu(D'_y) = D'_x$, and μ acts as the identity on all other vertices. It is easy to see that this is an automorphism of $C \wr D$, and it is clearly unnatural.
2. Suppose M is a partition of $V(C)$, the subdigraphs induced by the elements of M are externally related in C, and $\sigma : C \to C_M$ is an onto smorphism such that for each $A \in M$, there is an isomorphism $\sigma_A : \bar{A} \wr D \to \overline{\sigma^{-1}(A)} \wr D$. Further suppose that for some $B \in M$, σ_B is unnatural.

   Define $\sigma' : C \wr D \to C \wr D$ by $\sigma'|_{\bar{A} \wr D} = \sigma_A$ for each $A \in M$. It is straightforward to check that σ' is an automorphism, and since σ_B is unnatural, so is σ'.
3. Suppose that x is an inverting A-point. Define μ to be the identity on all vertices of $C \wr D$ that are not in $A \wr D$, and on all vertices of D'_x (as defined in the definition of an inverting A-point), while μ exchanges D_a with D'_a for all vertices $a \in V(A) \setminus \{x\}$. Then it is straightforward to verify that μ is an automorphism, and μ is clearly unnatural.

Sufficiency. Suppose that (1), (2) and (3) hold, while μ is an unnatural automorphism of $C \wr D$. We aim to construct a contradiction to condition (2). We will therefore proceed as follows:

1. construct a partition M of $V(C)$ and show that it is a partition;
2. show that the subgraphs induced by the elements of M are externally related in C;
3. show that M is in fact a *proper* partition;
4. define a mapping $\overline{\mu} : C \to C_M$ and show that it is an onto smorphism;
5. show that $\bar{A} \wr D \cong \overline{\overline{\mu}^{-1}(A)} \wr D$ for all $A \in M$; and
6. show that there is an unnatural isomorphism between $\bar{A} \wr D$ and $\overline{\overline{\mu}^{-1}(A)} \wr D$ for some $A \in M$.

(1) Let $M_1 = \{\{V(B_x)\} : x \in V(C) \text{ and } |V(B_x)| > 1\}$. Let $M_2 = \{\{x\} : x \in V(C)$ but $x \notin A$ for any $A \in M_1\}$, and let $M = M_1 \cup M_2$. To show that M is a partition, we need only show that the elements of M_1 are disjoint. Suppose to the contrary, that $x \in V(B_v) \cap V(B_w)$ with $v \neq w$ and $B_v, B_w \in M_1$. Then $U_{v,x} \neq V(\mu(D_v))$, so Corollary 3.11 asserts that $U_{w,x} = V(\mu(D_w))$. But this contradicts $|V(B_w)| > 1$.

(2) For every element $A \in M$, $\bar{A} = B_w$ for some $w \in V(C)$, so this is a direct consequence of Corollary 3.6.

(3) Now we show that M is a proper partition of $V(C)$. If not, then there is some vertex w of C such that $V(B_w) = V(C)$.

Case 1. $V(D_w) \subseteq V(\mu(D_w))$.

We cannot have $D_w = \mu(D_w)$. There must exist $x \neq w$ for which $x \in V(B_w)$, and since $V(\mu(D_x)) \cap V(\mu(D_w)) = \emptyset$, we must have $U_{w,x} \neq V(D_x)$. By Lemma 3.10, for any $y \neq x \in V(C)$, we must have $U_{w,y} = V(D_y) \subset V(\mu(D_w))$. By Corollary 3.8 (with y playing the role of v), there are some fixed colors k and k' such that for any $y \neq x \in V(C)$, the arcs from D_y to D_x all have color k, and the arcs from D_x to D_y all have color k'. Also, by Corollary 3.11, for any $v \neq w \in V(C)$, we must have $U_{v,x} = V(\mu(D_v)) \subset V(D_x)$, so Corollary 3.8 using μ^{-1} instead of μ, tells us that for any $v \neq w \in V(C)$, the arcs from $\mu(D_w)$ to $\mu(D_v)$ all have color k, and the arcs from $\mu(D_v)$ to $\mu(D_w)$ all have color k'. If $k = k'$, then w and x are k-twins, and Corollary 3.8 implies that the k-complement of D is disconnected, contradicting condition (1). So $k \neq k'$. But now C is the Z_3-join of $Y_1 \cong K_1$ (vertex w), $Y_3 \cong K_1$ (vertex x), and Y_2, the induced subgraph of C on the vertices $\{v \in V(C) : v \neq w, x\}$. Additionally, Corollary 3.8 implies that D is a (k, k') total order digraph, and since $V(D_w) \subset V(\mu(D_w))$, D is isomorphic to a proper subdigraph of itself. But this contradicts condition (ii).

Case 2. $V(D_w) \nsubseteq V(\mu(D_w))$.

Since $V(B_w) = V(C)$, we must have $U_{w,w} \neq \emptyset$; the case that we are in further implies $U_{w,w} \neq V(D_w)$. As in the previous case, Lemma 3.10, Corollary 3.11 and Corollary 3.8 allow us to conclude that there are fixed colors k and k' such that for any $v \neq w \in V(C)$, the arcs from D_v to D_w all have color k, the arcs from D_w to D_v all have color k', the arcs from $\mu(D_w)$ to $\mu(D_v)$ all have color k, and the arcs from $\mu(D_v)$ to $\mu(D_w)$ all have color k'. Together, these force $k' = k$. Furthermore, this shows that the subdigraph of C on $V(C) \setminus \{w\}$ is externally related. Notice also that since $V(B_w) = V(C)$, $\mu(D_w)$ is isomorphic to the C-join of $\{D'_v : v \in V(C)\}$, where $D'_v \cong D$ for every $v \neq w$, and D'_w is the induced subdigraph of $C \wr D$ on $U_{w,w}$. But this means that w is an inverting C-point of $C \wr D$, contradicting condition (3).

We conclude that M must indeed be a proper partition of $V(C)$.

(4) Now, μ induces a mapping $\overline{\mu} : C \to C_M$ defined by $\overline{\mu}(w) = \bar{A}$ where $A \in M$ such that $V(\mu(D_w)) \subseteq V(\bar{A} \wr D)$. It is clear from the definition of M and the fact that M is a partition, that such an A exists and is unique, so $\overline{\mu}$ is well-defined. Also, since μ is onto, so is $\overline{\mu}$. Note that $V(B_w) \subseteq V(\overline{\mu}(w))$ and $V(\mu(D_w)) \subseteq V(B_w \wr D)$, so $V(\mu(D_w)) \subseteq V(\overline{\mu}(w) \wr D)$, for all $w \in V(C)$. We claim that $\overline{\mu}$ is a smorphism. To see this, let $x, y \in V(C)$ with $\overline{\mu}(x) \neq \overline{\mu}(y)$. The following statements are equivalent, using the fact (see part (2) above) that every element A of M induces an externally related subdigraph:

- the arc from $\overline{\mu}(x)$ to $\overline{\mu}(y)$ in C_M has color k;
- all arcs from $\overline{\mu}(x)$ to $\overline{\mu}(y)$ in C have color k;
- all arcs from $\overline{\mu}(x) \wr D$ to $\overline{\mu}(y) \wr D$ in $C \wr D$ have color k;
- all arcs from $\mu(D_x)$ to $\mu(D_y)$ in $C \wr D$ have color k;
- all arcs from D_x to D_y in $C \wr D$ have color k; and
- the arc from x to y in C has color k.

(5) It is clear from the definition of $\overline{\mu}$ that μ restricted to $\overline{\overline{\mu}^{-1}(A)} \wr D$ is an isomorphism between $\bar{A} \wr D$ and $\overline{\overline{\mu}^{-1}(A)} \wr D$ for each $A \in M$.

(6) Since μ is an unnatural automorphism of $C \wr D$, there must be some B_x on which the restriction of μ to $\overline{\mu}^{-1}(B_x)$ is unnatural. But this contradicts condition (2), completing the proof of our theorem. □

4 Corollaries and Elucidation

The conditions included in the full statements of Theorems 3.1 and 3.2 require so much terminology as to make them difficult to read, much less to apply. We therefore provide some corollaries in which the terminology and conditions are significantly simpler.

Corollary 4.1 *Suppose that C is a finite color digraph, and D is not isomorphic to a proper subdigraph of itself. Then we have*

$$\mathrm{Aut}(C \wr D) \cong \mathrm{Aut}(C) \wr \mathrm{Aut}(D)$$

$$\Leftrightarrow$$

for every $k \in \{0, 1, \ldots, r\}$,

if C has a pair of k-twins, then the k-complement of D is connected.

Proof. Condition (i) of Theorem 3.2 cannot occur since Z is an infinite color digraph. Condition (ii) is not possible because D is not a proper subdigraph of itself. As to condition (3), the definition of an inverting A-point again requires that D be a proper subdigraph of itself.

The omission of condition (2) is less obvious. However, observe that the conclusion of Lemma 3.10 can only occur if D is isomorphic to a proper subdigraph of itself, since μ is an unnatural automorphism. Since condition (2) is not required in this lemma, it can be omitted in our theorem as long as D is not isomorphic to a proper subdigraph of itself.

□

In fact, without changing the proof, we can generalize this somewhat; we have included the less general version because the statement is simpler and applies more directly to common situations such as when $C \wr D$ is finite.

Corollary 4.2 *Let C be a color digraph that has no induced subdigraph G for which*

- *G is an externally related (k,k') total order digraph; and*
- *G has an induced subdigraph isomorphic to Z.*

Let D be a color digraph that is not isomorphic to a proper subdigraph of itself. Then we have

$$\mathrm{Aut}(C \wr D) \cong \mathrm{Aut}(C) \wr \mathrm{Aut}(D)$$
$$\Leftrightarrow$$
$$\textit{for every } k \in \{0,1,\ldots,r\},$$
$$\textit{if } C \textit{ has a pair of } k\textit{-twins, then the } k\textit{-complement of } D \textit{ is connected.}$$

Notice that taking the case when C and D are color graphs (with edges rather than arcs), gives the following result, which is also a corollary of Theorem 3.1. Although the original proof in [8] did not include colors, their inclusion does not affect the proof.

Corollary 4.3 *Let C be any color graph, and let D be color graph that is not isomorphic to a proper subdigraph of itself. Then we have*

$$\mathrm{Aut}(C \wr D) \cong \mathrm{Aut}(C) \wr \mathrm{Aut}(D)$$
$$\Leftrightarrow$$
$$\textit{for every } k \in \{0,1,\ldots,r\},$$
$$\textit{if } C \textit{ has a pair of } k\textit{-twins, then the } k\textit{-complement of } D \textit{ is connected.}$$

Before proceeding with other matters, we return to the point made earlier, that the definition of a "separable" (k,k') total order digraph is not very satisfying. In fact, its use in condition (i) of Theorem 3.2 verges on the circular, saying that we will not allow D to be a (k,k') total order digraph for which it is possible to construct the situation we wish to avoid: an unnatural automorphism in $C \wr D$. We have provided no insight into which (k,k') total order digraphs have this undesirable characteristic.

While it may be possible to weaken condition (i), we will now provide some examples of separable (k,k') total order digraphs, that serve to demonstrate some of the variety that exists within this class.

Example 4.4 Let G be the (k,k') total order digraph formed by taking the Z_3-join of $H_1 = K_1 = H_3$ and any color digraph H_2. Take D to be the infinite wreath product

$G \wr G \wr \ldots$. Then D is a separable (k, k') total order digraph. That D is a (k, k') total order digraph is clear; to see that it is separable, we show that $G \wr D$ has an unnatural isomorphism. Label the vertices of G that correspond to H_1 and H_3 by 1 and 3, respectively, and the vertices that correspond to H_2 by their labels in H_2. Similarly, within D_i, label the vertices of the outermost G that correspond to H_1 and H_3 by $D_{i,1}$ and $D_{i,3}$, respectively, and for any vertex α of H_2, label the vertices of the outermost G that correspond to this vertex, by $D_{i,\alpha}$. Notice that $G \wr D \cong D$, by some isomorphism, σ (say). Then the map μ defined by $\mu(D_{1,1}) = D_1$ (using the isomorphism given by σ^{-1}); $\mu(D_{1,\alpha}) = D_\alpha$ for any vertex α of H_2 (again using σ^{-1}); $\mu(D_{1,3}) = D_{3,1}$; $\mu(D_\alpha) = D_{3,\alpha}$ for any vertex α of H_2 (now using σ); $\mu(D_3) = D_{3,3}$ (again using σ) is an unnatural automorphism of $G \wr D$.

Example 4.5 Let S be the intersection of the rational numbers with the open interval $(0, 1)$, under the usual total order, and let D be the (k, k') total order digraph formed on this set (where no join has been performed). Then D is a separable (k, k') total order digraph. To see this, we show that $Z \wr D$ has an unnatural automorphism. There is an isomorphism σ that maps S onto two copies of itself, since S is countably infinite. Let μ be the map that takes D_i to itself for all $i < 0$, takes D_0 to $D_0 \cup D_1$ using σ, and takes D_i to D_{i+1} for all $i > 0$. This is an unnatural automorphism of $Z \wr D$.

However, in the finite case, it is possible to come up with a characterization of separable (k, k') total order digraphs.

Proposition 4.6 *Let D be a finite, separable (k, k') total order digraph. Then for some a, D can be formed as the wreath product of Z_a with some finite color digraph D'.*

Furthermore, in this case, $Z \wr D$ has an unnatural automorphism.

Proof. Let D be a finite, separable (k, k') total order digraph. By definition, $k \neq k'$, and there is some (k, k') total order digraph G, such that $G \wr D$ has an unnatural automorphism. For any subdigraph D' of D and any vertex v of G, we denote the subdigraph of D_v corresponding to D' by D'^v.

While there are any vertices of D not in $V(D^0) \cup \ldots \cup V(D^{j-1})$ (where this is the empty set if $j = 0$), recursively define D^j to be the smallest nonempty induced subdigraph of D such that for every $v \in V(D^j)$ and every $w \in V(D) \setminus (V(D^0) \cup \ldots \cup V(D^j))$, there is a k-arc from v to w and a k'-arc from w to v. Define i to be the number of these subdigraphs (i.e., one greater than the largest subscript used). Notice that this choice of $D^0, \ldots, D^{i-1}$ is uniquely determined by the structure of D. Notice also that since D is a total order digraph, by definition we must have $i \geq 2$.

Let μ be an unnatural automorphism of $G \wr D$. Then there exist vertices w, x, y of G for which $x \neq y$, and $x, y \in V(B_w)$. Since $i \geq 2$, we can in fact choose x and y to ensure that $V(D_x) \cap V(\mu(D^0_w) \neq \emptyset$, and $V(D_y) \cap \mu(D^j_w) \neq \emptyset$ for some $j \neq 0$. Notice that since all arcs from D^0 to D^j have color k, and the reverse arcs all have color k', it must be the case that all arcs from D_x to D_y have color k, and the reverse arcs have color k'.

Suppose that there were an additional vertex v of G for which $v \neq x, y$ and $v \in V(B_w)$. Since D is finite, we must have $U_{w,x} \neq V(D_x)$, $U_{w,y} \neq V(D_y)$, and $U_{w,v} \neq V(D_v)$. We can therefore apply Lemma 3.5 three times, with w playing its own role each time, and each of x, y and v taking on the role of v once. This allows us to conclude (among other things) that

- the arcs from x to v and from y to v have the same color;
- the arcs from x to y and from x to v have the same color; and
- the arcs from y to x and from y to v have the same color.

Combining these statements yields the conclusion that the arcs from x to y and from y to x have the same color, which forces $k = k'$. But this is a contradiction. We therefore conclude that for any vertex w of G, there are at most two distinct vertices x and y of G for which $x, y \in V(B_w)$. A similar argument shows that for any vertex x of G, there are at most two distinct vertices v and w of G for which $x \in V(B_v)$ and $x \in V(B_w)$.

We have shown that $V(\mu(D_w)) \subset V(D_x) \cup V(D_y)$. Since the arc from x to y has color k and the reverse arc has color k', there must be some $0 \leq i' < i$ such that $U_{w,x} = V(\mu(D_w^0)) \cup \ldots \cup V(\mu(D_w^{i'-1}))$, and $U_{w,y} = V(\mu(D_w^{i'})) \cup \ldots \cup V(\mu(D_w^{i-1}))$. Because of the uniqueness of the decomposition of D, we must have $U_{w,x} = V(D_x^{i-i'})) \cup \ldots \cup V(D_x^{i-1})$; also, $U_{w,y} = V(D_y^0) \cup \ldots \cup V(D_y^{i-i'-1})$. In fact, this shows that $\mu(D_w^j) \cong D_w^j \cong D_x^{i-i'+j}$ for every $0 \leq j \leq i'-1$; furthermore, $\mu(D_w^{i'+j}) \cong D_w^{i'+j} \cong D_y^j$ for every $0 \leq j \leq i-i'-1$. Rewriting the first of these statements as $D_x^j \cong D_w^{j+i'-i}$ for every $i-i' \leq j \leq i-1$, considering both statements in the general context of the color digraph D, and performing addition on the superscripts modulo i, we arrive at the conclusion that $D^j \cong D^{j+i'}$ for every $0 \leq j \leq i-1$. Therefore, if $d = \gcd(i, i')$, we have $D^j \cong D^{j+sd}$ for every $0 \leq j \leq d-1$ and for every $1 \leq s < i/d$. But this means that if we take D' to be the induced subdigraph of D on the vertices $V(D^0) \cup \ldots \cup V(D^{d-1})$, we have that D' is isomorphic to the induced subdigraph of D on the vertices $V(D^{sd}) \cup \ldots \cup V(D^{(s+1)d-1})$ for every $1 \leq s < i/d$, so $D \cong Z_{i/d} \wr D'$. This completes the proof of the first statement.

To prove the second statement, label the copies of D' in $Z \wr (Z_i \wr D')$ as $D'_{a,b}$, where $a \in \mathbb{Z}$ and $1 \leq b \leq i$ indicate which vertices of Z and Z_i (respectively) give rise to this copy of D'. Define μ by

$$\mu(D'_{a,b}) = \begin{cases} D'_{a,b+1} & \text{if } b < i, \text{ and} \\ D'_{a+1,1} & \text{if } b = i \end{cases} .$$

It is not hard to show that μ is an automorphism, and μ is clearly unnatural. □

Now we have a characterisation of when $\mathrm{Aut}(C \wr D) = \mathrm{Aut}(C) \wr \mathrm{Aut}(D)$, but on the surface of it, this gives us little information about what $\mathrm{Aut}(C \wr D)$ might look like, if it is not $\mathrm{Aut}(C) \wr \mathrm{Aut}(D)$. This is the question we consider in the next section.

5 What else could $\mathrm{Aut}(C \wr D)$ be?

Quite often, it is the case that even if $\mathrm{Aut}(C \wr D) \neq \mathrm{Aut}(C) \wr \mathrm{Aut}(D)$, there are nontrivial color digraphs C' and D' for which $C' \wr D' = C \wr D$ and $\mathrm{Aut}(C \wr D) = \mathrm{Aut}(C') \wr \mathrm{Aut}(D')$. Later in this section, we will determine precisely which color digraphs have this property.

Before doing so, however, we will provide a result that gives the form of $\mathrm{Aut}(C \wr D)$ in some generality. We will assume that D is finite, and that C has no induced subdigraph G for which

- G is an externally related (k, k') total order digraph; and
- G has an induced subdigraph isomorphic to Z.

Thus, Corollary 4.2 will apply.

We require a few more pieces of notation in this section.

Notation 5.1 In the next few results, it will prove convenient to have a special notation for the colour digraph on n vertices that has an arc of color k from every vertex to every other vertex (that is, the complete digraph on n vertices, all of whose arcs have color k). We denote this by K_n^k.

The following two pieces of notation will be used in both this section and the next.

Notation 5.2 Let Γ be a permutation group acting on the set Ω. Then for any partition $\mathcal{P}$ of the elements of Ω, we let $\mathrm{fix}_\Gamma(\mathcal{P})$ denote the subgroup of Γ that fixes every set $P \in \mathcal{P}$ setwise.

Notation 5.3 We denote the symmetric group acting on the elements of the set Ω by $\mathcal{S}_\Omega$, or if $\Omega = \{1, \ldots, n\}$, by $\mathcal{S}_n$.

By Corollary 4.2, if we have $\mathrm{Aut}(C \wr D) \neq \mathrm{Aut}(C) \wr \mathrm{Aut}(D)$, we must have some color k for which C has a pair of k-twins, and the k-complement of D is disconnected. Suppose that the k-complement of D is disconnected, and let D' be a connected component of the k-complement of D. Then we must have arcs of color k in both directions between every vertex of D' and every vertex of D that is not in D', and therefore the k'-complement of D is connected for every $k' \neq k$, even if $k = 0$. This has shown that if $\mathrm{Aut}(C \wr D) \neq \mathrm{Aut}(C) \wr \mathrm{Aut}(D)$, then the color k for which C has a pair of k-twins and the k-complement of D is disconnected, is unique. Henceforth, k will be used exclusively to denote this color.

With this fixed k, we consider the externally related k-classes of C. These form a partition of the vertices of C. We denote this partition by $\mathcal{P}$.

Let $\mathcal{B}$ be the set of connected components of the k-complement of D; we partition $\mathcal{B}$ into subsets $\mathcal{B}_1, \ldots, \mathcal{B}_m$ where all of the components in $\mathcal{B}_i$ are isomorphic for every $1 \leq i \leq m$, and m is the number of nonisomorphic components of the k-complement of D. For each $1 \leq i \leq m$, let $B_i \in \mathcal{B}_i$ be any one copy of the component in this set of isomorphic components. Then it is straightforward to see that

$$\mathrm{Aut}(D) = \bigtimes_{1 \leq i \leq m} \left(\mathcal{S}_{\mathcal{B}_i} \wr \mathrm{Aut}(B_i)\right),$$

a direct product of wreath products.

We are now ready to give the form of $\mathrm{Aut}(C \wr D)$.

Theorem 5.4 *For any color digraphs C and D, where D is finite and there are no colors k_1, k_2 for which C has an induced subdigraph G, where*

- *G is an externally related (k_1, k_2) total order digraph; and*
- *G has an induced subdigraph isomorphic to Z (where the colors of the total order in Z are k_1 and k_2.*

We must have

$$\mathrm{Aut}(C \wr D) = \big(\mathrm{Aut}(C) \wr 1_{\mathrm{Aut}(D)}\big) \left[\bigtimes_{1 \leq i \leq m} \left(\bigtimes_{P \in \mathcal{P}} \mathcal{S}_{\mathcal{B}_i \times P} \wr \mathrm{Aut}(B_i) \right) \right],$$

where $1_{\mathrm{Aut}(D)}$ denotes the identity element of $\mathrm{Aut}(D)$.

Before proving this theorem, some comments are appropriate.

There is redundancy in the group $\big(\mathrm{Aut}(C) \wr 1_{\mathrm{Aut}(D)}\big) \left[\bigtimes_{1 \leq i \leq m} \left(\bigtimes_{P \in \mathcal{P}} \mathcal{S}_{\mathcal{B}_i \times P} \wr \mathrm{Aut}(B_i) \right) \right]$. To see this, we clarify how the group $\big(\mathrm{Aut}(C) \wr 1_{\mathrm{Aut}(D)}\big) \left[\bigtimes_{1 \leq i \leq m} \left(\bigtimes_{P \in \mathcal{P}} \mathcal{S}_{\mathcal{B}_i \times P} \wr \mathrm{Aut}(B_i) \right) \right]$ acts on the vertices of $C \wr D$. For each of the m nonisomorphic connected components B_i of the k-complement of D, let B_i' denote the induced subgraph of D with the same vertices as B_i, so B_i' is isomorphic to the k-complement of B_i. Then $\bigtimes_{P \in \mathcal{P}} \mathcal{S}_{\mathcal{B}_i \times P} \wr \mathrm{Aut}(B_i)$ takes all of the vertices of C in each of the externally related k-classes of C in turn, and permutes all components isomorphic to B_i' in each copy of D that corresponds to these vertices of C. Since this is done to each of the m nonisomorphic connected components independently, this produces all of the direct products of wreath products. We then have $\mathrm{Aut}(C) \wr 1_{\mathrm{Aut}(D)}$ acting as usual on the vertices of $C \wr D$. The redundancy occurs because each of the m nonisomorphic components of the k-complement of D has been permuted independently within each externally related k-class of C, and then each copy of D is permuted as a set by the action of $\mathrm{Aut}(C) \wr 1_{\mathrm{Aut}(D)}$.

It is possible to remove this redundancy. Notice that every externally related k-class of C consists of a K_i^k, and if we delete the edges of this K_i^k, each of the vertices in this equivalence class has exactly the same in-neighbours and out-neighbours of every color, as every other vertex in the equivalence class. Therefore we have $\mathrm{fix}_{\mathrm{Aut}(C)}(\mathcal{P}) = \bigtimes_{P \in \mathcal{P}} \mathcal{S}_P$.

We could therefore write $\mathrm{Aut}(C \wr D)$ as

$$(\mathrm{Aut}_0(C) \wr 1_{\mathrm{Aut}(D)}) \left[\bigtimes_{1 \leq i \leq m} \left(\bigtimes_{P \in \mathcal{P}} \mathcal{S}_{\mathcal{B}_i \times P} \wr \mathrm{Aut}(B_i) \right) \right],$$

where $\mathrm{Aut}_0(C)$ is a permutation group for which $\mathrm{Aut}(C) = \mathrm{Aut}_0(C) \ltimes \mathrm{fix}_{\mathrm{Aut}(C)}(\mathcal{P})$. This notation has the advantage that it can be written as a semi-direct product: this group is in fact

$$(\mathrm{Aut}_0(C) \wr 1_{\mathrm{Aut}(D)}) \ltimes \left[\bigtimes_{1 \leq i \leq m} \left(\bigtimes_{P \in \mathcal{P}} \mathcal{S}_{\mathcal{B}_i \times P} \wr \mathrm{Aut}(B_i) \right) \right],$$

since $\left[\bigtimes_{1\le i\le m}\left(\bigtimes_{P\in\mathcal{P}} \mathcal{S}_{\mathcal{B}_i\times P} \wr \mathrm{Aut}(B_i)\right)\right] \lhd \mathrm{Aut}(C)\wr\mathrm{Aut}(D)$ (this will be shown in our proof) and the redundancy has been eliminated. However, difficulties in choosing the precise action of $\mathrm{Aut}_0(C)$ make this method of eliminating redundancy seem somewhat artificial, so we have left in the redundancy.

With these comments in mind, we proceed with the proof of the theorem.

Proof. Let $\mathcal{Q}$ be a partition of the vertices of D into sets of vertices, each of which induces a connected component of the k-complement of D. Then we let $\mathcal{Q}'$ be a partition of the vertices of $C\wr D$, where for each $Q\in\mathcal{Q}$, and for each $v\in V(C)$, there is a set $Q'_v\in\mathcal{Q}'$, namely $Q'_v=\{(v,w) : w\in Q\}$. We claim that the partition $\mathcal{Q}'$ is preserved by every element of $\mathrm{Aut}(C\wr D)$, by which we mean that if $g\in\mathrm{Aut}(C\wr D)$, $(v,w)\in Q'_v$, and $g((v,w))\in Q''\in\mathcal{Q}'$, then $g(Q'_v)=Q''$.

Towards a contradiction, suppose that there were some $g\in\mathrm{Aut}(C\wr D)$ that did not preserve the partition $\mathcal{Q}'$. Then there must be some $Q'_v\in\mathcal{Q}'$ for which there exist $Q',Q''\in\mathcal{Q}'$ with $Q'\ne Q''$, such that $g(Q'_v)\cap Q'\ne\emptyset$, and $g(Q'_v)\cap Q''\ne\emptyset$. Recall that each element of $\mathcal{Q}'$ is a set of vertices of $C\wr D$ in some copy of D that corresponds to the vertices of a connected component of the k-complement of D. Therefore, there exists some vertex v' of C for which $Q'\subset V(D_{v'})$. If $g(Q'_v)\subset V(D_{v'})$, then since the vertices of Q' form a connected component of the k-complement of D, $C\wr D$ must have every possible arc of color k in both directions between $g(Q'_v)\cap Q'$ and $g(Q'_v)\setminus Q'$. Since both of these sets are nonempty, this leads to the contradiction that $g(Q'_v)$ induces a disconnected subgraph of the k-complement of D.

If, on the other hand, $g(Q'_v)\not\subset V(D_{v'})$, then we may assume $Q''\subset V(D_{v''})$ for some $v''\ne v'$. If the two arcs between v and v' in C have the same color, k, say, then $C\wr D$ must have every possible arc of color k in both directions between $g(Q'_v)\cap D_{v'}$ and $g(Q'_v)\setminus D_{v'}$. Since both of these sets are nonempty, this again leads to the contradiction that $g(Q'_v)$ induces a disconnected subgraph of the k-complement of D. So the arcs between v and v' in C have two distinct colors. In the proof of Lemma 3.9, condition (1) of Theorem 3.2 is only used to show that we can choose w_0,x_0 and x_1 so that $U_{w_0,x_0}\ne V(D_{x_0})$ and $U_{w_0,x_1}\ne V(D_{x_1})$. But here D is finite, so the fact that $v',v''\in V(B_v)$ is enough to show that this holds for v taking the role of w_0, and v',v'' taking the roles of x_0 and x_1, and these vertices satisfy the premise of the second conclusion of Lemma 3.9. Notice that the first conclusion of Lemma 3.9 does not hold. Condition (ii) of Theorem 3.2 holds since D is finite, and the second conclusion of Lemma 3.9 cannot hold because of the restrictions on C. All of this together with the fact that g is an unnatural automorphism of $C\wr D$, produce a contradiction from Lemma 3.9. We conclude that the partition $\mathcal{Q}'$ is indeed preserved by every element of $\mathrm{Aut}(C\wr D)$.

With this fact in hand, it is straightforward to verify that

$$\left[\bigtimes_{1\le i\le m}\left(\bigtimes_{P\in\mathcal{P}} \mathcal{S}_{\mathcal{B}_i\times P} \wr \mathrm{Aut}(B_i)\right)\right] = \mathrm{fix}_{\mathrm{Aut}(C\wr D)}(\mathcal{P}).$$

Since $\mathrm{fix}_{\mathrm{Aut}(C\wr D)}(\mathcal{P})$ is the kernel of the projection of $\mathrm{Aut}(C\wr D)$ onto the partition $\mathcal{P}$, this group is in fact normal in $\mathrm{Aut}(C\wr D)$, as we claimed in the observations that preceded this proof.

Since $\mathrm{fix}_{\mathrm{Aut}(C\wr D)}(\mathcal{P}) \lhd \mathrm{Aut}(C \wr D)$, every automorphism in $\mathrm{Aut}(C \wr D)$ can be formed by combining an automorphism in $\mathrm{fix}_{\mathrm{Aut}(C\wr D)}(\mathcal{P})$ with an automorphism that permutes sets of the partition $\mathcal{P}$ according to some automorphism of C; as mentioned in our observations, $\mathrm{Aut}_0(C) \wr 1_{\mathrm{Aut}(D)}$ would provide a semi-direct product since redundancy would be eliminated, but we certainly have

$$\mathrm{Aut}(C \wr D) \leq (\mathrm{Aut}(C) \wr 1_{\mathrm{Aut}(D)}) \left[\bigtimes_{1\leq i\leq m} \left(\bigtimes_{P\in\mathcal{P}} \mathcal{S}_{\mathcal{B}_i\times P} \wr \mathrm{Aut}(B_i) \right) \right].$$

Since both of the groups that make up the product on the right have been shown to be subgroups of $\mathrm{Aut}(C \wr D)$, we have the desired equality. □

Although it is of interest to have determined this exact form of the automorphism group of any wreath product color digraph, the expression at which we have arrived is not always as enlightening as it could be. For many wreath products of color digraphs $C \wr D$, it turns out that if $\mathrm{Aut}(C \wr D) \neq \mathcal{S}_n$ for some n then it is possible to find nontrivial color digraphs C' and D' for which $C' \wr D' \cong C \wr D$, and $\mathrm{Aut}(C \wr D) = \mathrm{Aut}(C' \wr D') = \mathrm{Aut}(C') \wr \mathrm{Aut}(D')$. That is to say, that in these cases, if $\mathrm{Aut}(C \wr D) \neq \mathrm{Aut}(C) \wr \mathrm{Aut}(D)$, it merely means that we have made the wrong choices for C and D, the factors of our wreath product.

We will require a lemma.

Lemma 5.5 *Let C, C', D and D' be color digraphs. Suppose that $X = C \wr D = C' \wr D'$. Suppose further that there is some vertex v of C for which D_v is neither a union of copies of D', nor contained within a copy of D'.*

Whenever there is some color k, $0 \leq k \leq r$, for which the k-complement of D is disconnected, then C' has k-twins, and the k-complement of D' is disconnected.

Proof. Let w be a vertex of C' such that $V(D'_w) \cap V(D_v) \neq \emptyset$, $V(D'_w) \setminus V(D_v) \neq \emptyset$, and $V(D_v) \setminus V(D'_w) \neq \emptyset$. Let v' be a vertex of C such that $V(D_{v'}) \cap V(D'_w) \neq \emptyset$. All arcs from D_v to $D_{v'}$ have the same color, k, say.

By Lemma 3.7, all arcs from $V(D_v) \setminus V(D'_w)$ to $V(D_v) \cap V(D'_w)$ have color k also. Notice that this means that the k'-complement of D is connected for every $k' \neq k$. Thus, if there is some some color k' for which the k'-complement of D is disconnected, we must have $k' = k$, and if the color of the arcs from $V(D_v) \cap V(D'_w)$ to $V(D_v) \setminus V(D'_w)$ is not k, then the k-complement of D is also connected, a contradiction. So there are arcs of color k in both directions between $V(D_v) \setminus V(D'_w)$ and $V(D_v) \cap V(D'_w)$. In particular, for any vertex $w' \neq w$ of C' for which $V(D_v) \cap V(D'_{w'}) \neq \emptyset$, the arcs in both directions between D'_w and $D'_{w'}$ have color k.

This is enough to allow us to use Lemma 3.7, with D'_w and D_v taking on each others' roles, to conclude that all arcs in either direction between $V(D'_w) \cap V(D_v)$ and $V(D'_w) \setminus V(D_v)$, have color k. Since both of these sets are nonempty, the k-complement of D' is disconnected.

Finally, we establish that w and w' are k-twins. We have already shown that the arcs between w and w' in C' have color k. Let $w'' \in V(C')$, and suppose that the arc from w to w'' has color k'. Let v'' be any vertex of C for which $V(D_{v''}) \cap V(D'_{w''}) \neq \emptyset$. Then all arcs from D'_w to $D'_{w''}$ have color k', and the various nonempty intersections establish that this is equivalent to all arcs from D_v to $D_{v''}$ having color k', which in turn is equivalent to all arcs from $D'_{w'}$ to $D'_{w''}$ having color k'; that is, the arc from w' to w'' has color k'. We can reverse the direction of the arcs in this argument to complete the proof that w and w' are k-twins, as claimed. □

The next result characterises precisely which finite color digraphs C and D have the property that, if $\mathrm{Aut}(C \wr D) \neq \mathrm{Aut}(C) \wr \mathrm{Aut}(D)$, either $C \wr D \cong K_n^k$ for some color k and some n, or there are nontrivial color digraphs C' and D' for which $C' \wr D' = C \wr D$, and $\mathrm{Aut}(C' \wr D') = \mathrm{Aut}(C') \wr \mathrm{Aut}(D')$. Although assuming that C and D are finite makes the proof straightforward, this condition is not used very stringently and could probably be weakened.

Proposition 5.6 *Let C and D be finite color digraphs, C having n_1 vertices, and D having n_2 vertices, with $n_1 n_2 = n$, and $X = C \wr D$. The conditions on C and D that follow are both necessary and sufficient to ensure that*

$$\begin{gathered}\mathrm{Aut}(X) \neq \mathrm{Aut}(C) \wr \mathrm{Aut}(D) \Rightarrow \\ (\mathrm{Aut}(X) = \mathcal{S}_n, \\ \textit{or } \exists \textit{ nontrivial } C', D' \textit{ such that } C' \wr D' = X \textit{ and } \mathrm{Aut}(X) = \mathrm{Aut}(C') \wr \mathrm{Aut}(D'))\end{gathered}$$

The conditions are: For any color k for which C has k-twins and the k-complement of D is not connected, at least one of the following must hold:

1. *$D \cong K_{n_2}^k$ and $C \cong K_{n_1}^k$;*
2.
   - *$D = D'' \wr D'$ for some nontrivial D'' and D',*
   - *$\mathrm{Aut}(D) = \mathrm{Aut}(D'') \wr \mathrm{Aut}(D')$, and*
   - *if there is some k' for which C has k'-twins and there is some vertex v of D'' that forms a singleton component of the k'-complement of D'', then the k'-complement of D' is connected;*

   or
3. *$C = C' \wr C''$ for some nontrivial C' and C'', and $\mathrm{Aut}(C) = \mathrm{Aut}(C') \wr \mathrm{Aut}(C'')$.*

Proof. Sufficiency. Suppose that the conditions hold, and that $\mathrm{Aut}(X) \neq \mathrm{Aut}(C) \wr \mathrm{Aut}(D)$. Then by Corollary 4.1, there is some k for which C has k-twins and the k-complement of D is not connected. We break the proof down into cases, according to which of the three conditions holds.

Case 1. $D \cong K_{n_2}^k$ and $C \cong K_{n_1}^k$.

Then $X = C \wr D \cong K_n^k$, so $\mathrm{Aut}(X) = \mathcal{S}_n$, completing the proof in this case.

Case 2. $D \cong D'' \wr D'$ for some nontrivial D'' and D', $\mathrm{Aut}(D) = \mathrm{Aut}(D'') \wr \mathrm{Aut}(D')$, and: if there is some k' for which C has k'-twins and there is some vertex v of D'' that forms a singleton component of the k'-complement of D'', then the k'-complement of D' is connected.

We claim that $\mathrm{Aut}(X) = \mathrm{Aut}(C \wr D'') \wr \mathrm{Aut}(D')$. Since D'' and D' are nontrivial, so is C' where $C' = C \wr D''$, and (since wreath products are associative) clearly $X = C' \wr D'$, so establishing our claim will be sufficient to complete the proof in this case. Again, we will use the conditions in Corollary 4.1 to establish our claim.

Suppose that for some color k', C' has k'-twins, which we will call v_0 and v_1. Recall that $C' = C \wr D''$. If v_0 and v_1 are in the same copy of D'' within C', then choosing corresponding vertices v_0' and v_1' in D'', we must have v_0' and v_1' being k'-twins in D''. Now, since $\mathrm{Aut}(D) = \mathrm{Aut}(D'') \wr \mathrm{Aut}(D')$, Corollary 4.1 forces the k'-complement of D' to be connected.

If, on the other hand, v_0 and v_1 are in different copies of D'' within C', then the vertices v_0' and v_1' of C corresponding to these copies of D'' must have the property that v_0' and v_1' are k'-twins in C. Furthermore, since there are arcs of color k' in both directions between v_0 and the copy of D'' in C' that contains v_1, there must be arcs of color k' in both directions between v_1 and every other vertex in this copy of D''. So the vertex in D'' corresponding to v_1 will be the special vertex v described in this case, and we may therefore assume that the k'-complement of D' is connected.

We have shown that C' having k'-twins forces the k'-complement of D' to be connected, so by Corollary 4.1, $\mathrm{Aut}(X) = \mathrm{Aut}(C') \wr \mathrm{Aut}(D')$ and we are done.

Case 3. $C \cong C' \wr C''$ for some nontrivial C' and C'', and $\mathrm{Aut}(C) = \mathrm{Aut}(C') \wr \mathrm{Aut}(C'')$.

We claim that $\mathrm{Aut}(X) = \mathrm{Aut}(C') \wr \mathrm{Aut}(C'' \wr D)$. Since C' and C'' are nontrivial, so is D' where $D' = C'' \wr D$, and (since wreath products are associative) clearly $X = C' \wr D'$, so establishing our claim will be sufficient to complete the proof in this case. Again, we will use the conditions in Corollary 4.1 to establish our claim.

Since $\mathrm{Aut}(C) = \mathrm{Aut}(C') \wr \mathrm{Aut}(C'')$, we have that for any color k', C' having k'-twins implies that the k'-complement of C'' is connected. But if the k'-complement of C'' is connected, then the k'-complement of $C'' \wr D$, which is the same as the k'-complement of D', will also certainly be connected, so Corollary 4.1 again tells us that $\mathrm{Aut}(X) = \mathrm{Aut}(C') \wr \mathrm{Aut}(D')$ and we are done.

Necessity. We consider all of the ways in which the assumption

$$\begin{gathered}\mathrm{Aut}(X) \neq \mathrm{Aut}(C) \wr \mathrm{Aut}(D) \Rightarrow \\ (\mathrm{Aut}(X) = \mathcal{S}_n, \\ \text{or } \exists \text{ nontrivial } C', D' \text{ such that } C' \wr D' \cong X \text{ and } \mathrm{Aut}(X) = \mathrm{Aut}(C') \wr \mathrm{Aut}(D'))\end{gathered}$$

can be satisfied, and show that for each, the conditions must hold.

First, if $\mathrm{Aut}(X) = \mathrm{Aut}(C) \wr \mathrm{Aut}(D)$, then by Corollary 4.1, for every color k, C having k-twins implies that the k-complement of D is connected, so the premise of the conditions never occurs, and therefore the conditions are vacuously satisfied.

If $\mathrm{Aut}(X) = \mathcal{S}_n$, then there must be some color k for which $X \cong K_n^k$. Hence we must have $C \cong K_{n_1}^k$ and $D \cong K_{n_2}^k$. Notice that C has no k'-twins for any $k' \neq k$, so the premise of our conditions can only be satisfied by the color k. We have shown that in this case, condition (1) is satisfied whenever the premise holds.

Finally, if there exist nontrivial C' and D' such that $X \cong C \wr D = C' \wr D'$ and $\mathrm{Aut}(X) = \mathrm{Aut}(C') \wr \mathrm{Aut}(D')$, then since $\mathrm{Aut}(X) \neq \mathrm{Aut}(C) \wr \mathrm{Aut}(D)$, Corollary 4.1 tells us that there must be some color k for which the k-complement of D is disconnected. So by Lemma 5.5, if there were a copy of D that were neither a union of copies of D', nor contained within a copy of D', then C' has k-twins and the k-complement of D' is disconnected, but by Corollary 4.1, this is a contradiction. Hence any copy of D must either be a union of copies of D', or contained within a copy of D'.

Suppose first that every copy of D is a union of copies of D' (since D is finite, it is impossible to have some copies of D being unions of copies of D', while others are strictly contained in a copy of D'). Since $\mathrm{Aut}(X) = \mathrm{Aut}(C') \wr \mathrm{Aut}(D') \neq \mathrm{Aut}(C) \wr \mathrm{Aut}(D)$, the union must be nontrivial. Then since $C \wr D = C' \wr D'$, we must in fact have $D = D'' \wr D'$ for some nontrivial D'' (we already have D' nontrivial, by assumption), the first part of condition (2). Notice $C' = C \wr D''$.

Now, using Corollary 4.1, $\mathrm{Aut}(X) = \mathrm{Aut}(C') \wr \mathrm{Aut}(D')$ means that for any k', if C' has k'-twins then the k'-complement of D' is connected. Notice that if D'' has k'-twins then C' must have k'-twins, and therefore the k'-complement of D' must be connected. But then we can conclude (by Corollary 4.1) that $\mathrm{Aut}(D) = \mathrm{Aut}(D'') \wr \mathrm{Aut}(D')$, the second part of condition (2).

Suppose that there is some k' for which C has k'-twins and some vertex v of D'' that has arcs of color k' to and from every other vertex of D''. Then in C', take the copies of v in two copies of D corresponding to vertices in C that are k'-twins; these two vertices will be k'-twins in C'. So C' has k'-twins, and again the k'-complement of D' must be connected. This is precisely what remained to be shown of condition (2).

Finally, we suppose that every copy of D is contained within a copy of D', so every copy of D' is a union of copies of D, and again since $\mathrm{Aut}(X) = \mathrm{Aut}(C') \wr \mathrm{Aut}(D') \neq \mathrm{Aut}(C) \wr \mathrm{Aut}(D)$, the union must be nontrivial. Then since $C \wr D = C' \wr D'$, we must in fact have $D' = C''' \wr D$ for some nontrivial C''', and $C = C' \wr C'''$ (we already have C' nontrivial, by assumption).

Now, using Corollary 4.1, $\mathrm{Aut}(X) = \mathrm{Aut}(C') \wr \mathrm{Aut}(D')$ means that for any k', if C' has k'-twins then the k'-complement of D' is connected. Notice that the k'-complement of D' being connected forces the k'-complement of C''' to be connected, since $D' \cong C''' \wr D'$. But this has shown (using Corollary 4.1) that $\mathrm{Aut}(C) = \mathrm{Aut}(C') \wr \mathrm{Aut}(C''')$, and so condition (3) holds. □

It may not be easy to see precisely which color digraphs satisfy the condition given in Proposition 5.6. In fact, although it is possible to show that vertex-transitive color digraphs satisfy this condition, a direct proof of a stronger result turns out to be shorter.

Theorem 5.7 *For any finite vertex-transitive color digraph* $X \cong C \wr D$, *if* $\mathrm{Aut}(X) \neq \mathrm{Aut}(C) \wr \mathrm{Aut}(D)$ *then there are some natural numbers* $r > 1$ *and* $s > 1$, *and some color* k, *for which* $C \cong C' \wr K_r^k$, $D \cong K_s^k \wr D'$, *and* $\mathrm{Aut}(X) = \mathrm{Aut}(C') \wr (\mathcal{S}_{rs} \wr \mathrm{Aut}(D'))$.

Proof. By Corollary 4.1, since $\mathrm{Aut}(X) \neq \mathrm{Aut}(C) \wr \mathrm{Aut}(D)$, there is some color k for which C has k-twins and the k-complement of D is disconnected.

Since X (and therefore C) is vertex-transitive and finite, every externally related k-class of C has the same size, r (say), and since C has k-twins, we have $r > 1$. Therefore, every externally related k-class of C induces a subdigraph of C that is isomorphic to K_r^k for some $r > 1$. Since each class is externally related, we have $C \cong C' \wr K_r^k$ for some vertex-transitive color digraph C'.

Since D is also vertex-transitive, every connected component of the k-complement of D is isomorphic. If we give the name D' to the induced subdigraph of D that corresponds to the vertices in a connected component of the k-complement of D, we have $D \cong K_s^k \wr D'$, where s is the number of connected components of the k-complement of D (greater than 1, since the k-complement of D is disconnected).

Hence $X \cong C' \wr K_r^k \wr K_s^k \wr D' \cong C' \wr K_{rs}^k \wr D'$.

Notice that k is the only color for which the k-complement of $K_{rs}^k \wr D'$ is disconnected, and since each K_r^k was an externally related k-class of C, we see that C' cannot have k-twins. Hence by Corollary 4.1, $\mathrm{Aut}(X) = \mathrm{Aut}(C') \wr \mathrm{Aut}(K_{rs}^k \wr D')$.

Now, the only color k' for which K_{rs}^k has k'-twins, is $k' = k$, and since each D' corresponded to the vertices of a connected component of the k-complement of D, we must have the k-complement of D' connected. Hence by Corollary 4.1, $\mathrm{Aut}(K_{rs}^k \wr D') = \mathrm{Aut}(K_{rs}^k) \wr \mathrm{Aut}(D') = \mathcal{S}_{rs} \wr \mathrm{Aut}(D')$.

Combining the conclusions of the last two paragraphs, we have $\mathrm{Aut}(X) = \mathrm{Aut}(C') \wr (\mathcal{S}_{rs} \wr \mathrm{Aut}(D'))$, as desired. □

6 Isomorphisms of Wreath Products of Cayley Digraphs of Abelian Groups

In recent years, a great deal of work has been directed towards solving the *Cayley isomorphism* problem. That is, given any two isomorphic Cayley (di)graphs Γ and Γ' of a group G, is it true that there exists $\alpha \in \mathrm{Aut}(G)$ such that $\alpha(\Gamma) = \Gamma'$? If the answer to the preceding question is yes for every Γ' isomorphic to Γ, then we say that Γ is a *CI-(di)graph of G*. If any two isomorphic Cayley (di)graphs of G are isomorphic by a group automorphism of G, we say that *G is a CI-group with respect to (di)graphs.* This problem was first proposed in 1967 by Ádám [1] in a less general form when he conjectured that $\mathbb{Z}_n$ was a CI-group with respect to graphs. The problem was generalized by Babai in [3]. The reader is referred to [9] for a recent survey of this problem. Here, we will be concerned with the isomorphism problem for Cayley digraphs that can be written as a wreath product. Intuitively, if Γ_1 is a CI-(di)graph of G_1, and Γ_2 is a CI-(di)graph of G_2, then surely $\Gamma_1 \wr \Gamma_2$ is a CI-(di)graph $G_1 \times G_2$. This, however, is not true as the following example shows. Before turning to this example, we will need Babai's well-known characterization of the CI property [3] (we remark that Alspach and Parsons [2] also obtained this criterion, although in a less general form).

We require a few pieces of notation.

Notation 6.1 If G is a group, then G_L is the left regular representation of G. If $H \leq G$, we let $\bar{H}_L = \{g_L \in G_L : g \in H\}$.

Notation 6.2 We use $N_G(H)$ to denote the subgroup of G that normalizes H, where H is a subgroup of G.

Lemma 6.3 *For a Cayley (di)graph Γ of G the following are equivalent:*

1. *Γ is a CI-(di)graph,*
2. *given a permutation $\varphi \in \mathcal{S}_G$ such that $\varphi^{-1}G_L\varphi \leq \mathrm{Aut}(\Gamma)$, G_L and $\varphi^{-1}G_L\varphi$ are conjugate in $\mathrm{Aut}(\Gamma)$.*

Example 6.4 Let p be a prime. Then there exists a Cayley (di)graph Γ of $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$ such that $\Gamma = \Gamma_1 \wr \Gamma_1$, where Γ_1 is a CI-(di)graph of $\mathbb{Z}_p$ and Γ_2 is a CI-(di)graph of $\mathbb{Z}_{p^2}$, but Γ is not a CI-(di)graph of $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$.

Proof. We first claim that $\mathbb{Z}_p \wr (\mathbb{Z}_p \wr \mathbb{Z}_p)$ contains regular subgroups R_1 and R_2 isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$ that are not conjugate in $\mathrm{AGL}(1,p) \wr (\mathrm{AGL}(1,p) \wr \mathrm{AGL}(1,p))$.

Define $\tau_1, \tau_2, \rho_1, \rho_2 : \mathbb{Z}_p^3 \to \mathbb{Z}_p^3$ by
$\tau_1(i,j,k) = (i+1, j+b_i, k)$,
$\tau_2(i,j,k) = (i+1, j, k)$,
$\rho_1(i,j,k) = (i,j,k+1)$, and
$\rho_2(i,j,k) = (i, j+1, k+c_j)$,
where $b_i = 0$ if $i \neq p-1$ and $b_{p-1} = 1$, and $c_j = 0$ if $j \neq p-1$ and $c_{p-1} = 1$. It is straightforward to verify that $|\tau_1| = |\rho_2| = p^2$, $|\tau_2| = |\rho_1| = p$, and $R_1 = \langle \rho_1, \tau_1 \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_{p^2} \cong \langle \rho_2, \tau_2 \rangle = R_2$. Note that $\mathrm{AGL}(1,p) \wr (\mathrm{AGL}(1,p) \wr \mathrm{AGL}(1,p))$ admits a unique complete block system $\mathcal{B}$ consisting of p blocks of size p^2 formed by the orbits of $1_{\mathcal{S}_p} \wr (\mathrm{AGL}(1,p) \wr \mathrm{AGL}(1,p))$. Furthermore, $\mathrm{fix}_{R_1}(\mathcal{B}) = \langle \tau_1^p, \rho_1 \rangle$ and $\mathrm{fix}_{R_2}(\mathcal{B}) = \langle \rho_2 \rangle$. Let $\delta \in \mathrm{AGL}(1,p) \wr (\mathrm{AGL}(1,p) \wr \mathrm{AGL}(1,p))$. Then $\delta(\mathcal{B}) = \mathcal{B}$ so that $\mathrm{fix}_{\delta^{-1}R_2\delta}(\mathcal{B})$ is cyclic while $\mathrm{fix}_{R_1}(\mathcal{B})$ is not cyclic. Hence R_1 and R_2 are not conjugate in $\mathrm{AGL}(1,p) \wr (\mathrm{AGL}(1,p) \wr \mathrm{AGL}(1,p))$ as claimed.

It thus only remains to show that there exists Cayley (di)graphs of $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$ whose automorphism groups contain $\mathbb{Z}_p \wr (\mathbb{Z}_p \wr \mathbb{Z}_p)$ and are contained in $\mathrm{AGL}(1,p) \wr (\mathrm{AGL}(1,p) \wr \mathrm{AGL}(1,p))$. This though, is easy to accomplish using the literature. First, Alspach and Parsons [2] have determined necessary and sufficient conditions for a Cayley (di)graph of $\mathbb{Z}_{p^2}$ to be a CI-digraph of $\mathbb{Z}_{p^2}$ (including when the full automorphism groups contains $\mathbb{Z}_p \wr \mathbb{Z}_p$), and Gu and Li [4] have determined for which values of m all Cayley graphs of $\mathbb{Z}_{p^2}$ that are regular of degree m are CI-graphs. $\square$

Lemma 6.5 *Let G and H be groups and $J \leq \mathcal{S}_G$, $K \leq \mathcal{S}_H$ contain G_L and H_L respectively. Suppose that any two regular subgroups of J isomorphic to G are conjugate in J and any two regular subgroups of K isomorphic to H are conjugate in K. Let $\varphi \in \mathcal{S}_G \wr \mathcal{S}_H$ such that $\varphi^{-1}(G \times H)_L\varphi \leq J \wr K$. Then $\varphi^{-1}(G \times H)_L\varphi$ and $(G \times H)_L$ are conjugate in $J \wr K$. Furthermore, $\varphi^{-1}\bar{G}_L\varphi$ and $\bar{G}_L$ are also conjugate in $J \wr K$.*

Proof. It is straightforward to verify that $(G\times H)_L = \bar{G}_L \times \bar{H}_L$ so that $(G\times H)_L \le J \wr K$. As $\varphi \in \mathcal{S}_G \wr \mathcal{S}_H$, $\varphi(g,h) = (\sigma(g), \omega_g(h))$, where $\sigma \in \mathcal{S}_G$, $\omega_g \in \mathcal{S}_H$. Let $\mathcal{B}$ be the complete block system of $J\wr K$ formed by the orbits of $1_{\mathcal{S}_G} \wr K$. As any two regular subgroups of J isomorphic to G are conjugate in J, there exists $\delta \in J \wr K$ such that $\delta^{-1}\varphi^{-1}(G\times H)_L\varphi\delta/\mathcal{B} = (G\times H)_L/\mathcal{B} = \bar{G}_L$. Replacing $\varphi\delta$ by φ, we assume without loss of generality that $\varphi^{-1}(G\times H)_L\varphi/\mathcal{B} = G_L$. Hence $\varphi(g,h) = (g'_L\alpha(g), \omega_g(h))$ for some $g' \in G$ and $\alpha \in \mathrm{Aut}(G)$. Define $\psi : G\times H \to G\times H$ by $\psi(g,h) = (\alpha^{-1}(g), h)$. Then $\psi \in \mathrm{Aut}(G\times H)\cap(\mathcal{S}_G \wr \mathcal{S}_H)$. Furthermore, $\varphi^{-1}(G\times H)_L\varphi$ is conjugate to $(G\times H)_L$ in $J\wr K$ if and only if $\psi^{-1}\varphi^{-1}(G\times H)_L\varphi\psi$ is conjugate to $(G\times H)_L$ in $\psi^{-1}J\wr K\psi = \alpha J\alpha^{-1}\wr K$. Replacing $\alpha J\alpha^{-1}$ by J and $\varphi\psi$ by φ, we assume without loss of generality that $\alpha = 1$. As $\bar{G}_L \le J\wr K$, we may also assume that $g' = 1$. Hence $\varphi(g,h) = (g, \omega_g(h))$.

As any two regular subgroups of K isomorphic to H are conjugate in H, by standard arguments there exists $\delta \in 1_{\mathcal{S}_G} \wr K$ such that $\delta^{-1}\varphi^{-1}\bar{H}_L\varphi\delta|_B \le \bar{H}_L|_B$ for every $B \in \mathcal{B}$. By replacing φ with $\varphi\delta$, we assume that $\varphi^{-1}\bar{H}_L\varphi|_B \le \bar{H}_L|_B$ for every $B\in\mathcal{B}$, so that $\varphi(g,h) = (g, (h'_g)_L\alpha_g(h))$, where $h'_g \in H$, $\alpha_g \in \mathrm{Aut}(H)$. As $H_L|_B \in J\wr K$ for every $B \in \mathcal{B}$, we assume that $h'_g = 1_H$ for every $g \in G$, so that $\varphi(g,h) = (g, \alpha_g(h))$. Let $\ell \in G$, so that the map $(\ell, 1_H)_L \in (G\times H)_L$. Then

$$\begin{aligned}
\varphi^{-1}(\ell^{-1},1_H)_L^{-1}\varphi(\ell^{-1},1_H)_L(\ell,h) &= \varphi^{-1}(\ell,1_H)_L\varphi(1_G,h)\\
&= \varphi^{-1}(\ell,1_H)_L(1_G,\alpha_{1_G}(h))\\
&= \varphi^{-1}(\ell,\alpha_{1_G}(h))\\
&= (\ell,\alpha_\ell^{-1}\alpha_{1_G}(h)).
\end{aligned}$$

Setting k_ℓ to be the map defined by $k_\ell(\ell,h) = (\ell, \alpha_\ell^{-1}\alpha_{1_G})$ and $k_\ell(i,h) = (i,h)$ if $i \ne \ell$, we have that each $k_\ell \in J\wr K$ as $\varphi^{-1}(\ell^{-1},1_H)_L^{-1}\varphi(\ell^{-1},1_H)_L \in \mathrm{fix}_{J\wr K}(\mathcal{B})$. Setting $k = \Pi_{\ell\in G}k_\ell$, we see that $\varphi k(g,h) = (g, \alpha_{1_G}(h))$. It is then easy to see that $k^{-1}\varphi^{-1}(G\times H)_L\varphi k = (G\times H)_L$ and $k^{-1}\varphi^{-1}\bar{G}_L\varphi k = \bar{G}_L$. The result then follows. □

Theorem 6.6 *Let Γ_1 be a CI-digraph of H and Γ_2 be a CI-digraph of K, where H and K are abelian groups such that* $\gcd(|H|,|K|) = r$. *If every Sylow p-subgroup of both H and K is elementary abelian for every prime divisor p of r, then $\Gamma_1\wr\Gamma_2$ is a CI-digraph of $G = H\times K$.*

Proof. It is straightforward to verify that $(H\times K)_L = \bar{H}_L\times\bar{K}_L$ so that $\Gamma_1\wr\Gamma_2$ is a Cayley digraph of $H\times K$. Let $\varphi \in \mathcal{S}_G$ be such that $\varphi^{-1}G_L\varphi \le \mathrm{Aut}(\Gamma_1\wr\Gamma_2)$. We first show that there exists $\delta \in \mathrm{Aut}(\Gamma_1\wr\Gamma_2)$ such that $\langle G_L, \delta^{-1}\varphi^{-1}G_L\varphi\delta\rangle \le \mathrm{Aut}(\Gamma_1)\wr\mathrm{Aut}(\Gamma_2)$.

If $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\Gamma_1)\wr\mathrm{Aut}(\Gamma_2)$, we may take $\delta = 1$. We thus assume that $\mathrm{Aut}(\Gamma) \ne \mathrm{Aut}(\Gamma_1)\wr\mathrm{Aut}(\Gamma_2)$. By Theorem 5.7 we have that $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\Gamma'_1)\wr(\mathcal{S}_{rs}\wr\mathrm{Aut}(\Gamma'_2))$ for some $r,s > 1$, where Γ_1 and Γ_2 are appropriate wreath products. It is not then difficult to see that there exists $\delta \in \mathrm{Aut}(\Gamma)$ such that $\langle G_L, \delta^{-1}\varphi^{-1}G_L\varphi\delta\rangle \le (\mathrm{Aut}(\Gamma'_1)\wr\mathcal{S}_r)\wr(\mathcal{S}_s\wr\mathrm{Aut}(\Gamma'_2))$. We thus assume without loss of generality (replacing $\delta\varphi$ by φ) that $\varphi^{-1}G_L\varphi \le \mathrm{Aut}(\Gamma_1)\wr\mathrm{Aut}(\Gamma_2)$. Note that $\mathrm{Aut}(\Gamma_1)\wr\mathrm{Aut}(\Gamma_2)$ admits a unique complete block system $\mathcal{B}$ of $|G|$ blocks of size $|H|$ formed by the orbits of $\bar{K}_L$.

Let $r = p_1^{a_1} \cdots p_m^{a_m}$ be the prime power decomposition of r. Let P_i be a Sylow p_i-subgroup of H and Q_i be a Sylow p_i subgroup of K, $1 \leq i \leq m$. Then $H = H' \times \Pi_{i=1}^m P_i$, and $K = K' \times \Pi_{i=1}^m Q_i$, where $\gcd(|H'|, r) = 1$, $\gcd(|K'|, r) = 1$, and $\gcd(|H'|, |K'|) = 1$. Note that every Sylow subgroup of $G/(H' \times K')$ is elementary abelian by hypothesis. Now, as $\varphi^{-1} G_L \varphi \leq \mathrm{Aut}(\Gamma_1) \wr \mathrm{Aut}(\Gamma_2)$ and G is abelian, there exists $\hat{K} \leq G$ such that $\mathcal{B}$ is formed by the orbits of $\varphi^{-1}\hat{K}\varphi$. Note that $K' \leq \hat{K}$. Let $\hat{H} \leq G$ such that $\hat{H} \times \hat{K} = G$. Similarly, observe that $H' \leq \hat{H}$. Then $\varphi^{-1}\hat{H}_L \varphi/\mathcal{B} \leq \mathrm{Aut}(\Gamma_1)$ and $\varphi^{-1}\hat{H}_L^{-1}\varphi/\mathcal{B} \cong \varphi^{-1}\hat{H}_L\varphi$. As every Sylow subgroup of $G/(H' \times K')$ is elementary abelian and $K' \leq \hat{K}$, $H' \leq \hat{K}$, we have that $\hat{K} \cong K$ and $\hat{H} \cong H$. It is then easy to see that there exists $\alpha \in \mathrm{Aut}(G)$ such that $\alpha^{-1}\hat{K}\alpha = K$ and $\alpha^{-1}\hat{H}\alpha = H$. Note that Γ and $\varphi(\Gamma)$ are isomorphic by a group automorphism of G if and only if $\alpha^{-1}(\Gamma)$ and $\varphi(\Gamma)$ are isomorphic by a group automorphism of G. We may then, by replacing Γ with $\alpha^{-1}(\Gamma)$, assume that $\hat{K} = K$ and $\hat{H} = H$. But then $\varphi \in \mathcal{S}_H \wr \mathcal{S}_K$ so that by Lemma 6.5 G_L and $\varphi^{-1}G_L\varphi$ are conjugate in $\mathrm{Aut}(\Gamma_1) \wr \mathrm{Aut}(\Gamma_2)$. The result follows by Lemma 6.3. □

The following result is now immediate.

Corollary 6.7 *Let H and K be abelian groups such that every Sylow subgroup of H and K is elementary abelian. If Γ_1 is a CI-(di)graph of H and Γ_2 is a CI-(di)graph of K, then $\Gamma_1 \wr \Gamma_2$ is a CI-(di)graph of $H \times K$.*

Corollary 6.8 *Let H and K be abelian groups such that* $\gcd(|H|, |K|) = r$. *Then the following are equivalent:*

1. *whenever Γ_1 is a CI-digraph of H and Γ_2 is a CI-digraph of K, then $\Gamma_1 \wr \Gamma_2$ is a CI-digraph of $H \times K$,*
2. *if p divides r is prime, then every Sylow p-subgroup of H and K is elementary abelian.*

Proof. That (2) implies (1) follows directly from Theorem 6.6. To show that (1) implies (2), suppose that a Sylow p-subgroup of H or K is not elementary abelian for some prime $p|r$. Then G must contain a subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$. By Example 6.4, there a Cayley (di)graph Γ of $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$ which can be written as a wreath product of a Cayley (di)graph of $\mathbb{Z}_p$ and a Cayley (di)graph of $\mathbb{Z}_{p^2}$ and can also be written as a wreath product of a Cayley (di)graph of $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$. It is then not difficult to see that $|H| \cdot |K|/p^3$ disjoint copies of Γ is a Cayley (di)graph of $H \times K$ that is not a CI-(di)graph of $H \times K$ (as Γ is not a CI-digraph of $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$), a contradiction. □

It is possible that a stronger result is true. We would like to propose the following conjecture.

Conjecture 6.9 *Let H and K be abelian groups, Γ_1 a Cayley (di)graph of H, and Γ_2 a Cayley (di)graph of K. If Γ_1 is not a Cayley (di)graph of an abelian group with more elementary divisors than H and Γ_2 is not a Cayley (di)graph of an abelian group with more elementary divisors than K, then $\Gamma_1 \wr \Gamma_2$ is a CI-(di)graph of $H \times K$.*

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