Reconstructing permutations from cycle minors

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Abstract

The ith *cycle minor* of a permutation p of the set $\{1, 2, \ldots, n\}$ is the permutation formed by deleting an entry i from the decomposition of p into disjoint cycles and reducing each remaining entry larger than i by 1. In this paper, we show that any permutation of $\{1, 2, \ldots, n\}$ can be reconstructed from its set of cycle minors if and only if $n \geq 6$. We then use this to provide an alternate proof of a known result on a related reconstruction problem.

1 Background and Notation

For any positive integer n, let $[n]$ denote the set $\{1, 2, 3, \ldots, n\}$. Let S_n be the set of all permutations of $[n]$. Consider a permutation $p \in S_n$ and the corresponding sequence $p(1), p(2), \ldots, p(n)$, which we abbreviate $p_1 p_2 \ldots p_n$.

Definition. Let $n \geq 2$, $p \in S_n$ and $i \in [n]$. The ith *sequence reduction* of p, denoted $p \downarrow i$, is the permutation of $[n-1]$ formed by first deleting $p_k = i$ from the $p_1 p_2 \ldots p_n$ and then decreasing any number greater than i in the resulting sequence by 1.

For instance, $13425 \downarrow 3 = 1324$, because we first remove the 3 from 13425, leaving 1425, after which we decrease the 4 and the 5 by 1. We denote by $R(p)$ the set of all sequence reductions of p and by $M(p)$ the multiset of all sequence reductions of p. For example, $R(13425) = \{2314, 1234, 1324, 1342\}$ and $M(13425) = \{2314, 1234, 1324, 1324, 1342\}$.

Several reconstruction problems related to sequence reductions have been formulated. One such problem asks for which n can any permutation of length n be uniquely reconstructed from its set of sequence reductions. The analogous reconstruction problem for multisets of sequence reductions has also been investigated. Formally, these problems are equivalent to asking for which n is the restriction of R (or M, respectively) to S_n an injective map.

These problems are motivated by the famous Ulam Conjecture [7], which states that a graph with $n \geq 3$ vertices can be reconstructed from its multiset of induced $(n-1)$-vertex subgraphs. Harary [2] conjectured further that if $n \geq 4$, we can reconstruct a graph with n nodes from its set (ignoring multiplicity) of induced $(n-1)$-vertex subgraphs. The problem of reconstructing from sequence reductions is a natural analogue of the Ulam conjecture for permutations in the following sense. The inversion graph of a permutation $p \in S_n$ is the graph with vertices labeled $1, 2, \ldots, n$ and with an edge between vertices i and j with $i < j$ if and only if i is to the right of j in the sequence $p_1 p_2 \ldots p_n$. The $(n-1)$-vertex subgraphs of the inversion graph of a permutation p are isomorphic (ignoring the labels) to the inversion graphs of the corresponding sequence reductions of p.

The following theorem has been proven independently by Ginsburg [1], Ince [3], Raykova [5], and Smith [6].

Theorem 1.1. *Let $n \geq 5$ be a positive integer, and let p and q be two permutations in S_n. Then $R(p) = R(q)$ implies that $p = q$ (and thus $M(p) = M(q)$ implies that $p = q$).*

In addition, there are counterexamples for $n = 2$, 3, and 4. We have $M(3142) = M(2413)$, $M(312) = M(231)$, and $M(12) = M(21)$, and the same counterexamples hold for R.

In this paper we solve a natural variant on the problem of reconstructing permutations from their sequence reductions. We also use this variant to provide an alternate proof of Theorem 1.1.

1.1 Cycle Minors

Rather than considering a permutation $p \in S_n$ as a sequence consisting of the numbers in $[n]$, we consider the decomposition of p into disjoint cycles, i.e. a composition of disjoint cycles of the form $(i, p(i), p(p(i)), \ldots)$.

Definition. Let $n \geq 2$, $p \in S_n$ and $i \in [n]$. Then the ith *cycle minor* of p, denoted $p \Downarrow i$, is the permutation of $[n-1]$ formed by first deleting the entry i from the decomposition of p into disjoint cycles, and then subtracting 1 from any number greater than i.

For example, suppose $n = 9$ and $p = (1546)(279)(3)(8)$. The permutation p can be represented by the directed graph shown in Figure 1.

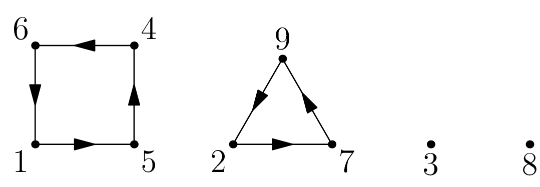

Figure 1: The directed graph associated with the permutation (1546)(279)(3)(8).

Then $p \Downarrow 5$ is the permutation $(145)(268)(3)(7)$, which has the directed graph shown in Figure 2, where new edges and labels are shown in red.

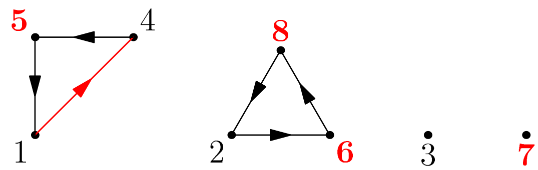

Figure 2: The cycle minor $(1546)(279)(3)(8) \Downarrow 5 = (145)(268)(3)(7)$.

Note that there are multiple ways of writing a given permutation as a product of disjoint cycles. For instance, the permutation $(21)(3)$ can also be written $(3)(12)$. By considering the representation of a permutation as a directed graph, it is clear that the definition of cycle minor is independent of such choices.

Definition. We denote the set of cycle minors of p by $\mathcal{C}(p)$ and the multiset of cycle minors of p by $\mathcal{M}(p)$.

We will also use the following conventions throughout the next section. If $p(a) = b$, we write $a \mapsto b$ *in* p. We say that a and b are *adjacent* in p if either $a \mapsto b$ or $b \mapsto a$ in p. We also say that a permutation p "contains" the k-cycle $(a_1 a_2 \ldots a_k)$, or the k-cycle is "in" p, if it appears in the decomposition of p into disjoint cycles.

2 Reconstruction from Cycle Minors

We now state our main result.

Theorem 2.1. *Suppose $n \geq 6$ and $p, q \in S_n$ such that $\mathcal{C}(p) = \mathcal{C}(q)$. Then $p = q$.*

Furthermore, there are counterexamples for $n = 2$, 3, 4, and 5:

$$\mathcal{C}((12)) = \mathcal{C}((1)(2)) = \{(1)\},$$

$$\mathcal{C}((123)) = \mathcal{C}((132)) = \{(12)\},$$

$$\mathcal{C}((13)(24)) = \mathcal{C}((14)(23)) = \{(2)(13), (1)(23), (3)(12)\},$$

$$\mathcal{C}((14253)) = \mathcal{C}((13524)) = \{(1423), (1342), (1324), (1243)\}.$$

The counterexample for $n = 5$ is the *only* pair of permutations in S_5 with the same set of cycle minors.

To prove Theorem 2.1, we first provide several preliminary lemmas.

Lemma 2.2. *Let $n \geq 3$. Suppose $p, q \in S_n$ and $\mathcal{C}(p) = \mathcal{C}(q)$. Then $\mathcal{C}(p \Downarrow n) = \mathcal{C}(q \Downarrow n)$ and $\mathcal{C}(p \Downarrow 1) = \mathcal{C}(q \Downarrow 1)$.*

Proof. First, notice that for any $p \in S_n$ and any $1 \leq i < j \leq n$, the permutations $p \Downarrow j \Downarrow i$ and $p \Downarrow i \Downarrow j-1$ are each formed by deleting i and j simultaneously from the cycle representation of p and subsequently subtracting 1 from any number that is between i and j and subtracting 2 from any number that is greater than j. Hence, we obtain

$$p \Downarrow i \Downarrow (j-1) = p \Downarrow j \Downarrow i \tag{2.1}$$

for any $1 \leq i < j \leq n$.

From the definition of $\mathcal{C}(p \Downarrow n)$ and using (2.1) repeatedly, we find

$$\begin{aligned}
\mathcal{C}(p \Downarrow n) &= \{p \Downarrow n \Downarrow i \mid 1 \leq i \leq n-1\} \\
&= \{p \Downarrow i \Downarrow (n-1) \mid 1 \leq i \leq n-1\} \\
&= \{p \Downarrow i \Downarrow (n-1) \mid 1 \leq i \leq n\} \\
&= \{r \Downarrow (n-1) \mid r \in \mathcal{C}(p)\}
\end{aligned}$$

Similarly, we have

$$\mathcal{C}(q \Downarrow n) = \{t \Downarrow (n-1) \mid t \in \mathcal{C}(q)\}$$

Since $\mathcal{C}(p) = \mathcal{C}(q)$ by assumption, it follows that $\mathcal{C}(p \Downarrow n) = \mathcal{C}(q \Downarrow n)$. A similar argument shows that $\mathcal{C}(p \Downarrow 1) = \mathcal{C}(q \Downarrow 1)$. □

Lemma 2.3. *Suppose $n \geq 2$ and $p, q \in S_n$ such that $p \Downarrow 1 = q \Downarrow 1$ and $p \Downarrow n = q \Downarrow n$. Then one of the following is true:*

(i) $p = q$.

(ii) $p = (1n) \circ q \circ (1n)$ *and either* $p(1) = n$ *or* $p(n) = 1$*; in other words,* 1 *and* n *are adjacent in* p *and interchanging them results in* q.

(iii) 1 *and* n *are fixed points of* p, $(1n)$ *is a* 2*-cycle in* q *and* $p(i) = q(i)$ *for all* $i \neq 1, n$.

(iv) 1 *and* n *are fixed points of* q, $(1n)$ *is a* 2*-cycle in* p *and* $p(i) = q(i)$ *for all* $i \neq 1, n$.

Proof. Let $s = p \Downarrow 1 = q \Downarrow 1$ and $t = p \Downarrow n = q \Downarrow n$. We consider several cases.

Case 1. Suppose that $n-1$ is a fixed point of s.

Let $m = p(n)$, so that $n \mapsto m$ in p, and assume $m \notin \{1, n\}$. Then $(n-1) \mapsto (m-1)$ in $p \Downarrow 1 = s$, which is a contradiction. Thus either $n \mapsto 1$ or $n \mapsto n$ in p. Similarly, if $k \mapsto n$ in p then $k = 1$ or $k = n$. Thus either n is a fixed point of p or $(1n)$ is a 2-cycle in p, and by an analogous argument, the same holds for q.

If n is a fixed point of both p and q, we see that p and q are otherwise identical since $p \Downarrow n = q \Downarrow n$. Thus (i) is satisfied.

If n is a fixed point of one of the permutations, say p, and $(1n)$ is a 2-cycle in q, then since 1 is a fixed point of $q \Downarrow n = p \Downarrow n$, the element 1 must be a fixed point of p as well. The remaining cycles are unchanged by removing n from p or q to form t, and thus either (iii) or (iv) is satisfied.

Finally, if both p and q contain the 2-cycle $(1n)$, then again the remaining cycles are unchanged by removing n from p or q to form t, and so p and q satisfy (i).

Therefore, if $n-1$ is a fixed point of s, p and q satisfy one of (i), (iii), or (iv).

Case 2. Suppose 1 is a fixed point of t.

Let $m = p(1)$, so that $1 \mapsto m$ in p, and assume $m \notin \{1, n\}$. Then $1 \mapsto m$ in $p \Downarrow n = t$, which is a contradiction. Thus either $1 \mapsto 1$ or $1 \mapsto n$ in p. Similarly, if $k \mapsto 1$ in p then $k = 1$ or $k = n$. Thus either 1 is a fixed point of p or $(1n)$ is a 2-cycle in p, and by an analogous argument, the same holds for q.

If 1 is a fixed point of both p and q, we see that p and q are otherwise identical since $p \Downarrow 1 = q \Downarrow 1$. Thus (i) is satisfied.

If 1 is a fixed point of one of the permutations, say p, and $(1n)$ is a 2-cycle in the other, q, then since $n-1$ is a fixed point of $q \Downarrow 1 = p \Downarrow 1$, the element n must be a fixed point of p as well. The remaining cycles are unchanged by deleting n from p or from q to form t, and thus either (iii) or (iv) is satisfied.

Finally, if both p and q contain the 2-cycle $(1n)$, then again the remaining cycles are unchanged by removing n from p or q to form t, and so p and q satisfy (i).

Therefore, if 1 is a fixed point of t, p and q satisfy one of (i), (iii), or (iv).

Case 3. Suppose $n-1$ is not a fixed point of s and 1 is not a fixed point of t.

Let $a = t^{-1}(1)$ and $b = t(1)$, so that $a \mapsto 1 \mapsto b$ in t. Since 1 is not a fixed point of t, neither a nor b can be equal to 1. Recall that t can be formed by deleting n from p or from q in cycle notation. It follows that either $a \mapsto 1 \mapsto b$, $a \mapsto 1 \mapsto n \mapsto b$, or $a \mapsto n \mapsto 1 \mapsto b$ in p, and the same is true of q.

Let $c = s^{-1}(n-1) + 1$ and $d = s(n-1) + 1$, so that $(c-1) \mapsto (n-1) \mapsto (d-1)$ in t. Since $n-1$ is not a fixed point of s, neither c nor d can be equal to n. Recall that s can be formed by deleting 1 from p or from q in cycle notation. It follows that either $c \mapsto n \mapsto d$, $c \mapsto 1 \mapsto n \mapsto d$, or $c \mapsto n \mapsto 1 \mapsto d$ in p, and the same is true of q.

Notice that $a, b \leq n-1$ and $c, d \geq 2$ since $s, t \in S_{n-1}$. Recall that neither a nor b is equal to 1 and neither c nor d is equal to n. Thus $a, b, c, d \notin \{1, n\}$.

Suppose 1 and n are not adjacent in p. Then $c \mapsto n \mapsto d$ in p. If 1 and n are adjacent in q, then $c \mapsto 1$ in $q \Downarrow n = t$, whereas $c \mapsto d$ in $p \Downarrow n = t$, which is a contradiction since $d \neq 1$. Thus 1 and n are not adjacent in q. It follows that $c \mapsto n \mapsto d$ in q. Since $p \Downarrow n = q \Downarrow n$, it follows that $p = q$, and we are in case (i).

If instead 1 and n are adjacent in p, then since 1 and n have distinct images under p and under p^{-1}, it follows that $a = c$ and $b = d$. Thus 1 and n are adjacent in q as well, for otherwise $a \mapsto 1 \mapsto b$ and $a \mapsto n \mapsto b$ in q, which is impossible since $a, b \notin \{1, n\}$. Thus 1 and n are adjacent in p and q, and since $p \Downarrow n = q \Downarrow n$, either $p = q$ or p can be formed by reversing the positions of 1 and n in q. Hence, either (i) or (ii) is satisfied, which completes the proof. □

We now have the tools required to prove Theorem 2.1.

Proof. We use induction on $n \geq 6$. The base case, $n = 6$, has been verified by computer, by checking that the 720 permutations in S_6 have distinct sets of cycle minors.

Let $n \geq 7$ and assume that for any $r, s \in S_{n-1}$, $\mathcal{C}(r) = \mathcal{C}(s)$ implies that $r = s$. Let $p, q \in S_n$, and assume $\mathcal{C}(p) = \mathcal{C}(q)$. Then $\mathcal{C}(p \Downarrow n) = \mathcal{C}(q \Downarrow n)$ and $\mathcal{C}(p \Downarrow 1) = \mathcal{C}(q \Downarrow 1)$ by Lemma 2.2. It follows from the inductive hypothesis that $p \Downarrow 1 = q \Downarrow 1$ and $p \Downarrow n = q \Downarrow n$.

Now, assume $p \neq q$. Then by Lemma 2.3, either $(1n)$ is a 2-cycle of p and 1 and n are fixed points of q (or vice versa) and p and q are otherwise identical, or 1 and n are adjacent in p and interchanging them results in q.

Suppose that the former is true, and without loss of generality suppose $(1n)$ is a 2-cycle of p and 1 and n are fixed points of q. Let a be an integer between 2 and $n-1$ inclusive. Then in $p \Downarrow a$, the elements 1 and $n-1$ form a 2-cycle. However, for any $b \in [n]$, either 1 or $n-1$ (possibly both) is a fixed point of $q \Downarrow b$. Thus $p \Downarrow a$ is an element of $\mathcal{C}(p)$ that is not an element of $\mathcal{C}(q)$, which is a contradiction.

Suppose instead that 1 and n are adjacent in p and interchanging them results in q. Without loss of generality, suppose further that $1 \mapsto n$ in p and $n \mapsto 1$ in q. If 1 and n form a 2-cycle in p then $p = q$, which is a contradiction. Therefore 1 and n are in a cycle of length at least 3.

Suppose one of p and q, say q, is an n-cycle. Then $p = (1n) \circ q \circ (1n)$ is an n-cycle as well. Thus either p and q are both n-cycles or neither is an n-cycle. We consider these two cases separately.

Case 1. Suppose neither p nor q is an n-cycle. Let k be the length of the cycle in p (and hence the length of the cycle in q) containing 1 and n. Let i be an element of p that is not in the cycle containing 1 and n, and let $r = p \Downarrow i$. We show that r is not a cycle minor of q.

Notice that $1 \mapsto (n-1)$ in a cycle of length k of r. Suppose $q \Downarrow i$ is a cycle minor of q for which 1 and $n-1$ occur in a cycle of length k. If i is a member of the k-cycle in q containing 1 and n, then either 1 or $n-1$ occurs in a cycle of length $k-1$ in $q \Downarrow i$, which is a contradiction. Hence i is not a member of the k-cycle of q containing 1 and n. Since $n \mapsto 1$ in q, it follows that $(n-1) \mapsto 1$ in $q \Downarrow i$.

Thus, any cycle minor of q having 1 and $n-1$ in a cycle of length k has $n-1 \mapsto 1$. Thus r cannot be a cycle minor of q, and we have a contradiction.

Case 2. Suppose p and q are both n-cycles. Then for any $i \notin \{1, n\}$, we have $1 \mapsto (n-1)$ in $p \Downarrow i$, but $(n-1) \mapsto 1$ in $q \Downarrow i$. Since $\mathcal{C}(p) = \mathcal{C}(q)$, either $q \Downarrow 1$ or $q \Downarrow n$ must have $1 \mapsto (n-1)$.

Suppose that $1 \mapsto (n-1)$ in $q \Downarrow 1 = p \Downarrow 1$. Then since $n \mapsto 1$ in q and $1 \mapsto n$ in p, it follows that $2 \mapsto n$ in q and $2 \mapsto 1 \mapsto n$ in p. Thus $1 \mapsto (n-1)$ in $p \Downarrow i$ for any $i \neq n$. Since $\mathcal{C}(p) = \mathcal{C}(q)$ and there are elements of $\mathcal{C}(q)$ in which $(n-1) \mapsto 1$, we must have $(n-1) \mapsto 1$ in $p \Downarrow n$. This is impossible because $2 \mapsto 1$ in $p \Downarrow n$, and $n-1 > 2$. Thus, we have a contradiction.

Suppose instead that $1 \mapsto (n-1)$ in $q \Downarrow n = p \Downarrow n$. Then since $n \mapsto 1$ in q and $1 \mapsto n$ in p, it follows that $1 \mapsto n-1$ in q and $1 \mapsto n \mapsto n-1$ in p. Thus $1 \mapsto (n-1)$ in $p \Downarrow i$ for any $i \neq 1$. Since $\mathcal{C}(p) = \mathcal{C}(q)$ and there are elements of $\mathcal{C}(q)$ in which $(n-1) \mapsto 1$, we must have $(n-1) \mapsto 1$ in $p \Downarrow 1$. But $n-1 \mapsto n-2$ in $p \Downarrow 1$, and $n-2 > 1$. Thus, we have a contradiction.

Having found a contradiction in all cases, we conclude that $p = q$. It follows by induction that for all $n \geq 6$, if $p, q \in S_n$ and $\mathcal{C}(p) = \mathcal{C}(q)$, then $p = q$. □

Corollary 2.4. *Suppose $n \geq 5$ and $p, q \in S_n$ such that $\mathcal{M}(p) = \mathcal{M}(q)$. Then $p = q$.*

Proof. If $\mathcal{M}(p) = \mathcal{M}(q)$ then $\mathcal{C}(p) = \mathcal{C}(q)$, so if $n \geq 6$ then $p = q$ by Theorem 2.1. For $n = 5$, we only need to verify that $\mathcal{M}((14253)) \neq \mathcal{M}((13524))$ since this is the only pair of permutations in S_5 for which reconstruction from the set of cycle minors fails, and indeed this is the case. $\square$

The permutations given above for $n = 2$, 3, and 4 that have the same set of cycle minors also have the same multisets of cycle minors, so 5 is the smallest value of n greater than 1 for which permutations can be reconstructed from their multisets of cycle minors.

3 Application to Sequence Reductions

Theorem 2.1 can be applied to the problem of reconstructing permutations from their sequence reductions.

Definition. Let $p = p_1 p_2 \dots p_n$ be a permutation, written in sequence form. Then the *associated cyclic permutation* of p, denoted $\hat{p}$, is the permutation having cycle representation $(p_1 p_2 \dots p_n)$.

For example, if $p = 31524$ (as a sequence), then $\hat{p} = (31524)$, the cyclic permutation which sends 3 to 1, 1 to 5, etc.

The following corollary to Theorem 2.1 provides a natural link between cycle minors and sequence reductions.

Corollary 3.1. *Suppose $n \geq 6$, $p, q \in S_n$ and $\mathcal{C}(\hat{p}) = \mathcal{C}(\hat{q})$. If there exists an index i such that $p(i) = q(i)$, then $p = q$.*

Proof. If $\mathcal{C}(\hat{p}) = \mathcal{C}(\hat{q})$ then $\hat{p} = \hat{q}$, meaning p and q are cyclic permutations of one another. In other words, there exists a positive integer k such that $p = q \circ (123 \dots n)^k$.

Suppose i is such that $p(i) = q(i)$. Note that $p(i) = q \circ (123 \dots n)^k(i) = q(i+k) = q(i)$, where the index $i+k$ is taken modulo n. Since q is a bijection, it follows that k is divisible by n, and so $p = q$. $\square$

Recall that $R(p)$ denotes the set of all sequence reductions of p. We now provide an alternate proof of Theorem 1.1 using Corollary 3.1, and the fact, shown by Ginsburg [1] and Ince [3], that the position of n in p is uniquely determined by $R(p)$ for $n \geq 5$.

Theorem 1.1. Let $n \geq 5$ and $p, q \in S_n$. If $R(p) = R(q)$, then $p = q$.

Proof. The case $n = 5$ has been verified by a computer search. Suppose $n \geq 6$ and $p, q \in S_n$ are such that $R(p) = R(q)$. Notice that $\hat{p} \Downarrow i = \widehat{p \downarrow i}$, so we have

$$\begin{aligned} \mathcal{C}(\hat{p}) &= \{\hat{p} \Downarrow i \mid 1 \leq i \leq n\} \\ &= \{\widehat{p \downarrow i} \mid 1 \leq i \leq n\} \\ &= \{\hat{s} \mid s \in R(p)\} \\ &= \{\hat{s} \mid s \in R(q)\} \\ &= \mathcal{C}(\hat{q}). \end{aligned}$$

Since $n \geq 5$, the position of n is the same in p and q by the result of Ginsburg and Ince, so by Corollary 3.1, we have $p = q$ as desired. □

4 Cycle k-minors

We have shown that a permutation in S_n can be reconstructed from its set of cycle minors for $n \geq 6$. The analogous result for sequence reductions has been extended to the case of *k-reductions*, where a k-reduction of a permutation p is a permutation that is the result of k successive sequence reductions starting from p. Reconstruction of permutations from their k-reductions has been studied in [3], [5], and [6].

Ince [3] and Raykova [5] have shown that for any k, permutations of n can be reconstructed from their sets of k-reductions whenever n is sufficiently large, and we ask if a similar theorem holds for cycle minors. We define a cycle k-minor as follows.

Definition. Let k and n be positive integers with $k < n$ and let $p \in S_n$. A *cycle k-minor* of p is any permutation in S_{n-k} formed by taking k successive cycle minors of p.

Alternatively, to form a cycle k-minor we delete some k numbers of the cycle representation of p and then replace the remaining numbers with the numbers $1, 2, \ldots, n-k$ so as to preserve the ordering. Let $\mathcal{C}_k(p)$ and $\mathcal{M}_k(p)$ be the set and multiset of cycle k-minors of p. The following conjecture is a natural generalization of Theorem 2.1.

Conjecture 4.1. *For any positive integer k, there exists a positive integer N_k such that for all $n \geq N_k$, permutations in S_n can be reconstructed from their sets of cycle k-minors.*

From computer calculations, it seems that we can reconstruct permutations from their set of cycle 2-minors if and only if $n \geq 7$, and from their set of cycle 3-minors if and only if $n \geq 10$.

For sufficiently large n, we can uniquely determine the sizes of the cycles of a permutation in S_n from its set of cycle k-minors as follows. A *partition* λ of a positive integer n is sequence $\lambda_1, \lambda_2, \ldots, \lambda_m$ of positive integers which satisfy $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ and $\sum_{i=1}^{m} \lambda_i = n$.

We can associate each permutation $p \in S_n$ having cycles of length $\lambda_1, \lambda_2, \ldots, \lambda_m$ in non-increasing order with this partition λ of n. For example, permutations in S_8 having decomposition into disjoint cycles of the form $(abc)(def)(gh)$ are associated with the partition $3, 3, 2$ of 8.

Recall that the conjugacy classes of S_n consist of all permutations having the same associated partition. Let λ be a partition of n, and let μ be a partition of $n-k$. We say that μ is a *k-minor* of λ if $\mu_i \leq \lambda_i$ for all i.

For any positive integer m, let $\rho(m)$ denote the smallest divisor of m that is greater than or equal to $\sqrt{m}$. In [4], we show the following.

Theorem 4.2. *Let n and k be positive integers with $k < n$. For $n \notin \{5, 12, 21, 32\}$, define*

$$g(n) = \min_{0 \leq t \leq n} \rho(n+2-t) - 2 + t$$

and also define $g(5) = 1$, $g(12) = 3$, $g(21) = 5$, $g(32) = 7$. Then partitions of n can be reconstructed from their sets of k-minors if and only if $k \leq g(n)$.

Clearly, the partition associated with a cycle k-minor of a permutation p is a k-minor of the partition associated with p. Thus we have the following corollary to Theorem 4.2.

Corollary 4.3. *The conjugacy class of a permutation can be reconstructed from its set of cycle k-minors whenever $k \leq g(n)$.*

In the case $k = 1$, this is not sufficient to reconstruct the permutation as well, since we can reconstruct partitions of $n \geq 3$ from their 1-minors, whereas we require $n \geq 6$ in order to reconstruct permutations from their cycle 1-minors. Nevertheless, this may be a useful intermediate step in settling Conjecture 4.1 and finding the values N_k.

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