Independence number of 2-factor-plus-triangles graphs

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Submitted: Jun 10, 2008; Accepted: Feb 18, 2009; Published: Feb 27, 2009
Mathematics Subject Classification: 05C69

Abstract

A *2-factor-plus-triangles graph* is the union of two 2-regular graphs G_1 and G_2 with the same vertices, such that G_2 consists of disjoint triangles. Let $\mathcal{G}$ be the family of such graphs. These include the famous "cycle-plus-triangles" graphs shown to be 3-choosable by Fleischner and Stiebitz. The independence ratio of a graph in $\mathcal{G}$ may be less than 1/3; but achieving the minimum value 1/4 requires each component to be isomorphic to the 12-vertex "Du–Ngo" graph. Nevertheless, $\mathcal{G}$ contains infinitely many connected graphs with independence ratio less than 4/15. For each odd g there are infinitely many connected graphs in $\mathcal{G}$ such that G_1 has girth g and the independence ratio of G is less than 1/3. Also, when 12 divides n (and $n \neq 12$) there is an n-vertex graph in $\mathcal{G}$ such that G_1 has girth $n/2$ and G is not 3-colorable. Finally, unions of two graphs whose components have at most s vertices are s-choosable.

1 Introduction

The Cycle-Plus-Triangles Theorem of Fleischner and Stiebitz [5] states that if a graph G is the union of a spanning cycle and a 2-factor consisting of disjoint triangles, then G is 3-choosable, where a graph is *k-choosable* if for every assignment of lists of size k to the vertices, there is a proper coloring giving each vertex a color from its list. Sachs [8] proved by elementary methods that all such graphs are 3-colorable. Both results imply an earlier conjecture by Du, Hsu, and Hwang [1], stating that a cycle-plus-triangles graph with $3k$ vertices has independence number k. Erdős [3] strengthened the conjecture to the more well-known statement that these graphs are 3-colorable. We return to the original topic of independence number but study it on a more general family of graphs.

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A *2-factor-plus-triangles graph* is a union of two 2-regular graphs G_1 and G_2 on the same vertex set, where the components of G_2 are triangles. Note that G_1 and G_2 may share edges. For such a graph G, we denote the vertex sets of the components of G_2 as $T_1, \ldots, T_k$, with $T_x = \{x_1, x_2, x_3\}$, and we refer to T_x as a "triple" to distinguish it from a 3-cycle in G_1. When G_1 is a single cycle, G is a cycle-plus-triangles graph.

Let $\mathcal{G}$ denote the family of 2-factor-plus-triangles graphs. It is easy to construct graphs in $\mathcal{G}$ that contain K_4 (see Figure 1, for example), so graphs in $\mathcal{G}$ need not be 3-colorable. Erdos [3] asked if a graph in $\mathcal{G}$ is 3-colorable whenever its factor G_1 is C_4-free. Fleischner and Stiebitz [6] answered this negatively, citing an infinite family of such graphs in $\mathcal{G}$ that are 4-critical, due to Gallai. In fact, graphs in $\mathcal{G}$ with $3k$ vertices may fail to have an independent set of size k, such as the graph in Figure 1 due to Du and Ngo [2]. Here we draw only G_1 and indicate the triples T_a, T_b, T_c, T_d using subscripted indices.

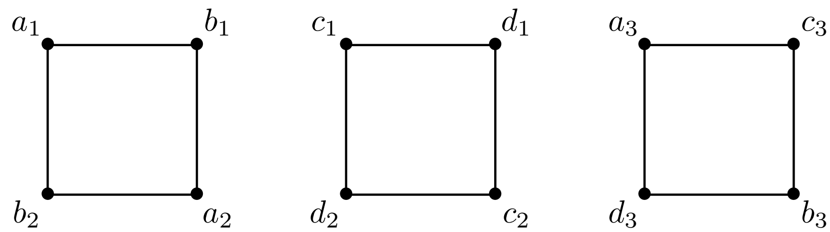

Figure 1: The Du-Ngo graph G_{DN}, omitting triangles on sets of the form $\{x_1, x_2, x_3\}$.

An *independent set* is a set of pairwise nonadjacent vertices. The *independence number* $\alpha(G)$ of a graph G is the maximum size of such a set in G.

Proposition 1.1. *The independence number of the Du-Ngo graph G_{DN} is* 3.

Proof. An independent set S in G_{DN} contains at most one vertex from each of the 4-cliques $\{a_1, b_1, a_2, b_2\}$ and $\{c_1, d_1, c_2, d_2\}$. Further, S contains two vertices of $\{a_3, b_3, c_3, d_3\}$ only if it avoids one of the 4-cliques. Thus $|S| \leq 3$, and $\{a_1, c_1, d_3\}$ achieves the bound. $\square$

The *independence ratio* of an n-vertex graph G is $\alpha(G)/n$. Proposition 1.1 states that the independence ratio of G_{DN} is $1/4$. Because graphs in $\mathcal{G}$ have maximum degree at most 4 and do not contain K_5, Brooks' Theorem implies that every graph in $\mathcal{G}$ has independence ratio at least $1/4$. We characterize the graphs achieving equality in this easy bound; they are those in which every component is G_{DN}. We produce larger independent sets for all other graphs in $\mathcal{G}$. We also construct infinitely many connected graphs in $\mathcal{G}$ with independence ratio less than $4/15$. However, we conjecture that for any t less than $4/15$, only finitely many connected graphs in $\mathcal{G}$ have independence ratio at most t.

In light of Erdős' question about 3-colorability of graphs in $\mathcal{G}$ when G_1 has no 4-cycle, we study the independence ratio under girth restrictions for G_1. For any odd g, we construct infinitely many connected examples in which the girth of G_1 is g and yet the independence ratio is less than $1/3$; it can be as small as $\frac{1}{3} - \frac{1}{g^2+2g}$ when $g \equiv 1 \bmod 6$. The number of vertices in each example is more than g^2, and we conjecture that the independence ratio of G is $1/3$ when G_1 has girth at least $\sqrt{|V(G)|}$. On the other hand, no girth threshold less than $|V(G)|$ can guarantee 3-colorability; when the number of vertices is a nontrivial multiple of 12, we construct examples where G_1 consists of just two cycles of equal length but G is not 3-colorable.

Finally, we show that if G is a union of two graphs whose components have at most s vertices, then G is s-choosable; this yields 3-choosability for graphs in $\mathcal{G}$ where the components of G_1 are all 3-cycles. This last result is an easy consequence of the s-choosability of the line graphs of bipartite graphs.

Our graphs have no multiple edges; when G_1 and G_2 share an edge, its vertices have degree less than 4 in the union. For a graph G and a vertex $x \in V(G)$, the *neighborhood* $N_G(x)$ is the set of vertices adjacent to x in G, and a *G-neighbor* of x is an element of $N_G(x)$. For $S \subseteq V(G)$, we let $N_G(S) = \bigcup_{x\in S} N_G(x)$. If A and B are sets, then $A - B = \{a \in A\colon\ a \notin B\}$.

2 Independence ratio at least $1/4$

The independence number of a graph is the sum of the independence numbers of its components. Therefore, to characterize the graphs in $\mathcal{G}$ with independence ratio $1/4$, it suffices prove that every connected graph in $\mathcal{G}$ other than G_{DN} has independence ratio larger than $1/4$. Let $\mathcal{G}' = \{G \in (\mathcal{G} - \{G_{DN}\})\colon\ G \text{ is connected}\}$.

Proving this is surprisingly difficult. We present an algorithm to produce a sufficiently large independent set for any $G \in \mathcal{G}'$. A simple greedy algorithm finds an independent set with almost $1/4$ of the vertices; it will be applied to prove the full result. This simple algorithm maintains an independent set I and the set S of neighbors of I.

Algorithm 2.1. Given an independent set I in G, let $S = N_G(I)$. While $I \cup S \neq V(G)$, choose $v \in V(G) - (I \cup S)$ to minimize $|N(v) - S|$, and add v to I and $N_G(v)$ to S.

Lemma 2.2. *If G is an n-vertex graph in $\mathcal{G}'$, then $\alpha(G) \geq (n-1)/4$. If G has an independent set I_0 with $3|I_0| > |N_G(I_0)|$, then $\alpha(G) > n/4$.*

Proof. Initialize Algorithm 2.1 with I as any single vertex in G; this puts at most 4 vertices in S. At each subsequent step, some vertex v outside $I \cup S$ has a neighbor in S, since G is connected and $N_G(I) = S$. Hence each step adds at most 3 vertices to S and 1 vertex to I. Therefore, $|S| \leq 3|I| + 1$ when the algorithm ends. Since $n = |I| + |S|$ at that point, we conclude that $|I| \geq (n-1)/4$.

If $3|I_0| > |N_G(I_0)|$, then initializing Algorithm 2.1 with $I = I_0$ (and $S = N_G(I_0)$) yields $|S| \leq 3|I| - 1$ at the end by the same computation, and hence $|I| \geq (n+1)/4$. $\square$

In order to push the independence ratio above 1/4, we will preface Algorithm 2.1 with another algorithm that will choose the initial independent set more carefully, seeking an independent set I_0 as in Lemma 2.2 or one that will lead to a gain later under Algorithm 2.1.

First we characterize how 4-cliques can arise in graphs in $\mathcal{G}$ (a *k-clique* is a set of k pairwise adjacent vertices).

Lemma 2.3. *A 4-clique in a graph G in $\mathcal{G}$ arises only as the union of a 4-cycle in G_1 and disjoint edges from two triples in G_2 (Figure 2 below shows such a 4-clique).*

Proof. Let X be a 4-clique in G. Since G_1 contributes at most two edges to each vertex, each vertex in X has a G_2-neighbor in X. In particular, no triple in G_2 is contained in X, and X must have the form $\{a_1, a_2, b_1, b_2\}$ for some T_a and T_b. To make X pairwise adjacent, a_1, b_1, a_2, b_2 in order must form a 4-cycle in G_1. $\square$

We define a substructure that yields a good independent set for the initialization of Algorithm 2.1. A *bonus 4-clique* in a graph in $\mathcal{G}$ is a 4-clique Q such that for some triple T_a contributing two vertices to Q, the vertices of $N_{G_1}(a_3)$ lie in the same triple. Figure 2 illustrates the definition.

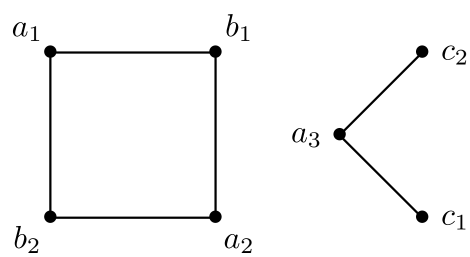

Figure 2: A bonus 4-clique

Lemma 2.4. *If an n-vertex graph G in $\mathcal{G}'$ has a bonus 4-clique, then $\alpha(G) > n/4$.*

Proof. Consider a bonus 4-clique, labeled as in Figure 2 without loss of generality. The set $\{b_1, a_3, c_3\}$ is independent, and its neighborhood is $\{a_1, a_2, b_2, b_3, c_1, c_2\} \cup N_{G_1}(c_3)$. Thus setting $I_0 = \{b_1, a_3, c_3\}$ in Lemma 2.2 yields the conclusion. $\square$

A *block* of a graph is a maximal subgraph that contains no cut-vertex. Two blocks in a graph share at most one vertex, and a vertex in more than one block is a cut-vertex. A *leaf block* of a graph G is a block that has at most one vertex shared with other blocks of G. We need a structural result to extract large independent sets from leaf blocks.

Lemma 2.5. *Let G be an n-vertex 4-regular graph in $\mathcal{G}'$. If G has no 4-clique, then G has an independent set I such that $3|I| > |N_G(I)|$ or such that $3|I| = |N_G(I)|$ and $|I| < n/4$.*

Proof. Every vertex of G lies in a triple, and every triple lies in a block of G. Since G is 4-regular, a leaf block contains a triple and at least one more vertex. A shortest path joining two vertices of the triple that uses a vertex outside the triple yields an even cycle with at most one chord. (Note: Erdős, Rubin, and Taylor [4] showed by a harder proof that all 2-connected graphs other than complete graphs and odd cycles have such a cycle.)

An independent set I with $|I| > n/4$ vertices satisfies $3|I| > |N_G(I)|$ and hence suffices.

We may assume that G has no 4-cycle, since G has no 4-clique and a 4-cycle in G with at most one chord has an independent set I with $3|I| = |N_G(I)|$ and $|I| = 2 \neq n/4$ (note that $3 \mid n$). If C is an even cycle in G having at most one chord, then at least one of the two maximum independent sets in C contains at most one vertex of such a chord and is independent in G. Let I be such an independent set.

Since each vertex of I has at least two neighbors on C and at most two outside it, $3|I| \geq |N_G(I)|$. We have found the desired set I unless $|I| = n/4$. In this case, let $T = V(G) - V(C)$. If I is not a maximal independent set, then $\alpha(G) > n/4$, so we may assume that every vertex of T has a neighbor in I. Since $I \subseteq V(C)$, each vertex in I has at most two neighbors in T. Hence each vertex of T has exactly one neighbor in I, and each vertex of I has two neighbors in T (and C has no chord).

Let u, v, w be three consecutive vertices on C, with $u, w \in I$. Let $\{x, x'\} = N_G(u) \cap T$ and $\{y, y'\} = N_G(w) \cap T$. If $xx' \notin E(G)$, then replacing u with $\{x, x'\}$ in I yields $\alpha(G) > n/4$. Hence we may assume that $xx' \in E(G)$, and similarly $yy' \in E(G)$. If v has a neighbor in $\{x, x', y, y'\}$, then G has a 4-cycle, which we excluded. Since G has no 4-clique, some vertex in $\{x, x'\}$ has a nonneighbor in $\{y, y'\}$, say $xy \notin E(G)$. Now replacing $\{u, w\}$ with $\{v, x, y\}$ in I yields $\alpha(G) > n/4$. □

We now present an algorithm to apply before Algorithm 2.1, as "preprocessing". The proof of Lemma 2.5 can be implemented as an algorithm used by Algorithm 2.6 when G has no 4-clique. Like Algorithm 2.1, Algorithm 2.6 maintains an independent set $I \subseteq V(G)$ and the set S of its neighbors. It produces a nonempty independent set I such that $3|I| > |S|$ or such that $3|I| = |S| < 3n/4$ and all vertices of 4-cliques lie in $I \cup S$.

After Algorithm 2.6, we apply Algorithm 2.1 starting with this set as I. Lemma 2.2 implies that if $3|I| > |S|$, then $\alpha(G) > n/4$. We will show in Theorem 2.8 that if $3|I| = |S|$, then the exhaustion of the 4-cliques during Algorithm 2.6 will guarantee the existence of a step in Algorithm 2.1 in which S gains at most two vertices. Thus again we will have $3|I| > |S|$ and $|I| > n/4$ at the end.

To facilitate the description of Algorithm 2.6, we introduce several definitions. A triple having two vertices in a 4-clique is a *clique-triple*. Two clique-triples that contribute two vertices each to the same 4-clique (see Lemma 2.3) are *mates*. If T_a intersects a 4-clique Q, but $I \cup S$ does not intersect $T_a \cup Q$, then T_a is a *free clique-triple*.

Algorithm 2.6. Given an n-vertex graph G in $\mathcal{G}'$, initialize $I = S = \varnothing$. Maintain $S = N_G(I)$. When we "stop", the current set I is the output.

Suppose first that G has no 4-clique. If $E(G_1) \cap E(G_2) \neq \varnothing$, then let I consist of one endpoint of such an edge and stop. Otherwise, G is 4-regular; let I be an independent set produced by the algorithmic implementation of Lemma 2.5, and stop.

If G has a bonus 4-clique, then define I as in Lemma 2.4 and stop.

If G has a 4-clique but no bonus 4-clique, then repeat the steps below until either $3|I| > |S|$ or $I \cup S$ contains all vertices of 4-cliques; then stop.

1. If a vertex outside $I \cup S$ has at most two neighbors outside S, add it to I and stop.
2. If there is a free clique-triple T_a with mate T_b such that S contains b_3 or some G_1-neighbor of a_3, then add $\{a_3, b_1\}$ to I and stop.
3. Otherwise, let T_a be a free clique-triple with mate T_b, and let $N_{G_1}(a_3) = \{c_3, d_3\}$. Since G has no bonus 4-clique, $c \neq d$. If $\{c_1, d_1, c_2, d_2\}$ is not a 4-clique in G, then add $\{a_3, b_1\}$ to I. If $\{c_1, d_1, c_2, d_2\}$ is a 4-clique in G, then add $\{a_3, b_1, c_3, d_1\}$ to I.

Lemma 2.7. *For $G \in \mathcal{G}'$, Algorithm 2.6 produces an independent set I with neighborhood S such that $3|I| > |S|$ or such that $3|I| = |S|$ and $I \cup S$ contains all 4-cliques in G.*

Proof. First suppose G has no 4-clique. If G is 4-regular, then Algorithm 2.6 uses the construction of Lemma 2.5 to produce I such that $3|I| > |S|$ or such that $3|I| = |S|$ and $|I| < n/4$ (and hence $I \cup S \neq V(G)$). If G is not 4-regular, then it finds such a set of size 1.

If G has a bonus 4-clique, then the independent set I is as in the proof of Lemma 2.4, with $3|I| > |S|$.

Therefore, we may assume that G has a 4-clique but no bonus 4-clique. In this case, the algorithm iterates Step 3 until it reaches a state where Step 1 or 2 applies or it runs out of free clique-triples.

To show that ending in Step 1 or 2 yields the desired conclusion, suppose that each instance of Step 3 maintains $3|I| \geq |S|$. In Step 1, we then add one vertex to I and at most two to S. In Step 2, we add $\{a_3, b_1\}$ to I and $\{a_1, a_2, b_2, b_3\} \cup N_{G_1}(a_3)$ to S, but S already contains at least one of these six vertices.

Hence we must show that Step 3 maintains $3|I| \geq |S|$. To avoid getting stuck by running out of free clique-triples before absorbing all 4-cliques into $I \cup S$, also we must maintain that every 4-clique not contained in $I \cup S$ intersects a free clique-triple.

These two properties hold initially. Suppose that they hold when we enter an instance of Step 3. We have mates T_a and T_b, with T_a being free. Since Step 2 does not apply, $b_3 \notin S$, so T_b also is free. Since G has no bonus 4-clique, $c \neq d$.

In the first case, $\{c_1, d_1, c_2, d_2\}$ is not a 4-clique, and we add $\{a_3, b_1\}$ to I. This adds $\{a_1, a_2, b_2, b_3\} \cup N_{G_1}(a_3)$ to S, gaining six vertices. The 4-clique $\{a_1, a_2, b_1, b_2\}$ has been absorbed. The vertices of other 4-cliques that might enter $I \cup S$ are those in $T_c \cup T_d$. Suppose that $\{c_1, c_2, x_1, x_2\}$ is a 4-clique, with T_x the mate of T_c. If T_x is not free before this instance of Step 3, then $x_3 \in S$, but now Step 2 would have applied instead of Step 3, with T_c as T_a and T_x as T_b. Since the addition to I does not affect x_3, afterwards T_x remains free. Similarly, the mate of T_d remains free if T_d is a clique-triple.

In the second case, $\{c_1, d_1, c_2, d_2\}$ is a 4-clique, and we add $\{c_3, d_1\}$ to I. This is an instance of the first case for the mates T_c and T_d unless $N_{G_1}(c_3) = \{a_3, b_3\}$. However, that requires $G = G_{DN}$, labeled as in Figure 1. Since $G \in \mathcal{G}'$, we find a 4-clique where the first case of Step 3 applies. $\square$

Theorem 2.8. *For $G \in \mathcal{G}'$, using the output of Algorithm 2.6 as initialization to Algorithm 2.1 produces an independent set having more than $1/4$ of the vertices of G.*

Proof. By Lemma 2.2, we may assume that the output of Algorithm 2.6 is an independent set I with neighborhood S such that $3|I| = |S|$ and every 4-clique is contained in $I \cup S$. Furthermore, if G has no 4-clique, then $I \cup S \neq V(G)$. To complete the proof, we show that with such an initialization, the final step of Algorithm 2.1 adds at most two vertices to S (hence strict inequality holds at the end).

We claim that also $I \cup S \neq V(G)$ when G has a 4-clique and Algorithm 2.6 ends with $3|I| = |S|$. We noted in the proof of Lemma 2.7 that ending in Step 1 or 2 yields $3|I| > |S|$, so ending with $3|I| = |S|$ requires ending in Step 3. On the last step, we have free mates T_a and T_b, and we add $\{a_3, b_1\}$ to I and $\{a_1, a_2, b_2, b_3\} \cup N_{G_1}(a_3)$ to S. If this exhausts $V(G)$, then $N_{G_1}(a_3) = V(G) - (I \cup S) - (T_a \cup T_b)$ before the final step. The other vertices of the triples containing the vertices of $N_{G_1}(a_3)$ are already in S. These two vertices lie in the same triple; otherwise, each has at most two neighbors outside S before the last step, and Step 1 would apply. On the other hand, if they belong to the same clique, then $\{a_1, a_2, b_1, b_2\}$ is a bonus 4-clique, which would have been used at the start.

Hence we may assume that at least one vertex remains outside $I \cup S$ when we move to Algorithm 2.1. We claim that at most two vertices are added to S in the final step of Algorithm 2.1. If three vertices are added to S, then let x be the vertex added to I, with neighbors u, v, w added to S. Choosing one of $\{u, v, w\}$ instead of x would also add at least three vertices to S, since we chose v to minimize $|N(v) - S|$. This implies that $\{u, v, w, x\}$ is a 4-clique in G. This possibility is forbidden, since all vertices contained in 4-cliques are added to $I \cup S$ during Algorithm 2.2. $\square$

Corollary 2.9. *Every 2-factor-plus-triangles graph has independence ratio at least $1/4$, with equality only for graphs whose components are all isomorphic to G_{DN}.*

3 Constructions

The Du-Ngo graph G_{DN} is the only graph in $\mathcal{G}'$ with independence ratio $1/4$. In this section, we construct a sequence of graphs with independence ratio less than $4/15$.

Figure 3 shows a 27-vertex graph G in $\mathcal{G}'$ with $\alpha(G) = \frac{1}{4}(27 + 1)$. Note that G is connected. An independent set I has at most six vertices in the subgraph inside the dashed box (at most two from each "column" of 4-cycles). Also, I has at most one vertex in the remaining 3-cycle $[x_3, y_3, z_3]$ in G_1. Hence $\alpha(G) \leq 7 = (27 + 1)/4$, and $\{a_1, b_3, c_1, d_3, e_1, f_3, x_3\}$ achieves the upper bound.

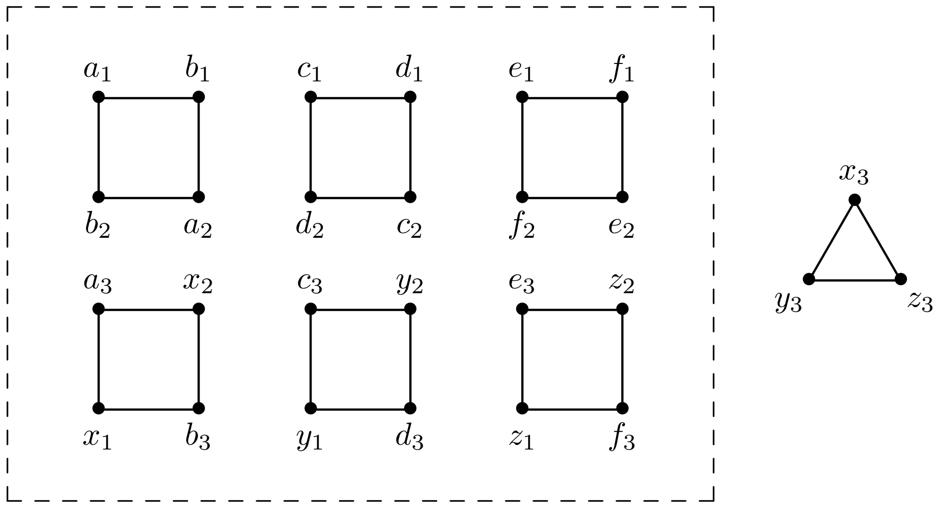

Figure 3: A graph in $\mathcal{G}'$ with independence number $(n+1)/4$

One may ask whether infinitely many graphs G in $\mathcal{G}'$ satisfy $\alpha(G) = (|V(G)|+1)/4$, or at least with $\alpha(G) \leq (|V(G)|+c)/4$ for some constant c. We conjecture that no such constant exists; in fact, we conjecture the following stronger statement.

Conjecture 3.1. *For every $t < 4/15$, only finitely many graphs in $\mathcal{G}'$ have independence ratio at most t.*

This conjecture is motivated by the following theorem, which shows that the conclusion is false when $t \geq 4/15$. To avoid confusion with our earlier use of G_1 and G_2, we use Q_i and R_i to index sequences of special graphs in this construction.

Theorem 3.2. *For $i \geq 0$, there is a graph $Q_i \in \mathcal{G}$ with independence ratio $\frac{4(2^i)-5/3}{15(2^i)-6}$.*

Proof. We first construct a rooted graph R_i for $i \geq 0$. Then Q_i will be built from three disjoint copies of R_i by adding a 3-cycle on the roots. With v denoting the root of R_i, let $R_i' = R_i - v$. We construct R_i with n_i vertices such that

1. $n_i = 15(2^i) - 6$ and R_i is connected,
2. R_i decomposes into a 2-factor on R_i' and $n_i/3$ disjoint triangles, and
3. $\alpha(R_i') = 4(2^i) - 2$, with a maximum independent set avoiding the neighbors of v.

We show R_0 in Figure 4 with root c_3. This graph is connected, has $15(2^0) - 6$ vertices, and is the union of a 2-factor on R_0' and triangles with vertex sets T_a, T_b, and T_c. An independent set in R_0' has at most one vertex from each 4-clique, and $\{a_1, b_3\}$ is an independent set of size 2 avoiding T_c, so $\alpha(R_0') = 4(2^0) - 2 = 2$.

For $i \geq 1$, start with two disjoint copies of R_{i-1}, having roots c_3 and d_3. Add triples T_x and T_y on six new vertices. Augment the union of the 2-factors in the copies of R_{i-1}'

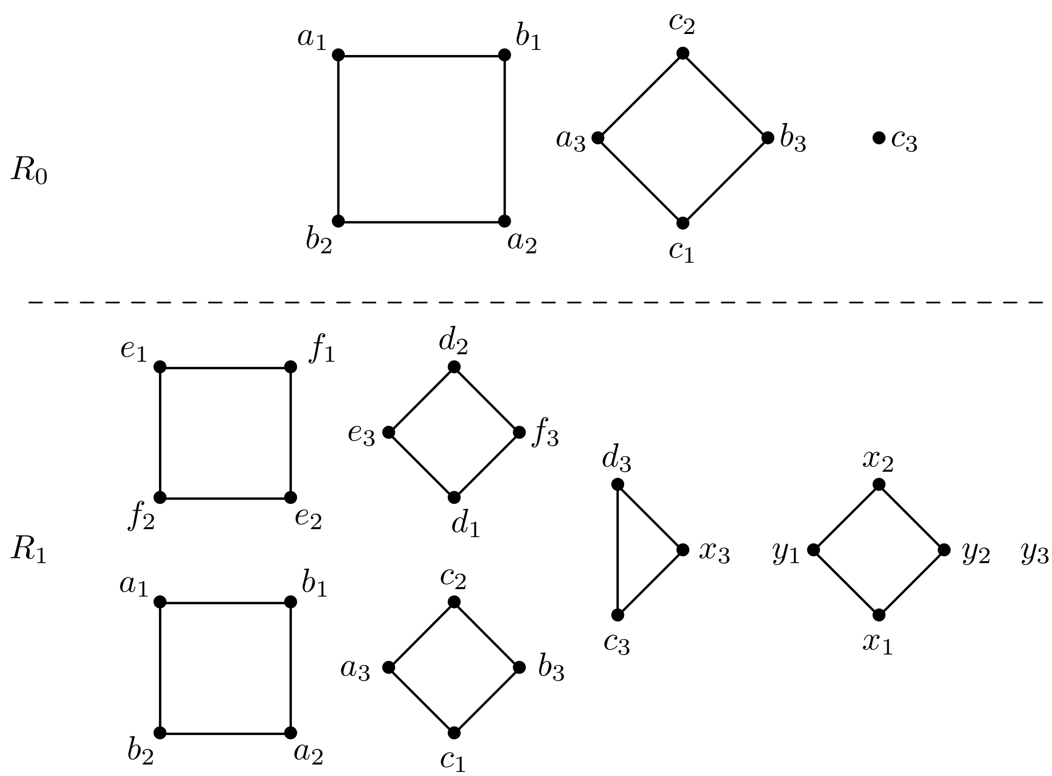

Figure 4: The graphs R_0 and R_1

by adding the 3-cycle $[c_3, d_3, x_3]$ and the 4-cycle $[x_1, y_1, x_2, y_2]$. Leave y_3 as the root in the resulting graph R_i. Figure 4 shows R_1.

Doubling and adding six vertices shows inductively that $n_i = 15(2^i) - 6$. By construction, R_i is the union of a 2-factor on R_i' and $n_i/3$ disjoint triangles. For connectedness, note that inductively each vertex in a copy of R_{i-1} has a path to its root, and using the added 3-cycle, 4-cycle, and triples yields a path from each vertex to the root of R_i.

It remains to check property (3). Let I be an independent set in R_i'. Maximizing the contributions to I from the two copies of R_{i-1}' yields $|I| \le 2\alpha(R_{i-1}') + 2 = 4(2^i) - 2$. Furthermore, since R_{i-1}' has a maximum independent set avoiding the neighbors of the root of R_{i-1}, we can use c_3 and x_1 as the two added vertices from R_i', thereby forming a maximum independent set in R_i' that avoids T_y.

In forming Q_i by adding a 3-cycle on the roots of three disjoint copies of R_i, we obtain a connected 2-factor-plus-triangles graph. We can obtain maximum contribution from the three copies of R_i' obtained by deleting the roots without using any neighbor of the roots. Hence $\alpha(Q_i) = 3\alpha(R_i') + 1 = 12(2^i) - 5$. With Q_i having $3n_i$ vertices, we obtain the independence ratio claimed. $\square$

In light of Erdős' question concerning the 3-colorability of graphs in $\mathcal{G}$ when 4-cycles are excluded from G_1, it is natural to ask whether this additional condition guarantees independence ratio 1/3. The answer is no. For every odd g, we construct infinitely many graphs in $\mathcal{G}'$ with independence ratio less than 1/3 formed using a 2-factor that has girth g. When $g \equiv 1 \mod 6$, the smallest graph in our family has g^2+2g vertices; this suggests the following conjecture, which by our construction would be asymptotically sharp.

Conjecture 3.3. *Every n-vertex graph in $\mathcal{G}'$ with girth at least $\sqrt{n}$ has an independent set of size at least $n/3$.*

Our construction was motivated by an arrangement of triples on a 7-cycle, where two of the triples have one element off the cycle. This arrangement, shown in Figure 5, is due to Sachs (see [6]). We use it to build examples with girth 7. For larger g congruent to 1 modulo 6, we construct an arrangement on a g-cycle. A special list allows us to enlarge the arrangement by multiples of 6.

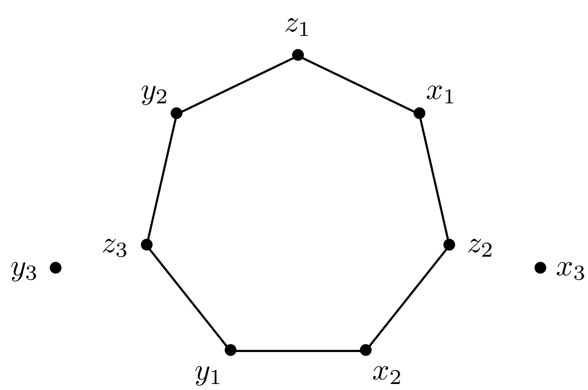

Figure 5: The graph H'_7

Definition 3.4. An *a,b-brick* is a list of six characters plus two holes called *notches*: $(a_1, \square, b_1, a_2, b_2, a_3, \square, b_3)$. An a,b-brick can link to a c,d-brick by starting the c,d-brick at the second notch in the a,b-brick. The last element of the a,b-brick fits into the first notch in the c,d-brick. The link leaves notches in the second and next-to-last positions.

A *starter brick* is a list of seven characters plus two notches that has the form $(y_1, \square, y_2, z_1, x_1, z_2, x_2, \square, z_3)$. For $g = 6j+1$, let H'_g consist of two special vertices x_3 and y_3 plus the cycle of length g whose vertices in order are named by a cyclic arrangement having a starter brick and $a^{(i)}, b^{(i)}$-bricks for $1 \le i \le j-1$, linked together in order. The $a^{(1)}, b^{(1)}$-brick links to the second notch of the starter brick, and the $a^{(j-1)}, b^{(j-1)}$-brick links at its end to the first notch of the starter brick. In the degenerate case $j = 1$, the starter brick links to itself, producing the graph H'_7 shown in Figure 5. For each symbol q, the vertices of $\{q_1, q_2, q_3\}$ in H'_g form a triangle. Note that H'_g has $g+2$ vertices.

The remaining theorems in this section rest on the following simple lemma.

Lemma 3.5. *Let I be an independent set intersecting triples T_a and T_b in a graph G in $\mathcal{G}$. If T_a and T_b form an a,b-brick in G_1, and I contains the vertex in a notch of the a,b-brick, then I also contains the vertex farthest from it in the a,b-brick.*

Proof. An a,b-brick has the form $(a_1, \Box, b_1, a_2, b_2, a_3, \Box, b_3)$. If I contains the vertex in the first notch, then I omits a_1 and b_1. Since I must intersect T_a, we have $b_2 \notin I$. Hence I must contain b_3 to intersect T_b. $\Box$

Theorem 3.6. *For each odd g, there are in $\mathcal{G}'$ infinitely many graphs with girth g whose independence ratio is less than $1/3$.*

Proof. First suppose that $g = 6j+1$. For $k \geq 1$, we construct such a graph $H_{g,h,k}$ with $(g+2)hk$ vertices. Start with hk copies of the graph H'_g of Definition 3.4, where h is odd and at least 3. The vertices having the three subscripted copies of a given label form a triple, with x_3 and y_3 lying outside the cycle as in Figure 5. Each copy of H'_g requires an additional superscript in the labels to distinguish its vertices from those of other copies.

Number the copies 0 through $hk-1$. For $0 \leq i \leq k-1$, add a cycle on the vertices representing x_3 in copies $hi+1$ through $hi+h \pmod{hk}$ of H', and add a cycle on the vertices representing y_3 in copies hi through $hi+h-1$ of H'. This completes the graph $H_{g,h,k}$; note that it has $(g+2)hk$ vertices and is a 2-factor-plus-triangles graph.

Since H'_g has an x_3, y_3-path, the cycles on the copies of x_3 and y_3 make it possible to reach each copy of H' from any other. Hence $H_{g,h,k}$ is connected.

Each cycle in the 2-factor forming $H_{g,h,k}$ has length g or h. A cycle of length h contributes at most $(h-1)/2$ vertices to an independent set; we apply this to the cycles through the copies of x_3 and y_3. There are $2k$ such cycles, contributing at most $k(h-1)$ vertices. In addition, we claim that the g-cycle in each copy of H'_g contributes at most $2j$ vertices to an independent set; note that $2j = (g-1)/3$. If this claim is true, then

$$\alpha(H_{g,h,k}) \leq hk\frac{g-1}{3} + k(h-1) = hk\frac{g+2}{3} - k < hk\frac{g+2}{3} = \frac{1}{3}|V(H_{g,h,k})|.$$

The inequality would be too weak if the g-cycle could contribute $2j+1$ vertices.

To prove the claim, note that the g-cycle contains the vertices of $2j-1$ full triples (including one in the starter brick) plus $\{x_1, x_2, y_1, y_2\}$. To contribute more than $2j$ vertices, we must find an independent set having an element from each full triple, plus one of $\{x_1, x_2\}$ and one of $\{y_1, y_2\}$.

Suppose that such an independent set I exists. Since the last vertex of each brick fits into the first notch of the next brick, $z_3 \in I$ implies $b_3^{(j-1)} \in I$, and $y_1 \in I$ implies $a_1^{(1)} \in I$, by applying Lemma 3.5 iteratively to each ordinary brick. In the first case, $b_3^{(j-1)} \in I$ forbids having a vertex from $\{y_1, y_2\}$. In the second case, $x_2, z_3 \notin I$, and I cannot have two elements in $\{z_1, x_1, z_2\}$. Both arguments apply in degenerate form when $k = 0$.

In the remaining case, $z_3, y_1 \notin I$. Here one from each of T_x, T_y, T_z must be chosen nonconsecutively from the string $(y_2, z_1, x_1, z_2, x_2)$, and this is not possible. This completes the argument for $g \equiv 1 \mod 6$.

When $g \not\equiv 1 \pmod 6$, we set h to be g and let the first value higher than g that is congruent to 1 modulo 6 play the role of g in the construction above. Since k is arbitrary, the family is still infinite. $\square$

To form the smallest example constructed in Theorem 3.6 when $g \equiv 1 \mod 6$, set $h = g$ and $k = 1$. The resulting graph $H_{g,g,1}$ has girth g and has $g^2 + 2g$ vertices. Letting $n = |V(H_{g,g,1})|$, we have an n-vertex example where G_1 has girth $\sqrt{n+1} - 1$ and the independence ratio (of $H_{g,g,1}$) is less than 1/3. When $g \not\equiv 1 \mod 6$ and we must use $H'_{g'}$ for some g' larger than g, we use even more vertices. This motivates Conjecture 3.3.

Although girth at least $\sqrt{n}$ in G_1 may be enough to force an independent set of size $n/3$ in G, it does not force 3-colorability. Surprisingly, no threshold for the girth in terms of n forces this except n itself, where G becomes a cycle-plus-triangles graph. Note that if the girth of an n-vertex 2-regular graph G_1 is not n, then it is at most $n/2$.

Theorem 3.7. *If $n = 24+12k$ with $k \geq 0$, then there is an n-vertex 2-factor-plus-triangles graph G such that G_1 consists of two $n/2$-cycles and G is not 3-colorable.*

Proof. We use $a^{(i)}, b^{(i)}$-bricks as in Theorem 3.6, but for this theorem the starter bricks have 12 symbols plus two notches. We use two starter bricks:

$$(z_1, \square, z_2, u_1, z_3, u_2, v_3, w_3, y_2, x_3, y_1, x_2, \square, y_3)$$

$$(\hat{z}_2, \square, \hat{z}_3, v_1, w_1, \hat{z}_1, v_2, w_2, u_3, \hat{y}_2, x_1, \hat{y}_3, \square, \hat{y}_1)$$

Let G_1 consist of cycles C and $\hat{C}$, where C consists of the first starter brick and $a^{(i)}, b^{(i)}$-bricks for $1 \leq i \leq k$, and $\hat{C}$ consists of the second starter brick and $\hat{a}^{(i)}, \hat{b}^{(i)}$-bricks for $1 \leq i \leq k$, linked in order as in Theorem 3.6. The triples for u, v, w, x create connections between the two cycles, but all other triples are confined to C or to $\hat{C}$. When $k = 0$, each starter brick links into itself to form a 12-cycle. (Examples with n vertices and girth $n/2 - 6r$ arise by using $k - r$ ordinary bricks in C and $k + r$ ordinary bricks in $\hat{C}$; the same argument applies.

Suppose that the resulting graph G has a proper 3-coloring f. Each color class is an independent set having one vertex in each triple. Simplifying notation, let b_3 and a_1 denote the vertices in the first and second notches of the starter brick in C, respectively, while $\hat{b}_3$ and $\hat{a}_1$ denote those vertices in $\hat{C}$. Without loss of generality, we may assume that $f(a_1) = 1$. Repeatedly applying Lemma 3.5 yields $f(z_1) = 1$. Now we may assume that $f(b_3) = 3$; repeatedly applying Lemma 3.5 yields $f(y_3) = 3$.

If the neighbors in G_1 of a vertex α belong to the same triple, then the third member of that triple must have the same color as α. Hence $f(x_3) = f(y_3) = 3$, and $f(u_1) = f(z_1) = 1$. Also, if a vertex next to α and another member of the triple containing α have distinct colors, then $f(\alpha)$ is the third color. Hence $f(x_2) = 2$ and $f(z_2) = 2$. Once we color two members of a triple, the third has the third color. Hence $f(x_1) = 1$ and $f(z_3) = 3$. If two neighbors of α have distinct colors, then α has the third color. Hence $f(y_1) = 1$. Now $f(y_2) = 2$.

Since $f(z_3) = 3$ and $f(u_1) = 1$, we have $f(u_2) = 2$, and then $f(u_3) = 3$. Now $f(x_1) = 1$ and $f(u_3) = 3$ imply $f(\hat{y}_2) = 2$, and hence $f(\hat{y}_3) = 3$ and $f(\hat{y}_1) = 1$. This leaves $f(\hat{a}_1) = 2$. Iterating Lemma 3.5 now yields $f(\hat{z}_2) = 2$ and $f(\hat{b}_3) = 1$. Now $f(\hat{z}_3) = 3$ and $f(\hat{z}_1) = 1$.

We have now determined the colors of all vertices in the starter bricks except those in the triples T_v and T_w. For all other vertices in these bricks, the color matches the subscript. The relevant remaining segments are (u_2, v_3, w_3, y_2) and $(\hat{z}_3, v_1, w_1, \hat{z}_1, v_2, w_2, u_3)$. Color 2 is forbidden from $\{v_3, w_3\}$. Hence it appears on one of $\{v_1, v_2\}$ and one of $\{w_1, w_2\}$. However, the subscripts on its appearances differ. If $f(v_1) = f(w_2) = 2$, then $f(w_1) = f(v_2) = 3$ (since $f(\hat{z}_1) = 1$), and then $f(v_3) = f(w_3)$. If $f(v_2) = f(w_1) = 2$, then $f(w_2) = f(v_1) = 1$ (since $f(\hat{z}_3) = f(u_3) = 3$), and again $f(v_3) = f(w_3)$. Hence the coloring cannot be completed. □

4 Triangles-Plus-Triangles Graphs

Although some 2-factor-plus-triangles graphs are not 3-colorable, some (such as cycle-plus-triangles graphs) are 3-choosable. Another such class occurs at the other "extreme", when the cycles in the 2-factor are 3-cycles. That is, the union of two graphs on the same vertex set whose components are all triangles is 3-choosable.

We prove a more general statement in terms of the numbers of vertices in the components of two subgraphs whose union is G. Our main tool is the theorem of Galvin [7] about list coloring of the line graphs of bipartite graphs: if G is a bipartite multigraph with maximum degree k, then the line graph of G is k-choosable.

Proposition 4.1. *If G_1 and G_2 are graphs whose components have at most s vertices, then $G_1 \cup G_2$ is s-choosable.*

Proof. Let $G = G_1 \cup G_2$. By adding isolated vertices to G_1 and/or G_2 as needed, we may assume that $V(G_1) = V(G_2) = V(G)$ without changing G. For each $v \in V(G)$, let $L(v)$ be a set of s available colors. Form a graph H with a vertex for each component of G_1 and a vertex for each component of G_2. For each vertex of G, place an edge in H joining the vertices representing the components containing it in G_1 and G_2 (H is the "intersection graph" of the components in G_1 and G_2). By construction, H is bipartite. The degree of a vertex in H is the number of vertices in the corresponding component of G_1 or G_2.

Each edge of H corresponds to a vertex v in G. Assign to this edge the list $L(v)$. Since H is bipartite and has maximum degree at most s, Galvin's Theorem implies that we can choose a proper edge-coloring of H from the lists. This assigns colors to the vertices of G from their lists so that vertices in the same component of G_1 or in the same component of G_2 have distinct colors. Hence it is a proper coloring of G. □

In particular, every triangles-plus-triangles graph is 3-choosable.

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