The $\{4, 5\}$ isogonal sponges on the cubic lattice

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Submitted: Aug 28, 2008; Accepted: Feb 4, 2009; Published: Feb 13, 2009
Mathematics Subject Classifications: 52B70, 05B45, 51M20

Abstract

Isogonal polyhedra are those polyhedra having the property of being vertex-transitive. By this we mean that every vertex can be mapped to any other vertex via a symmetry of the whole polyhedron; in a sense, every vertex looks exactly like any other. The Platonic solids are examples, but these are bounded polyhedra and our focus here is on infinite polyhedra. When the polygons of an infinite isogonal polyhedron are all planar and regular, the polyhedra are also known as sponges, pseudopolyhedra, or infinite skew polyhedra. These have been studied over the years, but many have been missed by previous researchers. We first introduce a notation for labeling three-dimensional isogonal polyhedra and then show how this notation can be combinatorially used to find all of the isogonal polyhedra that can be created given a specific vertex star configuration. As an example, we apply our methods to the $\{4, 5\}$ vertex star of five squares aligned along the planes of a cubic lattice and prove that there are exactly 15 such unlabeled sponges and 35 labeled ones. Previous efforts had found only 8 of the 15 shapes.

1 Introduction

Convex polyhedra with *regular polygons* as faces and with *all vertices alike* have been known and studied since antiquity. The ones with all faces congruent are called regular or Platonic, while allowing different kinds of polygons as faces leads to uniform or Archimedean polyhedra. The aim of the present note is to study the analogues of these classical polyhedra obtained by replacing "convex" with "*acoptic*" (that is, self-intersection free) as well as admitting infinite numbers of faces. Such polyhedra have been studied in the past. The best known examples are the three regular *Coxeter-Petrie polyhedra* [7], in which six squares, four regular hexagons, or six regular hexagons meet at each vertex. However, even though these types of polyhedra have a long history of study, a consistent notation and descriptive terminology remains lacking. We hope to provide such a framework here. After having done so, it will then be possible to give a coherent review of the previous research, which we do in Section 5. (We note, however, that our methods apply equally well to non-acoptic polyhedra; our decision to limit ourselves here to acoptic polyhedra is done primarily for reasons of visual clarity: infinite polyhedra that are non-acoptic wrap around themselves in hopelessly confusing shapes. We definitely do not intend to imply that non-acoptic isogonal polyhedra are less mathematically valid for study. Indeed, one of us has reported on non-acoptic isogonal prismatoids in previous work [16]).

The meaning of "vertices that are all alike" can reasonably be interpreted in several ways. On the one hand, it can be taken as saying that the *star* of each vertex (that is, the family of faces that contain the vertex) is congruent to the star of every other vertex. Another possible interpretation is that the polyhedron has sufficiently many *symmetries* (geometric isometries) to make sure that every vertex star can be mapped to any other vertex star by a symmetry of the whole polyhedron. This is the definition of an *isogonal* polyhedron. One can hazard to guess that the ancients had the former meaning in mind, while the isogonality condition is frequently imposed in more recent discussions. (There are other interpretations as well, but they are not relevant to our present inquiry.) Although the two concepts of "alike" are logically distinct, they lead to the same family of five regular (Platonic) polyhedra. (We note that here, and throughout the sequel, we consider two polyhedra as being the same if one can be obtained from the other by a similarity transformation.) But for polyhedra often called "Archimedean" or "uniform" the situation is different. Requiring that the vertices form one orbit under symmetries (uniform polyhedra) yields one polyhedron fewer than if only congruence of stars is required (Archimedean polyhedra); the "additional" one is the *pseudorhombicuboctahedron*, also known as "Miller's mistake." (Many presentations commit the error of conflating the two meanings [17].) For *infinite* acoptic polyhedra with regular polygons as faces, the difference between the two definitions is analogous to that between *finite* uniform and Archimedean polyhedra, but in the infinite case the two notions entail even greater differences than in the finite case.

In order to make our exposition precise we need to introduce several concepts and an appropriate notation.

Platonic polyhedra are those with congruent regular convex polygons as faces, and congruent vertex stars. The family of all such polyhedra having p-gons as faces and q faces in each vertex star will be denoted by $\mathcal{P}(p,q)$. Here, and in the case of the other families we consider, if the specific value of p or q is not relevant to the discussion, we replace it with $\bullet$; for example, $\mathcal{P}(4,\bullet)$ denotes the family of all Platonic polyhedra with square faces. Additional restrictions, such as finite, infinite, or convex, can be indicated using the particular words. We shall be interested here in a subset of Platonic polyhedra, the *isogonal Platonic polyhedra.* The family of all such isogonal Platonic polyhedra will be denoted by $\mathcal{P}\{p,q\}$, and is a subfamily of $\mathcal{P}(p,q)$. In this notation, the three Coxeter-Petrie polyhedra are infinite members of $\mathcal{P}\{4,6\}$, $\mathcal{P}\{6,4\}$, and $\mathcal{P}\{6,6\}$, respectively.

Similarly, *Archimedean polyhedra* have, as faces, convex regular polygons of at least two kinds, and congruent vertex stars. Assuming the q faces in each vertex star have, in cyclic order, $p_1, p_2, \ldots, p_q$ sides, the family is denoted by $\mathcal{A}(p_1, p_2, \ldots, p_q)$. *Uniform polyhedra* form the subfamily $\mathcal{A}\{p_1, p_2, \ldots, p_q\}$ of $\mathcal{A}(p_1, p_2, \ldots, p_q)$, and consist of those polyhedra with all vertex stars equivalent by symmetries of the polyhedron. With these definitions the pseudorhombicuboctahedron is seen as belonging to $\mathcal{A}(3,4,4,4)$ but not to $\mathcal{A}\{3,4,4,4\}$. Note that since Platonic polyhedra must have all polygons alike and Archimedean polyhedra must have at least two different kinds, the two families are disjoint. This latter point simplifies the discussion.

To attain some familiarity with these definitions, let us consider the particular case of four squares incident with each vertex; that is, the family $\mathcal{P}(4,4)$. It is easy to verify that the only possible vertex stars consist of two pairs of coplanar squares, inclined at the common edges of the pairs at an angle τ to each other, where $-\pi < \tau < \pi$ (see Figure 1a). Moreover, the only vertex star possible for each of the vertices at the endpoints (the *distal* vertices) of the common edges just mentioned is a straight continuation of that edge, so that the polyhedron must contain two-way infinite strips of squares (Figure 1b), meeting at the angle τ. Hence the whole polyhedron is characterized by its intersection with a plane perpendicular to the common direction of all the infinite strips. A few examples with $\tau = \pi/2 = 90^\circ$ are shown in Figure 2. The polyhedra that correspond to (a) and (b) are in $\mathcal{P}\{4,4\}$, while the ones in (c) and (d) are in $\mathcal{P}(4,4)$, but not in $\mathcal{P}\{4,4\}$. In fact, it is easy to prove that the three polyhedra in (a) and (b) of Figure 2 are the only ones in $\mathcal{P}\{4,4\}$, but that infinite sequences of zeros and ones (using sequences of no more than two consecutive zeros or ones, to maintain the acoptic property) may be represented by Platonic polyhedra of the types in (d) – therefore $\mathcal{P}(4,4)$ contains infinitely many members.

The above short discussion described all polyhedra in which each vertex star contains four squares, with the angle $\tau = \pi/2 = 90^\circ$. For other values of τ it is equally easy to find a similar characterization of the possibilities; in particular, for $\tau = 0^\circ$ the only polyhedron that arises is the square tiling of the plane.

For this article we restrict attention to the case in which five squares are incident with each vertex and the polyhedra are isogonal; in other words, polyhedra in $\mathcal{P}\{4,5\}$. In addition, we also restrict our study to acoptic polyhedra; that is, those that have no self-intersections. Finally, the vertex stars with five squares that come into consideration are determined by the five dihedral angles at the edges where adjacent squares meet. These angles can be reduced to depend on only two parameters, but there seems to be no published account on the precise dependence, by which we mean the possible quintuples of resultant values, or the number of possibilities for a given set of parameters. We shall not consider the general situation, although our methods could deal with any particular

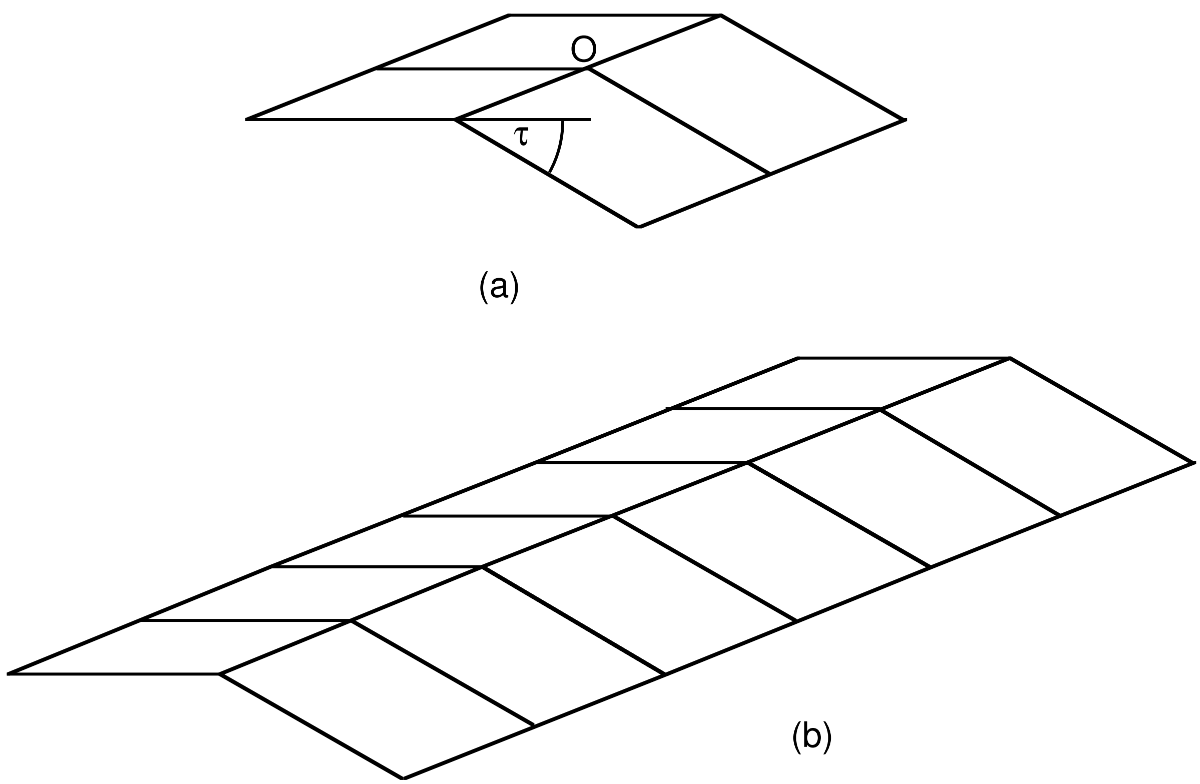


Figure 1: (a) A vertex star with four squares, and the characteristic angle τ. The angle τ in (a) is counted positive if the situation is as shown, and negative if the two coplanar squares are directed upward. (b) The faces adjacent to the two-edge segment of the vertex star form two infinite planar strips.

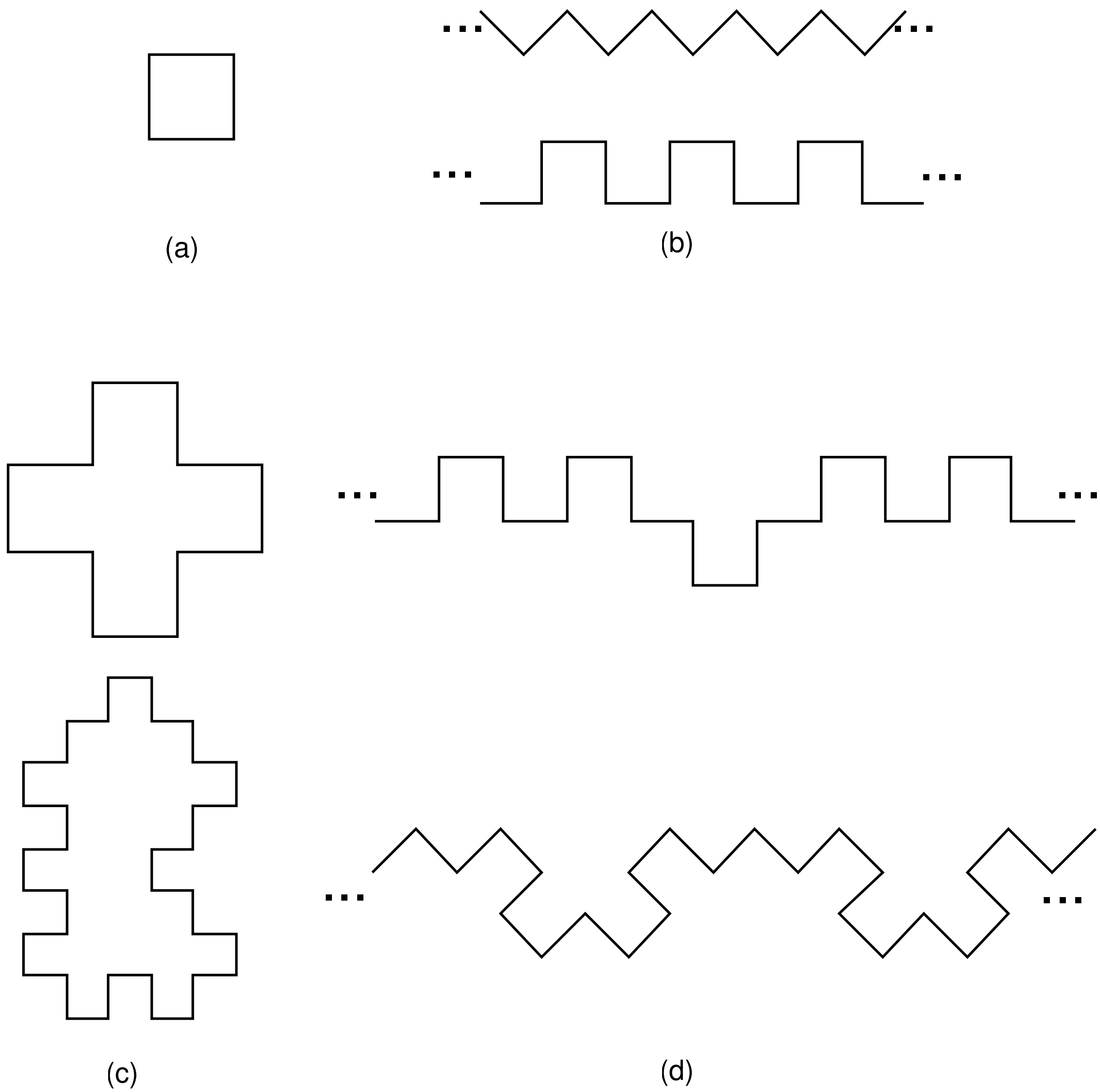


Figure 2: Cross-sections of uniform (in (a) and (b)) and Archimedean but not uniform (in (c) and (d)) polyhedra with four square faces in each vertex star, and with adjacent pairs of coplanar squares perpendicular to each other.

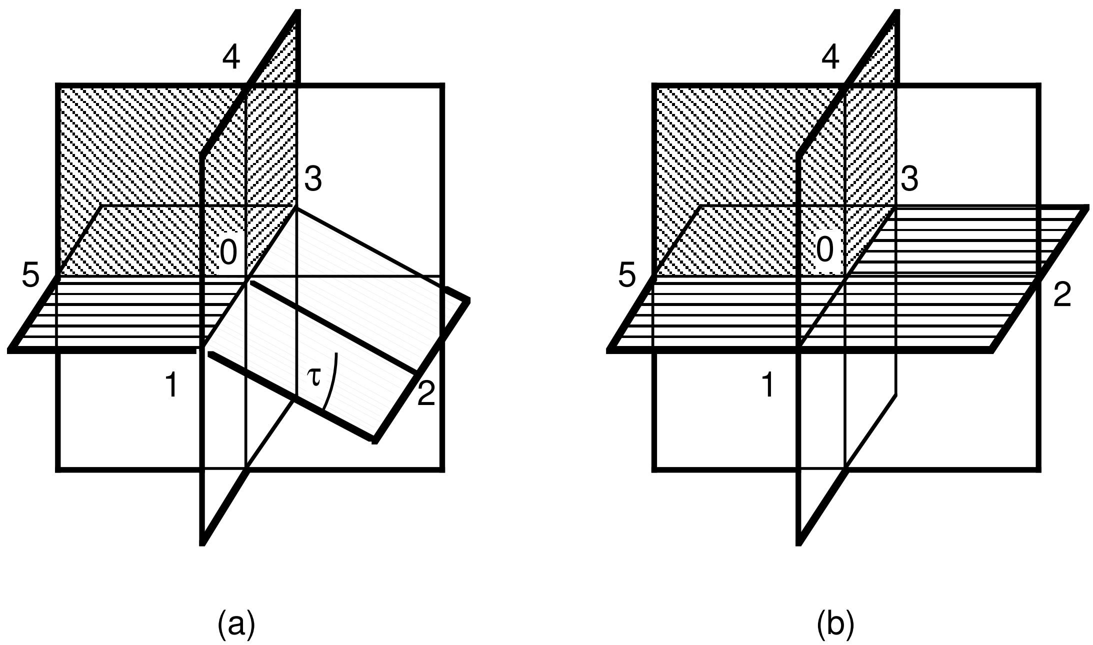


Figure 3: The angle τ in (a) is counted positive if the situation is as shown, and negative if the two coplanar squares are directed upward. In (b), we have $\tau = 0$.

case. By restricting one of the angles to 180°, the vertex star remains dependent on an angle τ, with $-\pi/2 < \tau < \pi$, as illustrated in Figure 3. We shall assume that $\tau = 0°$; in other words, that any two adjacent squares are either coplanar or enclose an angle of $\pi/2 = 90°$. This constitutes our third and final restriction on the vertex star we consider here. Equivalently, the vertices of the polyhedra we consider are at the points of the cubic (integer) lattice in 3-space, the faces are some of the squares of that lattice, and the edges have length 1; this explains the title of the article. It is easy to verify that the polyhedra we consider here must be periodic (repeatable by translations) in at least two independent directions. This is in contrast to the example in Figure 2(a), which is periodic in one direction only. The only reason the polyhedra are not all periodic in three dimensions is because some of them extend infinitely in only two dimensions.

In order to deal with the seemingly straightforward question of finding the different polyhedra possible under the rather strict limitations we impose, we must develop considerable machinery. Thus it seems justified to provide here a short explanation for the need of such elaborate tools. Our goals include finding how many different isogonal polyhedral shapes are possible under the restrictions that each vertex star contains five squares, adjacent squares being either coplanar or perpendicular. As we prove, there are precisely fifteen. However, we know of no *direct* way of finding them all, or of proving directly that there are no others. The difficulty of the task is best illustrated by the fact that neither of the two previous attempts (by Wachman *et al.* [25] in 1974 and by Wells [27] in 1977) came even close to this goal. It seems that – in close analogy to the situation concerning isogonal plane tilings – one has to proceed by a two-step approach. First, investigating a more general (essentially combinatorial) variant of the problem leads to an enumeration of possible "candidates" for the polyhedra we seek. Then each one of these *combinatorial* "candidate polyhedra" can be investigated as to its realizability by an actual *geometric* polyhedron. These steps are discussed in detail below.

2 Notation

We first describe how we encode by symbols the various polyhedra that we wish to consider. The notation explained here is appropriate for all types of isogonal polyhedra, as is the method for finding them that we will describe in the next section. In particular, even though our focus here (as well as almost all of the previously published research) is on sponges made up only of regular polygons, our notation and methods work equally well on isogonal polyhedra that contain non-regular polygons. As examples of the notation, though, we repeat that we are restricting attention to *infinite acoptic isogonal polyhedra*, having *square* faces, with *five* squares in each vertex star and with adjacent faces either *perpendicular or coplanar* (aligned with the cubic lattice). That is, that $\tau = 0°$ in the notation of Figure 3. For brevity, extending the terminology of [8] beyond purely regular polyhedra, we refer to infinite isogonal polyhedra with regular polygons for faces as *sponges*. Furthermore, simplifying the general notation of Section 1, if all faces are n-gons and k meet at each vertex, we shall denote them by the generic symbol $\{n, k\}$. Throughout this paper only, if $n = 4$ we shall also assume that the notation $\{4, k\}$ implies that the vertices are at points of the integer lattice. We note that not all members of $\mathcal{P}\{4, 5\}$ satisfy this condition.

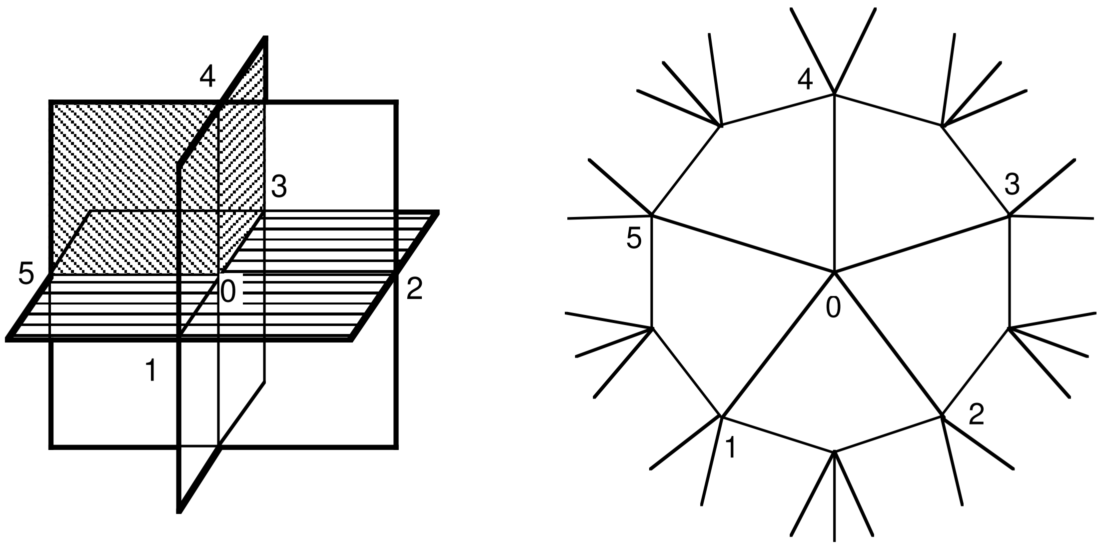
In the case of isogonal (and other) tilings of the plane (see [18], [20] section 6.3), it is convenient to introduce the concepts of *marked* tilings, and their *incidence symbols*. Analogously, it is useful to deal with *marked* (or *labeled*) *sponges* and their incidence symbols. This enables one to use combinatorial approaches to enumerate all marked sponges; then geometric considerations determine the enumeration of unmarked sponges, which constitute the polyhedral shapes. The notation here is an expansion of that used for planar tilings, which cannot cover the wealth of possibilities that arise in three dimensions.

We are concerned with acoptic polyhedra, and these are *orientable*. This means that each face has two sides (as does the entire sponge); we shall describe one of the sides as *red*, the other as *black*. The assumed isogonality of the sponges requires us to consider the isometries that may map one vertex star to another (or to itself). While there are multiple such isometries, some of which depend on the characteristics of the vertex star, three of them can be considered fundamental, with any others being constructible from the three basic ones. The first one is a reflection across a plane (not necessarily of symmetry); the second is a rotation around an axis through the central vertex (a turn); and the third is a rotation around an axis perpendicular to the axis through the central vertex (a flip). An example of a dependent (constructible) isometry, that could be called an "inversion" (turning inside out), would be where opposing pairs of edges emanating from the central vertex change places with each other. This isometry can only exist when the vertex star has an even number of edges, and can be constructed by combining a reflection and a flip. Of the three isometries, the turn and the flip are orientation-preserving (rigid *motions*), while the reflection is orientation-reversing (*mirror* isometry). On the other hand, the reflection and the turn are color-preserving, while the flip is color-reversing.

An *incidence symbol* for a sponge consists of two parts. The first part is the *vertex symbol*. This is a labeling of the edges of a chosen vertex star V that can be used to similarly label, in a consistent manner, the edges of *all* the vertex stars because of their equivalence due to isogonality. The labeling of the vertex star V can depend on whether or not there are symmetries of the sponge that map the vertex star V onto itself in a non-trivial way. It should be noted that there exist strategies other than the one described here for assigning vertex symbols to vertex stars that may produce different symbols. Some of these symbols may or may not be more intuitively representational of the vertex structure, and we make no claim that the method described here is superior. However, the method here can always be guaranteed to work. It should also be noted that the choice of starting edge and other arbitrary choices described below may also produce different symbols; however, these can always be shown to be mere equivalents of each other.

To begin the creation of a vertex symbol, we (arbitrarily) choose the red sides of the faces forming a vertex star V as the side of the vertex star to label. Next, again by convention, we choose the counterclockwise orientation around V on its red side as the direction of "positively increasing" edges. Then we (arbitrarily) select one edge of V as the first and label it a^+. In the case of the $\{4,5\}$ sponges considered here, we assume that the chosen edge is the one that corresponds to the edge 04 in Figure 4(a), and that we have chosen as the red side of the vertex star the side visible in that diagram. (When the vertex star exhibits symmetries, some of the arbitrary choices above may produce just such "natural" choices.) Proceeding counterclockwise around V we label the remaining edges b^+, c^+, d^+, and so on until all of the edges are labeled. Thus, the vertex symbol of the $\{4,5\}$ vertex V would be $a^+b^+c^+d^+e^+$. If V admits non-trivial symmetries, the labeling is modified so that all edges of V in the same orbit get the same label. In the case of the $\{4,5\}$ star (Figure 4(a)) only one non-trivial symmetry is possible, a reflection of the vertex star across the plane containing the edge 04 and bisecting the angle between the edges 01 and 02. This is incorporated into the vertex symbol as follows. If an edge labeled x^+ is mapped onto a different edge by a reflection, that edge is labeled x^-. If an edge labeled x^+ is mapped onto itself by a reflection, it is labeled x without any superscripts. Hence, in the case under consideration, the only other possible vertex symbol, besides $a^+b^+c^+d^+e^+$, is $a\ b^+c^+c^-b^-$ (ignoring equivalents due to different choices of starting edge).

Other symmetries of V (that are possible in some sponges) may require additional handling. In the case of a simple turn, the edge labels just begin again. For example, in the regular $\{4,6\}$ Coxeter-Petrie sponge the vertex star can be rotated two edges forward as an isometry, giving it two orbits, so the vertex symbol would be $a^+b^+a^+b^+a^+b^+$. If an edge $x^+/x^-/x$ can be mapped into a different one via a flip, the flipped edge is labeled



(a) (b)

Figure 4: The $\{4,5\}$ vertex star and the "flattened" diagram of its neighbors.

$x^{\wedge+}/x^{\wedge-}/x^{\wedge}$. Thus, the same (highly symmetric) $\{4,6\}$ vertex star above can ultimately be labeled $a\ a^{\wedge}a\ a^{\wedge}a\ a^{\wedge}$, which indicates all of its different kinds of symmetries. When two polygons in a vertex star have a 180° dihedral angle (they are coplanar), special situations are possible. As with reflection, where an edge x^+ mapped onto itself is labeled x, in the case of a flip that maps an edge onto itself $x^+/x^-/x$ and $x^{\wedge+}/x^{\wedge-}/x^{\wedge}$ are merged to create $x^{*+}/x^{*-}/x^*$. Finally, because now two degrees of freedom are present (reflection state and flip state), it is possible that a coplanar edge might be simultaneously both x^+ and $x^{\wedge-}$ but neither x^- nor $x^{\wedge+}$. In this case, the symbol '&' is used to represent this combination, so that $x^+/x^{\wedge-}$ together is represented as $x^{\&+}$. Similarly, $x^{\&-}$ represents the combination of x^- and $x^{\wedge+}$. Note that an edge can never have both reflective and non-reflective symmetry, but it can have both reflected and flipped symmetry; in such a case it would be labeled x^*.

The second part of the incidence symbol is the *adjacency symbol*. This expresses and records how the two labels that each edge receives (from the two vertex stars that contain it) are related. The adjacency symbol contains as many entries as are required to specify the adjacency for each distinct edge label in the vertex symbol. For example, if the vertex symbol were $a^+b^+c^+d^+e^+$, then five symbols would be required in the adjacency symbol; if the vertex symbol were $a\ b^+c^+c^-b^-$, then only three symbols would be required. Each label in the adjacency symbol represents the label given to its paired vertex symbol edge by the other vertex star incident with it. Given the definitions of the various vertex symbol edge notations, certain restrictions apply on which adjacency symbols may be legitimate for a specific vertex symbol. For example, in the case of the $\{4,5\}$ sponges we are considering, if the vertex symbol is $a^+b^+c^+d^+e^+$ this (along with a consideration of the dihedral angles involved) implies that a^+ must be paired with one of a^+, a^-, $b^{\wedge+}$, $b^{\wedge-}$, $e^{\wedge+}$, or $e^{\wedge-}$, which becomes the first entry in the adjacency symbol. Similarly, the second entry of the adjacency symbol that corresponds to b^+ must be one of $a^{\wedge+}$, $a^{\wedge-}$, b^+, b^-, e^+, or e^-. In each case the pairing must be consistent by isogonality, must be mutual, and must be sign and color (side) consistent. Thus, if an edge is labeled a^+ at one end and $b^{\wedge-}$ at the other end, then an edge with label a^- or $a^{\wedge+}$ at one end must have $b^{\wedge+}$ or b^- at the other, and similarly for the other cases. The third entry corresponds to the edge labeled c^+; it must be one of c^+, c^-, $c^{\wedge+}$, $c^{\wedge-}$, d^+, d^-, $d^{\wedge+}$, or $d^{\wedge-}$. The same possibilities are required for the fourth entry, which corresponds to the edge labeled d^+, while the fifth e^+ entry's possibilities must match those of the b^+ entry. The mutuality of the entries in the two parts of the incidence symbol implies that the letters in the adjacency symbol form a permutation of a, b, c, d, e.

On the other hand, if the vertex symbol is $a\ b^+c^+c^-b^-$, then the first entry in the adjacency symbol can only be a, while the other two entries must be among b^+ or b^-, and c^+, c^-, $c^{\wedge+}$, or $c^{\wedge-}$, respectively. The mutuality connects the second and third entries to the fourth and fifth entries, thus the last two entries are optional in the written symbol.

This completes the discussion of the notation used to specify isogonal polyhedra. Two major points derive from using this notation in a search for sponges. The first is that it allows an "identifier" to be assigned to a polyhedron that clearly distinguishes it from another polyhedron. One no longer needs to study photographs or diagrams to know whether two cited sponges are the same or not. The second, and more powerful, advantage is that every sponge can be assigned an incidence symbol, and there can only be a finite number of them for any particular vertex star. Thus by combinatorially compiling a list of all possible symbols, then checking each one to see if it corresponds to an actual sponge, a list of sponges can be produced that will then be known to be complete. As we discuss in our historical review, attempts made without using such a combinatorially labeled approach have often failed to find a complete set of sponges.

Therefore, as just stated, a list of all combinatorially possible symbols becomes the starting list of candidates for geometric realizability. However, the above conditions still permit a very large number of potential incidence symbols. This number can be drastically reduced by the observation illustrated in Figure 4(b) for the $\{4,5\}$ vertex star of Figure 4(a). It expresses the fact that the vertex stars adjacent to a central vertex star are also adjacent to each other in a circuit. This observation will be used below in a technique that screens and eliminates possible combinatorial candidates without having to fully consider their geometric constructability.

3 Methods

Our determination of the possible $\{4,5\}$ sponges was actually carried out in two different ways. In the first, using lots of sheets of paper with diagrams like the one in Figure 4(b), the different combinatorial candidate incidence symbols were determined by hand. The possibility of geometric realization was then explored by making cardboard models. The alternative determination was carried out using computers to investigate the possible incidence symbols and their geometric realizations in 3-dimensional space. Some readers may consider only the manual results described here as a proper proof, though the computer method was considerably faster and much easier. We first discuss the manual method, then the computer one. We repeat that the methods below, while described specifically for the $\{4,5\}$ sponges, are applicable to other types of isogonal polyhedra as well.

In order to explain the method of finding candidates for incidence symbols of $\{4,5\}$ sponges, we start by looking at the neighbors of a given vertex. "Flattening out" such a neighborhood as in Figure 4(b), we label the edges issuing from that vertex according to the vertex symbol $a^+b^+c^+d^+e^+$ near the vertex, and then first consider the possible labels at the vertex situated at the other (distal) end of the edge labeled a^+. (The choice for the order of considering the edge adjacencies is arbitrary; the order described here is simply the one we chose.) As mentioned earlier, this can be any label from among a^+, a^-, $b^{\wedge+}$, $b^{\wedge-}$, $e^{\wedge+}$, or $e^{\wedge-}$. We treat them one by one. In Figure 5(a) we have selected a^+, and this then determines the labels at all edges incident with that vertex; in order to avoid clutter, we show only the two labels (b^+ and e^+) that are relevant to the discussion. Next we need to select the labels for the distal ends of the two edges marked by a $\bullet$. Again there are several possibilities, and we choose to pursue here in detail only two. Selecting b^+ for the position on the left, the knowledge of the vertex symbol and of the mutuality of adjacency symbols determines all of the labels shown in Figure 5(b), with the labels at places indicated by a $\bullet$ again open to different choices. We shall pursue these other possibilities in Figure 6, but first we deal with the alternative choice of b^- on the left in Figure 5(a). As indicated in Figure 5(c), this choice immediately forces all of the other labels and we arrive at the adjacency symbol $a^+e^-d^+c^+b^-$, as a *candidate* for a sponge.

Returning to the original selection of b^+ in Figure 5(b), the edges marked by a $\bullet$ remain to be chosen. Excluding flipped vertices for the moment, there are four possible choices: c^+, c^-, d^+, and d^-. These are specified in the four parts of Figure 6. In each case we are left with a single conclusion. The choices in (a) and (d) do not lead to any adjacency symbols since there is no allowable way of selecting labels at positions marked by a $\bullet$. On the other hand, the choices in (b) and (c) lead to the adjacency symbol candidates $a^+e^+c^-d^-b^+$ and $a^+e^+d^+c^+b^+$.

Naturally, there are often additional multiple choices, but the number is never too large for a complete manual determination.

The above examples did not demonstrate flipping any of the vertex stars. To illustrate how this works, let us instead select $b^{\wedge-}$ as the first entry of the adjacency symbol. In Figure 7(a) we see that this forces several additional labels, until we reach the edges marked by a $\bullet$. Two of the possible choices are indicated in Figures 7(b) and (c), but both still leave undecided the edges marked by a $\bullet$. Further investigation shows that, using the first choice of Figure 7(b), the only possible completions are the candidate adjacency symbols $b^{\wedge-}a^{\wedge-}c^+d^{\wedge-}e^+$ and $b^{\wedge-}a^{\wedge-}c^{\wedge-}d^+e^+$. On the other hand, the choice in Figure 7(c) can be completed in four ways: $b^{\wedge-}a^{\wedge-}c^-d^+e^-$, $b^{\wedge-}a^{\wedge-}c^-d^-e^-$, $b^{\wedge-}a^{\wedge-}c^-d^{\wedge+}e^-$, and $b^{\wedge-}a^{\wedge-}c^-d^{\wedge-}e^-$. Including flipped vertices for the choices following from Figure 5(b)

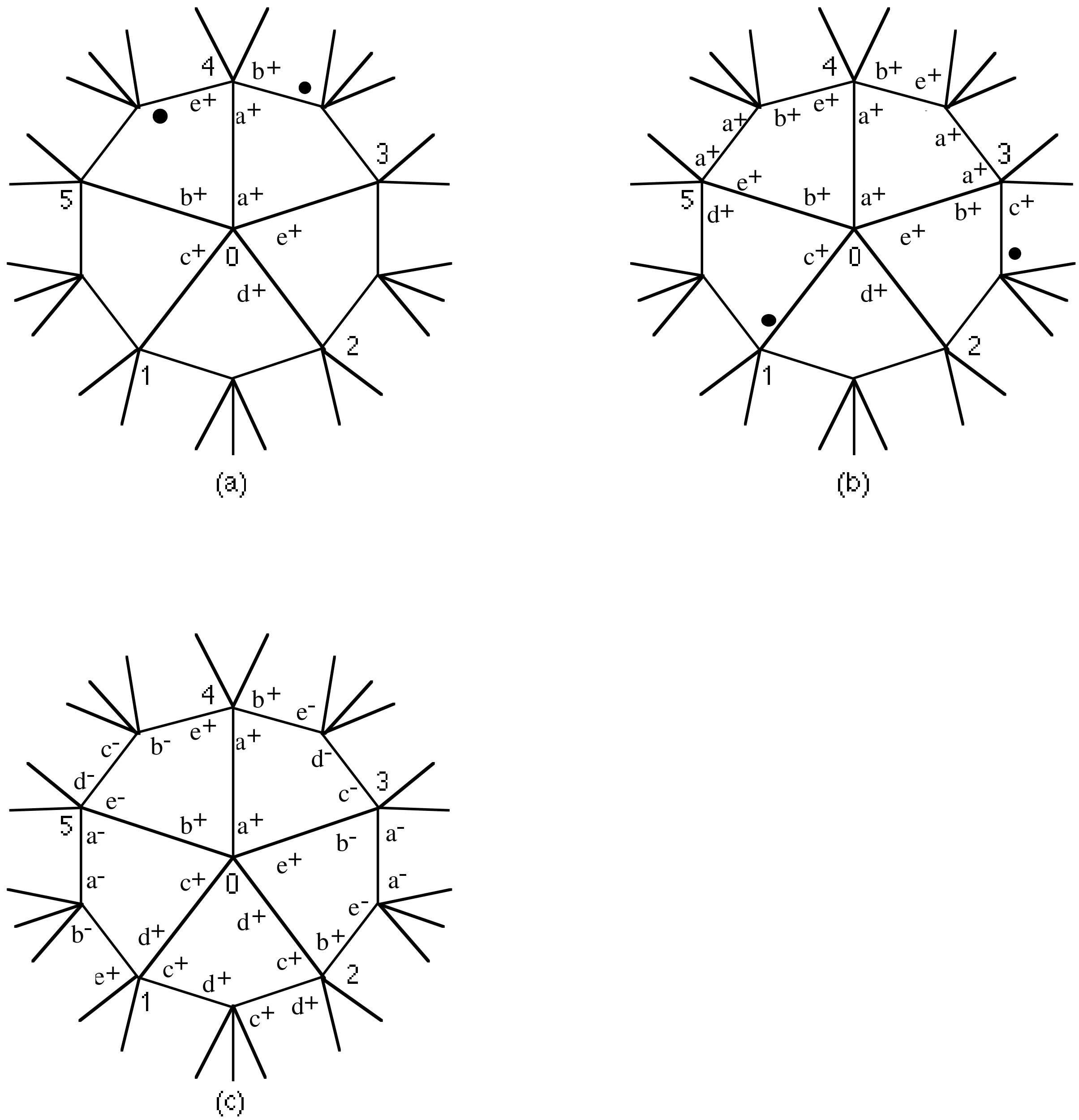


Figure 5: The first two steps of the elimination method. After the first choice in (a) of the edge adjacent to a^+, the choice of e^+ for the adjacent edge of b^+ leads to further choices in (b) while the choice of e^- immediately leads to a successful conclusion in (c). Note the reversed order of labels for reflected (minus) vertices compared to unreflected vertices.

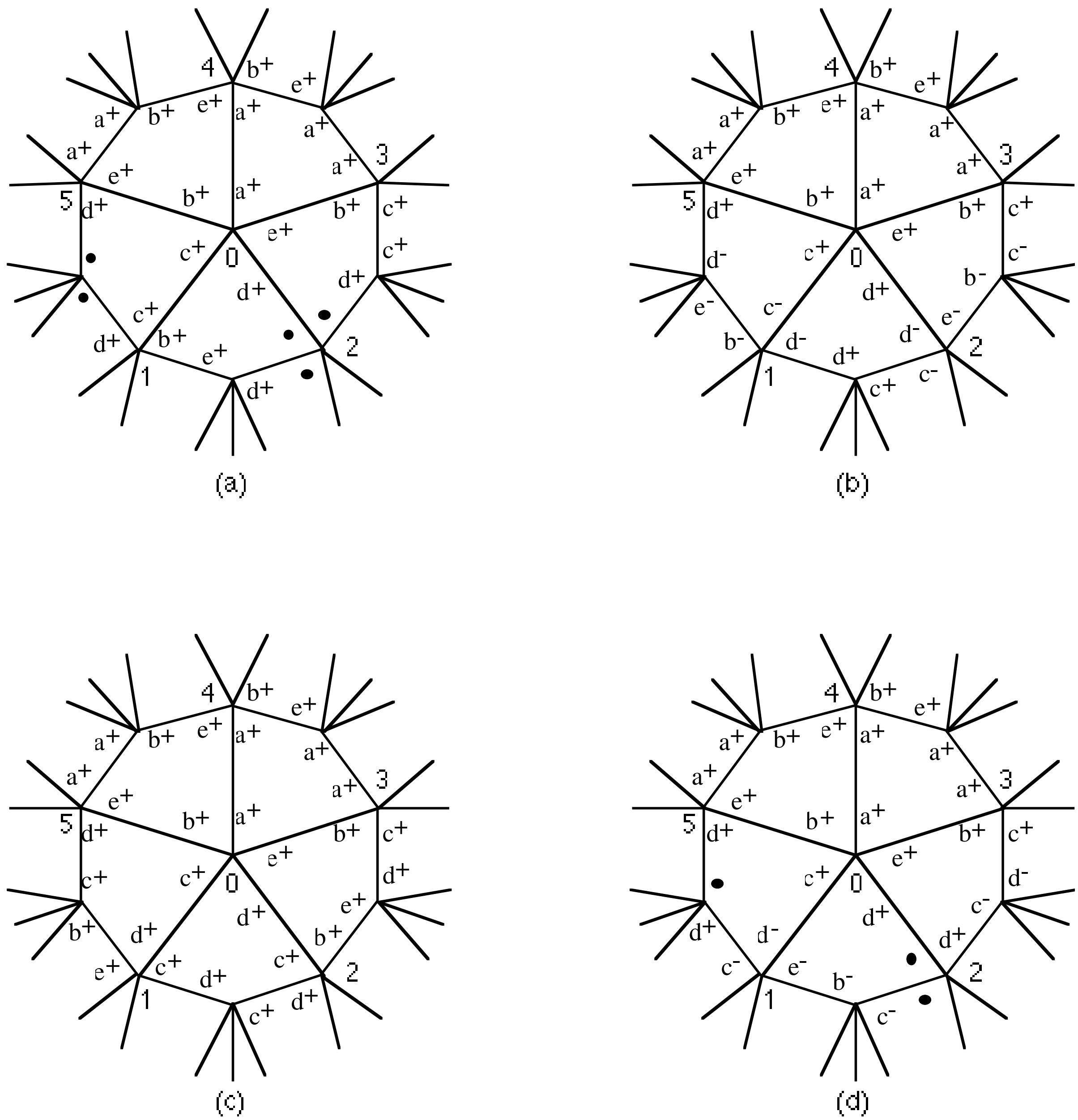


Figure 6: The four results of the choices starting from Figure 5(b). Figures (b) and (c) can be completed successfully while (a) and (d) cannot: in both of the latter cases, at least two different edges of the same vertex must be adjacent to d^+.

leads to the candidate symbols $a^+e^+c^{\wedge+}d^{\wedge+}b^+$ and $a^+e^+d^{\wedge-}c^{\wedge-}b^+$.

Several remarks need to be made at this time.

First, to the *counterclockwise* orientation we assumed for the vertex symbol starting with the red side of the vertex star, there corresponds the *clockwise* orientation of the black side of the vertex star. This explains the labels in Figure 7. (This is similar to the reversal of orientation also required for reflected vertex stars.)

Next, we note that several incidence symbols may differ only *inessentially*. Since mirror images of any sponge are considered as essentially the same, the directional orientation of the edges on the red side of the vertex star may be reversed; in general, this may result in a different adjacency symbol. Also, the side of the vertex star designated as red was chosen arbitrarily; hence another two symbols may be found for the same sponge. For example, if the adjacency edge choice above for the a^+ edge had been $e^{\wedge-}$ instead of $b^{\wedge-}$, then six candidate adjacency symbols would also have been found, but starting with $e^{\wedge-}$ instead of $b^{\wedge-}$. However, these would all be duplicates of the six found above, since they differ simply by the original choice of which edge to call b^+ (which direction to proceed around the central vertex) after starting from the same a^+ edge.

After all possibilities for symbols have been exhausted according to the process described above and all such inessential duplicates have been eliminated, a list of 36 candidate incidence symbols remain. These must next be tested via the use of cardboard models (or the computer program) to verify that each symbol represents a constructible polyhedron. It turns out that four of these 36 cannot be constructed (those with incidence symbols $a^+b^-c^+d^-e^-$, $a^+b^-c^{\wedge+}d^-e^-$, $a^+b^-c^{\wedge-}d^-e^-$, and $b^{\wedge-}a^{\wedge-}c^+d^{\wedge-}e^+$). As the latter symbol was one of the examples from above, we will continue with it to show how this test rejects the symbol.

We will consider the edges surrounding the square in space defined by the a^+ and c^+ edges emanating from the central vertex. (We state again that we know of no way to know in advance which edge(s) will cause a failure: all edges must be tested. For brevity, we have omitted the successful tests.) Starting with the c^+ edge, its adjacent vertex must also label that same edge as c^+, according to the adjacency symbol. The edge of that vertex parallel to the a^+ edge of the original vertex will also be a^+. A third vertex adjacent to this second a^+ edge must label its edge $b^{\wedge-}$, also according to the adjacency symbol. Finally, the edge of that third vertex that is parallel to the c^+ edge of the original vertex will be e^-. Next, starting around the square in space with the a^+ edge of the original vertex, its adjacent vertex must label that same edge as $b^{\wedge-}$, and the edge of that second vertex that is parallel to the c^+ edge of the original vertex will also be e^-. However, even though this final edge is simultaneously found to be e^- by following the path around the square from both directions, the two vertices that meet along this edge are not properly aligned: the two faces of each vertex star adjacent to this edge are not coincident. Instead, one vertex has been rotated 180° from the other and the polyhedron cannot be constructed. The "flattened" test is guaranteed to work because it follows a path around a known polygon that is part of the specified vertex star and links two consecutive edges emanating from the central vertex. However, in an instance such as the above where the two edges are not consecutive, it is first of all not

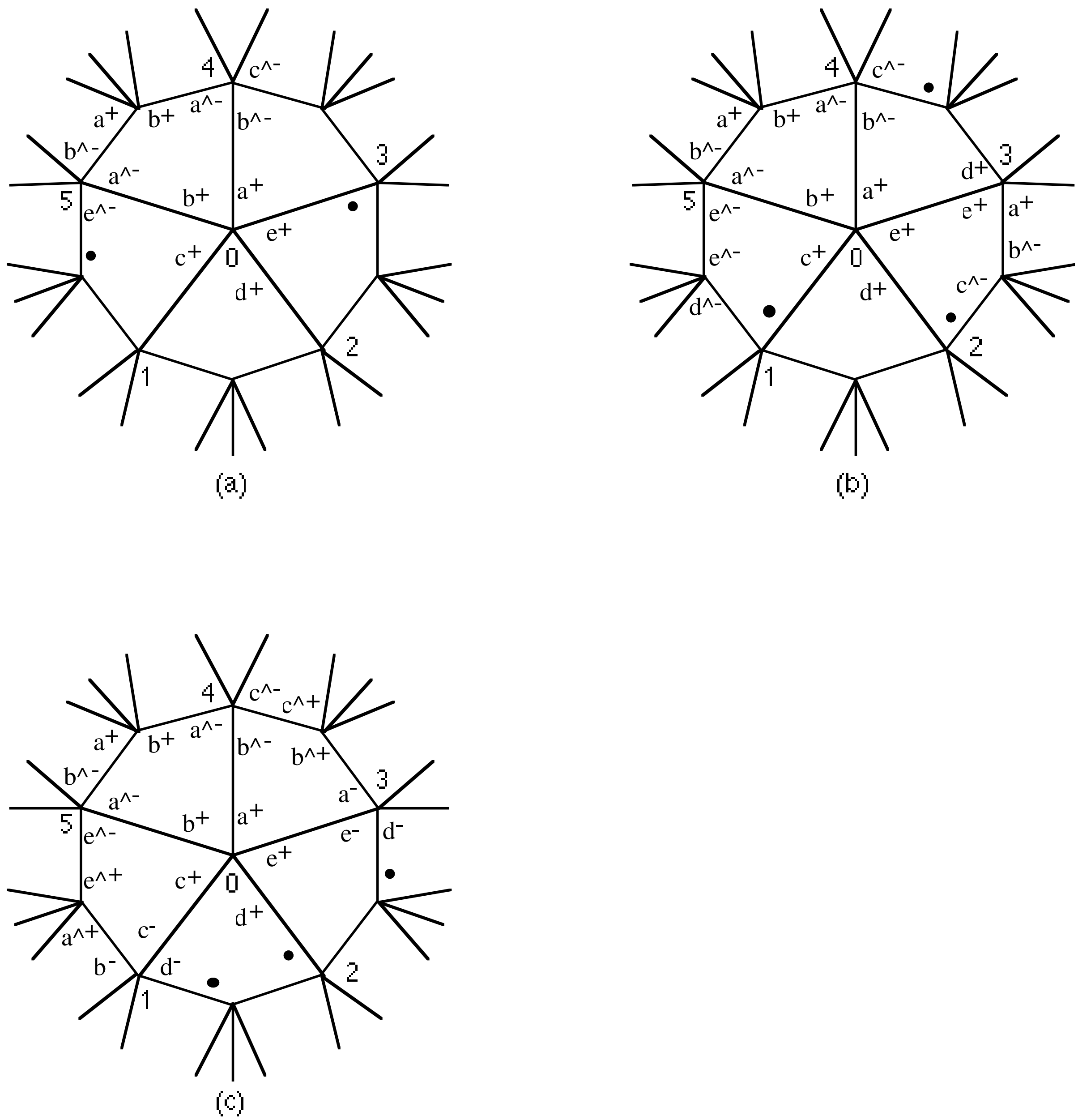


Figure 7: Demonstration of the method when choosing a flipped vertex adjacency. In (a), $b^{\wedge-}$ has been chosen to be adjacent to a^+ leaving choices to be made for the edges adjacent to e^+. In (b), e^+ was chosen as the adjacent edge while in (c), e^- was chosen. Both of the latter choices still leave undetermined adjacencies. Note the reversal of order of the edge labels for flipped vertices as compared to unflipped vertices.

guaranteed that there will even be edges that meet when traveling around a particular polygonal path through space, but because all dihedral angle information has been lost during the flattening process the test cannot predict whether the vertices will or will not properly align when they meet. For this reason, the “flattened” test is only a necessary condition for polyhedron existence and acts as a simple, preliminary filter for the only known true test, which is to actually attempt to construct the polyhedron.

We now turn to a description of the computerized methods. In one sense they are essentially the same as the manual methods, except automated, but there are some complications that arise from the computer's lack of human intelligence to recognize “obvious” opportunities and/or mistakes. We discuss these issues in more detail below.

The input data to the computer program consists of the (x,y,z) coordinates for each vertex of each polygon in the vertex star, along with a vertex symbol to be associated with the edges of the vertex star coordinates. The reason the vertex symbol is required is that recognizing the geometrical symmetries of an arbitrary vertex star is very difficult for a computer program. We do have a supplemental program that tests an arbitrary vertex star for reflections, turns, and flips to help point out existing symmetries but, still, the primary judge of a valid vertex symbol remains the responsibility of the human user of the program. (Analyzing the symmetry group of the vertex star is a very useful technique when compiling the list of all possible vertex symbols.) Also, making the vertex symbol a program input allows the user to make the (arbitrary) choices of which are the red and black sides, which edge is a, and which direction represents the ‘+’ orientation, for which there may be aesthetically pleasing “natural” choices that a computer program would not be able to recognize. Therefore, the end result is that vertex stars with symmetries that support multiple vertex symbols must each be run through the program separately.

It is also important to note that running all possible vertex symbols separately is a requirement, rather than an option. That is, it is not enough to try using only what may appear to be the most general vertex symbol. As an example, the dodecahedron, whose vertex star comprises three pentagons, cannot be constructed using the completely asymmetric vertex symbol $a^+b^+c^+$: some symmetry is required within the vertex symbol. On the other hand, as will be seen, the $\{4,5\}$ vertex star analyzed here in its asymmetric form can construct many more sponges than can be constructed when using just its associated symmetric vertex symbol.

Another advantage relevant to the manual but not the computer method can be seen above by the pre-elimination of certain edge adjacencies on examining the geometric vertex stars. For example, it is immediately apparent that the c edges cannot be adjacent to the b edges because their dihedral angles are different. But it takes more consideration to recognize that only a^+ or a^- and not $a^{\wedge+}$ or $a^{\wedge-}$ can match with an a^+ edge. A computer program can similarly pre-check the dihedral angles, and ours does so. But while a program can determine whether two vertex stars, in specific positions, are arranged properly so that a specified pair of edges match up correctly, it is quite a different thing to supply two unaligned vertex stars and ask a computer program to check whether there are *any* possible alignments. Therefore, instead of trying to pre-eliminate impossible adjacencies, the computer program simply checks all of them. Because of the speed of modern computers this is not a problem at all and, moreover, it eliminates any possibility of a potential adjacency being overlooked, which could be possible if a list of allowable adjacencies were provided to the program as input.

Thus, the basic algorithm of the program can be described quite simply as follows. Given a prescribed vertex star and its vertex symbol, generate all combinatorially possible adjacency symbols and then check them, one at a time, to see if any valid polyhedra can be created. (In practice, each adjacency symbol is checked as it is produced, so that large lists of candidates do not need to be stored in internal computer memory.)

The first tests simply check the incidence symbol for consistency according to the rules for pair-wise symmetry (if a is adjacent to b, then b must be adjacent to a), reflective symmetry (if the a-b adjacency is reflective, then so must the b-a adjacency be), and so on. Then, given a consistent incidence symbol, the next test is to check that the dihedral angles of the vertex star match for all of the proposed edge adjacencies. The third check is to apply the "flattened" test, as described above, in exactly the same manner as for the manual method. Finally, the last test is to attempt to geometrically construct the polyhedron, also following the same ideas as in the manual method, but now done via computational geometry calculations rather than using cardboard models. A final parameter supplied to the program is a specified rectangular prism volume; if the program can fill this volume with the polyhedron without any errors then the polyhedron is considered valid and is recorded. A very valuable advantage of the computer program over the manual method is that, because all of the (x,y,z) coordinates of the points are known at this time, the program can produce a three-dimensional VRML97 (Virtual Reality Modeling Language, 1997) model of the polyhedron for viewing by web browsers or stand-alone VRML97 viewing software. VRML97 is the current name for what used to be called VRML 2.0. The new standard now for displaying 3-D objects on the Web is X3D, but X3D has been designed to also read VRML97 files, so these images should still be viewable for some time. Web browser plug-ins to display and manipulate the VRML97 objects in 3-D can be found on the Web using any of the common search engines. Once installed on a computer, the software enables a person to rotate and examine the polyhedron on the computer screen as if they were holding a physical model in their hands, without any of the cardboard, tape, or patience required by the manual method.

A summary description of the computer algorithm is shown in Figure 8 in pseudocode.

We mentioned earlier the possibility of duplicate incidence symbols. These can occur when using either the manual or computer method, though they can often be avoided in the manual method by recognizing that certain vertex symbols are merely duplicates and eliminating them early in the process. However, these are always a concern when using the computer method because the program reports in its final output every possible incidence symbol, and these will include all duplicates. But they can be removed in a straightforward manner. Consider a situation where a vertex star has a symmetry, but it has been marked so as to remove that symmetry. Using the $\{4,5\}$ vertex star analyzed here as an example, the $a^+b^+c^+d^+e^+$ vertex symbol overrides the reflective symmetry contained in the geometric vertex star, as recognized by its symmetric symbol $a\ b^+c^+c^-b^-$. We showed above, starting in Figure 7, that the incidence symbol $[a^+b^+c^+d^+e^+;\ b^{\wedge-}a^{\wedge-}c^{\wedge-}d^+e^+]$

```
Create the initial central vertex at the origin
Add it to a first-in-first-out aligned candidate vertex queue
Loop
    Get the next candidate vertex from the queue
    If an accreted vertex already exists at its position then
        If the two vertices are equal then
            Discard the candidate vertex
        Else
            The polyhedron is invalid
        End if
    Else
        Loop through all edge vertices of the candidate vertex
            If an accreted vertex already exists at that position then
                If the accreted vertex does not align with the candidate vertex then
                    The polyhedron is invalid
                End if
            End if
        End loop

        If the polyhedron is still valid then
            Accrete the candidate vertex to the polyhedron
            Loop through all vacant edge vertices of the candidate vertex
                If the position is within the specified volume then
                    Create a new aligned vertex at that position
                    Add the new candidate vertex to the queue
                End if
            End loop
        End if
    End if
End loop when no more candidate vertices exist or if the polyhedron is invalid

If the polyhedron is valid then
    Record the polyhedron
End if
```

Figure 8: The computer vertex accretion algorithm to generate isogonal polyhedra.

passed all tests and therefore represents a $\{4,5\}$ sponge. If we had instead labeled the b edge in the reverse orientation, so that the edge now labeled e^+ would be b^+, and so on, our vertex symbol would become $a^+e^+d^+c^+b^+$ (temporarily keeping the edges in the same symbol position). Using this new labeling of the edges, our adjacency symbol would change to $e^{\wedge-}a^{\wedge-}d^{\wedge-}c^+b^+$, since we *only changed the names* of the edges, not their adjacency relationships. When we then rearrange the edges (equally in both vertex and adjacency symbols), we arrive at the vertex symbol $[a^+b^+c^+d^+e^+;\, e^{\wedge-}b^+c^+d^{\wedge-}a^{\wedge-}]$, which is different from our original one. Yet it is the same sponge. On the other hand, if we apply this same relabeling to this new incidence symbol, we arrive back at our original one. (This is because the symmetry is a two-way reflection.) In other cases the symbols will not change and will map back onto themselves, and in some vertex stars with more than two-way symmetries there will be a chain of connections. Therefore, because of this relabeling technique, it is always possible to eliminate the duplicate incidence symbols produced by the program. (In fact, we also have a short utility program that does this for us.) While eliminating the duplicates is somewhat of an inconvenience, one advantage is that, for every vertex symbol that maps into a different symbol, there must be another one that maps back into the original. This fact helps serve as a self-validation of the program results, showing that no vertex symbols were left out.

As for validation of the program, we first performed normal software testing procedures using simple, well-known vertex stars such as the Platonic solids. Another test was to submit the vertex stars for the planar isogonal tilings results reported by Grünbaum and Shephard [18]. In this case, only adjacency symbols without $x^\wedge$ edges were considered in order to correspond to the two-dimensional situation. The program correctly found exactly the same 91 tilings originally reported, up to symbol isomorphism. A third test was to replicate the results found by Hughes Jones [21] using vertex stars constructed from equilateral triangles arranged according to the possible paths when traversing a cuboctahedron. Again, the program found exactly the same shapes, but additionally listed the possible labelings for those shapes that possessed a symmetry. Finally, as regards the $\{4,5\}$ vertex star specifically being analyzed here, the results from the computer program and the manual results described above matched exactly.

4 Results

After all of the screening tests have been performed and the non-constructible and duplicate vertex symbols eliminated, what remains is a list of 32 incidence symbols with the asymmetric vertex symbol $a^+b^+c^+d^+e^+$ (N1 through N32), and three additional ones using the symmetric vertex symbol $a\, b^+c^+c^-b^-$ (S1, S2, S3). Observe that after substituting a for a^+ and a^-, c^- for d^+, and b^- for e^+ in each of the asymmetric vertex symbols, some of them reduce to one of the three symmetric sponge symbols but some become inconsistent. The inconsistent ones therefore represent truly asymmetric sponges, while the others represent labeled versions of the three symmetric sponges. We show the final results in Table 1.

Therefore we have proven the following result:

Symbol #	Shape #	Vertex symbol	Adjacency symbol
N1	1	$a^+b^+c^+d^+e^+$	$a^-b^+c^-d^+e^-$
N2	2		$a^-b^+c^-d^-e^-$
N3	3		$a^-b^+c^-d^{\wedge+}e^-$
N4	4		$a^-b^+c^-d^{\wedge-}e^-$
N5	5		$a^-b^-c^+d^-e^-$
N6	6		$a^-b^-c^-d^{\wedge+}e^-$
N7	7		$a^-b^-c^-d^{\wedge-}e^-$
N8	8		$b^{\wedge-}a^{\wedge-}c^-d^+e^-$
N9	9		$b^{\wedge-}a^{\wedge-}c^-d^-e^-$
N10	10		$b^{\wedge-}a^{\wedge-}c^-d^{\wedge+}e^-$
N11	11		$b^{\wedge-}a^{\wedge-}c^-d^{\wedge-}e^-$
N12	12		$b^{\wedge-}a^{\wedge-}c^{\wedge-}d^+e^+$
S1	13	$a\ b^+c^+c^-b^-$	$a\ b^+c^-c^+b^-$
N13		$a^+b^+c^+d^+e^+$	$a^-b^+c^-d^-e^+$
N14			$a^+b^+d^+c^+e^+$
N15			$a^-e^-c^-d^-b^-$
N16			$a^+e^-d^+c^+b^-$
S2	14	$a\ b^+c^+c^-b^-$	$a\ b^-c^-c^+b^+$
N17		$a^+b^+c^+d^+e^+$	$a^+b^-c^-d^-e^-$
N18			$a^-b^-c^-d^-e^-$
N19			$a^+b^-d^+c^+e^-$
N20			$a^-b^-d^+c^+e^-$
N21			$a^+e^+c^-d^-b^+$
N22			$a^-e^+c^-d^-b^+$
N23			$a^+e^+d^+c^+b^+$
N24			$a^-e^+d^+c^+b^+$
S3	15	$a\ b^+c^+c^-b^-$	$a\ b^-c^{\wedge+}c^{\wedge-}b^+$
N25		$a^+b^+c^+d^+e^+$	$a^+b^-c^{\wedge+}d^{\wedge+}e^-$
N26			$a^-b^-c^{\wedge+}d^{\wedge+}e^-$
N27			$a^+b^-d^{\wedge-}c^{\wedge-}e^-$
N28			$a^-b^-d^{\wedge-}c^{\wedge-}e^-$
N29			$a^+e^+c^{\wedge+}d^{\wedge+}b^+$
N30			$a^-e^+c^{\wedge+}d^{\wedge+}b^+$
N31			$a^+e^+d^{\wedge-}c^{\wedge-}b^+$
N32			$a^-e^+d^{\wedge-}c^{\wedge-}b^+$

Table 1: The 35 different incidence symbols and 15 different geometric sponges.

Theorem 1 *If squares are restricted to lie only along the planes of a cubic lattice, then there are 35 different incidence symbols for labeled* $\{4,5\}$ *sponges that lead to exactly 15 different geometric* $\{4,5\}$ *sponges.*

The 15 unlabeled geometric $\{4,5\}$ sponges are shown in Figures 9, 10 and 11. While some of the sponges may at first glance appear identical, N10 and N11 in particular, a careful examination shows that they are all different. Files for displaying the different $\{4,5\}$ sponges in VRML97 format can be retrieved at:

<http://hdl.handle.net/1773/4603>

Note that the images in Figures 9-11 are not shown using the red/black color scheme in order to try to make the 2-dimensional images more representative of the appearances of the 3-dimensional objects. However, the VRML97 images are displayed with the red/black vertex star colors as described above.

5 Historical and other remarks

Perhaps the most noticeable aspect of the history of sponges is the nearly total absence of continuity and connection between the works of various authors, and none of them appear to have considered the possibility that it might be possible to isogonally connect their congruent vertex stars in different ways to create the same polyhedron (*i.e.*, labeled sponges). About the only reference found in most (but not all) papers and books on the topic is the 1937 paper by Coxeter [7]. In it Coxeter describes the three regular $\{4,6\}$, $\{6,4\}$, and $\{6,6\}$ sponges, and proves that there are no other regular sponges. The term 'sponge', attributed to Andreas, was first described by Coxeter only later, in 1939 [8].

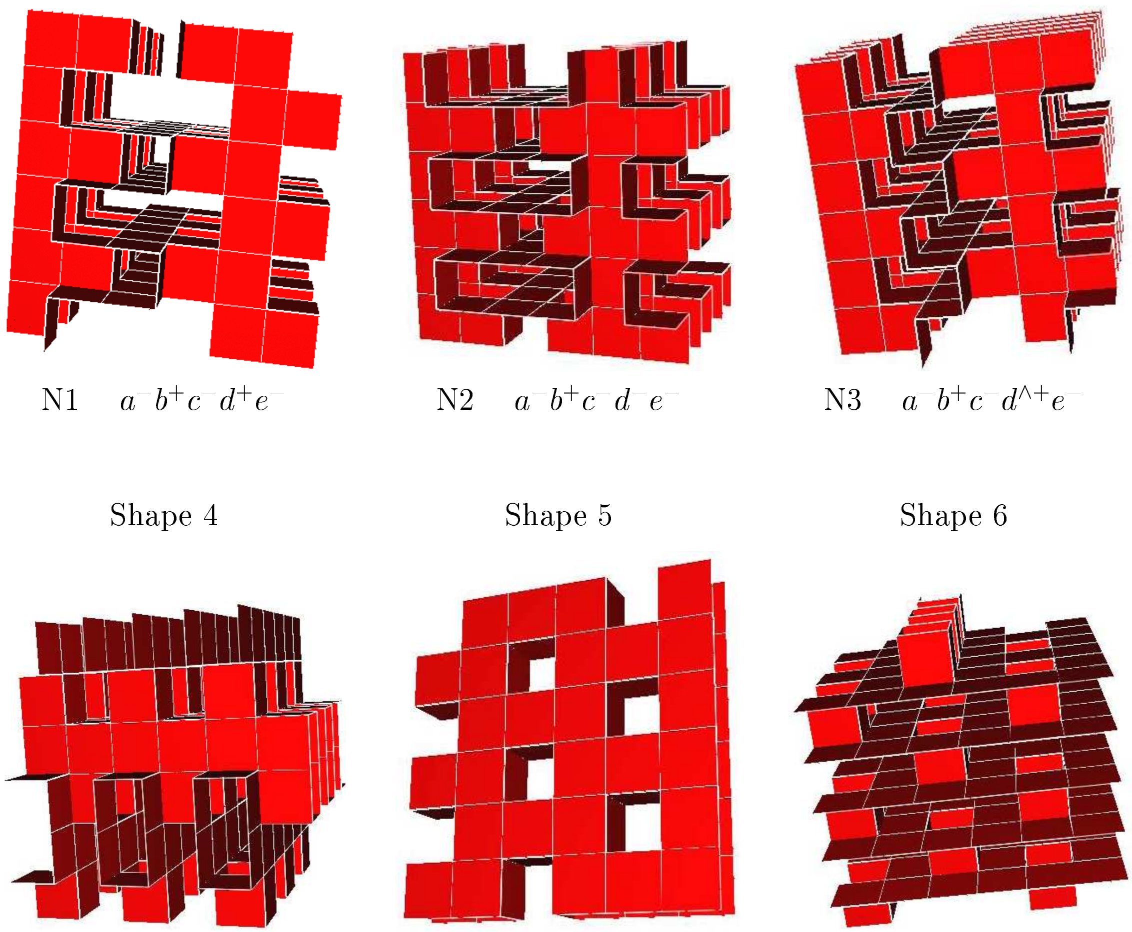
In 1950, ApSimon [1] described two triangle-faced infinite polyhedra, one in $\mathcal{P}\{3,12\}$ and another in $\mathcal{P}\{3,9\}$. He also described an infinite polyhedron in $\mathcal{P}(3,8)$ but not in $\mathcal{P}\{3,8\}$, as well as (in [2] and [3]) several other infinite polyhedra having certain kinds of symmetries.

In 1967, Gott [12] described nine sponges: the three regular sponges, as well as (in words but without diagrams) two sponges based on a bending of the regular $\{4,6\}$ vertex star, and one each in $\mathcal{P}\{3,8\}$, $\mathcal{P}\{3,10\}$, $\mathcal{P}\{4,5\}$, and $\mathcal{P}\{5,5\}$. The last one is most remarkable, and had not been previously reported by any other researchers. The two triangle-faced polyhedra are distinct from the ones found by ApSimon. The $\{4,5\}$ sponge is the one denoted here as S2 and shown in Figure 11.

One of the papers [26] in a series published by Wells in *Acta Crystallographica*, starting in 1954 and ending in 1976, deals with sponges. In this paper (published in 1969) are stereo pair photographs of models of several $\{3,\bullet\}$ and $\{4,5\}$ sponges. Most of the results of the series are collected in his later book [27], but some of the reproductions there are of poorer quality than these original ones.

In his note [23] from 1970, which is devoted to a study of infinite periodic minimal surfaces, Schoen mentions infinite uniform polyhedra, and provides some illustrations. The main topic is quite old, going back at least to an 1865 paper of Schwarz [24]. The

Shape 1 Shape 2 Shape 3



N1 $a^-b^+c^-d^+e^-$ N2 $a^-b^+c^-d^-e^-$ N3 $a^-b^+c^-d^{\wedge +}e^-$

Shape 4 Shape 5 Shape 6

N4 $a^-b^+c^-d^{\wedge -}e^-$ N5 $a^-b^-c^+d^-e^-$ N6 $a^-b^-c^-d^{\wedge +}e^-$

Figure 9: The first six (N1-N6) of the twelve $\{4,5\}$ sponges of type N.

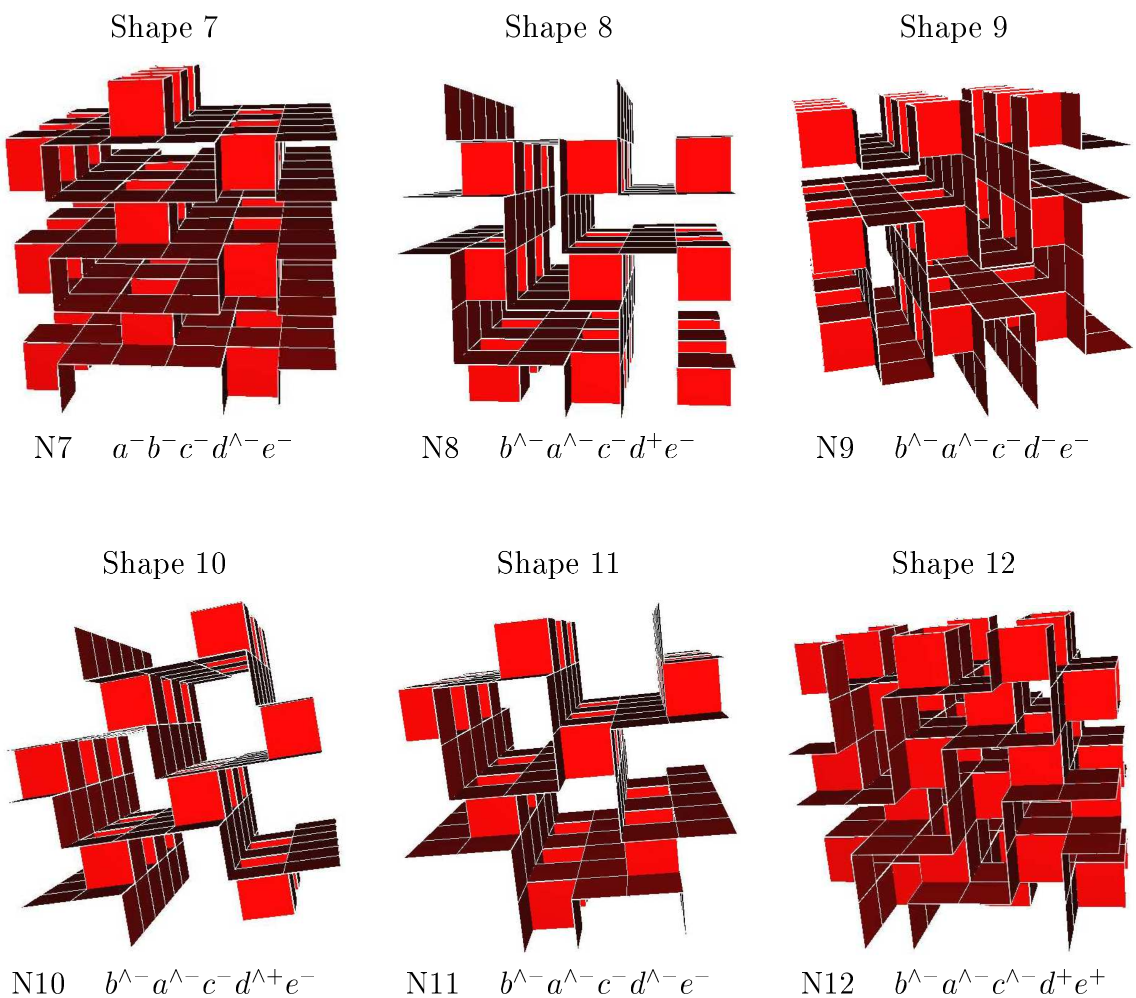
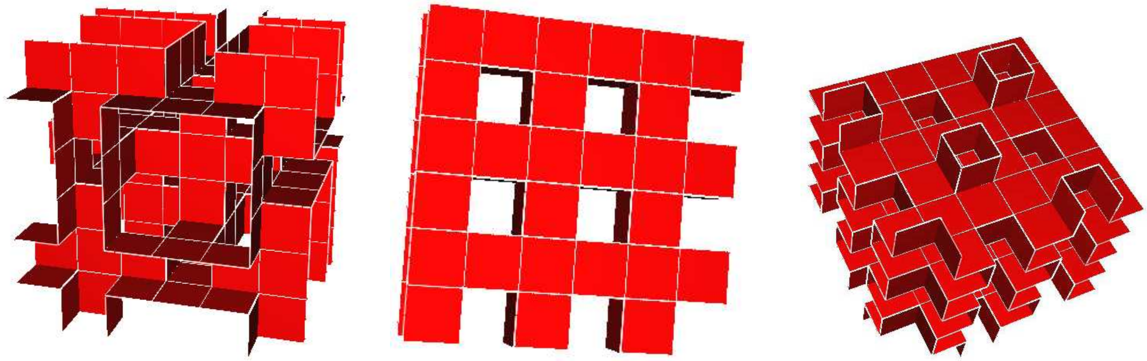


Figure 10: The second six (N7-N12) of the twelve $\{4, 5\}$ sponges of type N.

Shape 13 Shape 14 Shape 15



S1 $a\ b^{+}c^{-}c^{+}b^{-}$ S2 $a\ b^{-}c^{-}c^{+}b^{+}$ S3 $a\ b^{-}c^{\wedge+}c^{\wedge-}b^{+}$

Figure 11: The three $\{4,5\}$ sponges of type S.

connection between minimal surfaces and uniform polyhedra appears in various other writings as well (besides [23] see, for example, Goodman-Strauss and Sullivan [11] and the references given in these papers). For a splendid catalog of minimal surfaces (without mention of polyhedra) and a long list of references, see Lord and Mackay [22].

Quoting Coxeter [7] and papers by Wells (later summarized in his book [27]), Williams briefly considers infinite uniform polyhedra [28].

A collection of more than one hundred infinite uniform polyhedra is the 1974 work of Wachman, Burt, and Kleinmann [25]. It contains seven of our $\{4,5\}$ sponges, three others in $\mathcal{P}\{4,5\}$ with pairs of dihedral angles other than 90° or 180°, and one in $\mathcal{P}(4,5)$ only. It also includes five $\{4,6\}$ sponges, including one of the two described by Gott with some faces at 60° dihedral angles, and thirteen $\mathcal{P}\{3,\bullet\}$ sponges, as well as a large number of infinite polyhedra with more than one kind of regular polygon as faces. Of the seven matching ours here, their numbers 1, 2, 3, 4, 5, 8, and 10 correspond to our S2, N5, S3, N7, N11, N6, and S1 sponges, respectively. (The missing numbers are the non-$\{4,5\}$ sponges described above.) No other source has anything approaching the richness of this collection. It is regrettable that none of the mathematical reviewing journals took any note of this publication.

In 1977 there appeared two works relevant to the topic considered here. One is the paper by Grünbaum [14] where several new kinds of regular polyhedra are considered, and which gives a number of references to infinite uniform polyhedra. The other is the book by Wells [27] which, as mentioned above, collects and extends the results of several of his papers; it has a chapter on what we define as Platonic sponges. Wells presents nine triangle-faced and six square-faced sponges, plus the three regular Coxeter-Petrie sponges. In the two adjoining chapters he also describes several with more than one kind of regular polygon as faces.

In the 1979 paper by Grünbaum and Shephard [19], the beginnings of the application of incidence symbols to infinite collections of polygons in periodic surfaces in 3-dimensional space were described. This was one of the ingredients that led to the enumeration approaches of the present paper.

None of the works discussed so far makes any claim on completeness (except that Coxeter [7] proves that there are no other *regular* sponges besides the three he found).

In 1993, Grünbaum [15] described a new $\{4,5\}$ sponge, the one listed here as N2. It was the discovery of this previously unknown sponge, after so many previous reports appeared to have exhausted the $\{4,5\}$ possibilities, that was a prime motivator for finding a systematic method to search for all isogonal polyhedra based on a given vertex star and that ultimately led to the work we describe here. At that time Grünbaum also proposed five conjectures about isogonal polyhedra composed of regular polygons. Conjecture 1, that there were no more $\{3,\bullet\}$ sponges than those already discovered, was proven false by Hughes Jones in 1995. The $\{4,5\}$ results presented in this paper show Conjecture 5, that there are no further flexible uniform polyhedra than those discovered at that time, to also have been false. The remaining three conjectures still appear to be holding up, though: (2) there are no uniform polyhedra with more than 12 faces at a vertex; (3) if a uniform polyhedron has more than eight faces at a vertex then they must all be triangles; and (4) there are no uniform polyhedra with all face polygons having more than six sides.

The first paper to report a complete enumeration of a specific type of sponge is Hughes Jones [21] in 1995. Hughes Jones considers infinite isogonal polyhedra formed by triangles that lie in the planes of a tiling of 3-space by regular tetrahedra and octahedra, which meets our definition of $\{3,\bullet\}$ sponges. In these sponges every vertex star consists of triangles, the free edges of which form a Hamiltonian path on the edges of a (uniform) cuboctahedron, which is usually denoted as (3.4.3.4). By a combinatorial analysis analogous to the one presented above for the $\{4,5\}$ sponges, and subsequent verification of their geometric realizability, Hughes Jones proves that there are precisely 26 sponges of this class of $\{3,\bullet\}$. More specifically, there is a single $\{3,7\}$ sponge, three $\{3,8\}$ sponges, thirteen $\{3,9\}$ sponges, and nine $\{3,12\}$ sponges. Without giving any details, Hughes Jones claims to know of eleven other polyhedra in $\mathcal{P}\{3,\bullet\}$ but not in the cuboctahedron class under consideration, and that six of them can be found in Wells' [27] book. Using our methods, and starting with the same set of vertex stars, we also found the same list of sponges reported by Hughes Jones. However, we also found that some of these vertex stars (specifically, a few of the non-reflectively symmetric ones) form chiral pairs when the vertex star is reflected. That is, the starting set of vertex stars must also include their reflected versions in order to obtain a truly complete set of $\{3,\bullet\}$ cuboctahedron-based sponges. We hope to discuss this topic of chiral pair sponges in more detail in a future publication.

The only other published work with a complete enumeration of all sponges of a specific kind is the recent paper by Goodman-Strauss and Sullivan [11]. Using an approach completely different from the one followed here, the authors show that there are precisely six $\{4,6\}$ cubic lattice sponges. (We obtained the same result during the early stages of our work on this paper. Also, we and other researchers ([12], [25], [27]) have found additional $\{4,6\}$ sponges not restricted to a cubic lattice alignment. We hope to discuss these further in the future as well.) They also investigate more general polyhedra in

$\mathcal{P}(4,6)$, and connections to periodic minimal surfaces. Sadly enough, the review of [11] in Mathematical Reviews (2004k:52020) does not mention the description within this paper of the five non-regular $\{4,6\}$ cubic lattice sponges. Even worse, since it is totally misleading, is the review in the Zentralblatt (1048.52008).

The recent book [6] by Conway, Burgiel, and Goodman-Strauss considers several sponges under a different point of view; all those shown can be found in [25] and [27].

* * * * *

Other than our own Web pages [10], there is still very little other material on sponges and related objects on the Web as of this date. Most of what appears seems to originate from the book by Wells [27]. Two pages by Dutch [9] deal with infinite isogonal polyhedra. The first shows the three regular Coxeter-Petrie polyhedra and two from Gott's [12] paper, his $\{3,8\}$ and $\{3,10\}$ sponges. The second page shows the two sponges labeled by Wells as (4,5)-4tb and (4,5)-8ti, which are actually two views of the same sponge, one being seen as the complement of the other. He also includes six examples in $\mathcal{P}\{3,\bullet\}$ described by Wells. They are shown in very attractive color graphics, and accompanied by an explanation of their construction.

A larger collection that attempts to include all of the ones described by Wells as well as some others including Gott's, is by Green [13]. She also provides some explanations and some simple VRML models. Bulatov [5] shows some infinite isogonal polyhedra taken from Green's pages, but also describes some interesting infinite classes of very complex but finite, non-acoptic isogonal polyhedra. Finally, a good starting point for references and links to infinite isogonal polyhedra is the entry in Wikipedia under 'Infinite skew polyhedron', though its page as of this writing has only a few pictures of sponges, all of which are from sources discussed here already (primarily those described by Coxeter-Petrie, Gott, and Wells).

* * * * *

It is worth mentioning that several of the $\{4,\bullet\}$ sponges are not rigid. By this we mean that if the sponges are considered as constructed of rigid squares connected by hinges along their edges, a physical model of the sponge can move as a mechanism. The only mention of this possibility we found in the literature is a comment by Coxeter (in [4], p. 153) concerning the regular sponge $\{4,6\}$: "...to make $\{4,6\}$, use rings of four squares. This last model, however, is not rigid; it can gradually collapse, the square holes becoming rhombic. (In fact, J. C. P. Miller once made an extensive model and mailed it, flat in an envelope)." The other $\{4,6\}$ sponges are rigid. Many of the $\{4,5\}$ sponges described here are movable, though not always while remaining isogonal or acoptic. Considering the asymmetric ones, all can be isogonally constructed with a different τ angle (see Figure 3) except N12, but shapes N1, N2, ... N5 are no longer acoptic, leaving only shapes N6, N7, ... N11 as both flexible and isogonal. (Setting $\tau \neq 0$ violates the symmetry of the vertex star so that none of the symmetric sponges can be constructed.) As an example, the version of N7 for $\tau = \pi/4$, having tunnels with regular octagonal cross-section, is shown on page 20 of Wachman *et al.* [25].

Acknowledgements

We wish to thank the anonymous reviewer for comments that improved our discussion. Also, we are grateful to Ann Lally and University of Washington Libraries for establishing an online digital archive for storing our VRML files and for their assistance to us in entering them into the archive.

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