k-cycle free one-factorizations of complete graphs

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Abstract

It is proved that for every $n \geq 3$ and every even $k \geq 4$, where $k \neq 2n$, there exists one-factorization of the complete graph K_{2n} such that any two one-factors do not induce a graph with a cycle of length k as a component. Moreover, some infinite classes of one-factorizations, in which lengths of cycles induced by any two one-factors satisfy a given lower bound, are constructed.

1 Introduction

A *one-factor* of a graph G is a regular spanning subgraph of degree one. A *one-factorization* of G is a set $F = \{F_1, F_2, \ldots, F_n\}$ of edge-disjoint one-factors such that $E(G) = \bigcup_{i=1}^{n} E(F_i)$. Evidently, the union of two edge-disjoint one-factors is a two-factor consisting of cycles of even lengths.

The exact number $N(2n)$ of all pairwise non-isomorphic one-factorizations of the complete graph K_{2n} is known only for $2n \leq 14$; namely $N(4) = N(6) = 1$, $N(8) = 6$, $N(10) = 396$, cf. [14], $N(12) = 526,915,620$ [8], and $N(14) = 1,132,835,421,602,062,347$ [10]. Moreover, Cameron [4] proved that $\ln N(2n) \sim 2n^2 \ln(2n)$ for sufficiently large n. Therefore, any investigations (including enumeration) regarding all one-factorizations of K_{2n} are deemed reasonable if they are restricted to a subclass which satisfies some additional properties. One of the obvious requirements concerns an isomorphism of graphs induced by pairs of one-factors. In this way, a question arises regarding the existence of uniform (perfect) one-factorizations. A one-factorization is *uniform* when the union of any two one-factors is isomorphic to the same graph H. In particular, if H is connected (i.e. a Hamiltonian cycle), then a one-factorization is called *perfect*.

Perfect one-factorizations of complete graphs were introduced by Kotzig [11] and in known notation by Anderson [2]. Only three infinite classes of perfect one-factorizations are known, namely when $2n-1$ is prime [11, 3] and when n is prime [1]. All other known examples of perfect one-factorizations of K_{2n} have been found using various methods, cf. [16, 17]. Perfect one-factorization conjecture, which claims the existence of perfect one-factorizations for every even order of the complete graph, is far from proven. Perfect one-factorizations are very rare among all one-factorizations; this argument is supported by a comparison of known numbers, $P(2n)$, of all perfect pairwise non-isomorphic one-factorizations of K_{2n}, with $N(2n)$. There are $P(4) = P(6) = P(8) = P(10) = 1$, $P(12) = 5$, cf. [16, 17], $P(14) = 23$ [7] and $P(16) \geq 88$ [15]. Uniform one-factorizations other than those which are perfect have been investigated far less, cf. [5, 14]. In fact, there are only three known infinite classes and several sporadic examples of uniform non-perfect one-factorizations.

In this context, weaker properties regarding lengths of cycles which are required to exist, or which are forbidden in the union of any two one-factors, may be considered. A one-factorization $F = \{F_1, F_2, \ldots, F_n\}$ of G is said to be *k-cycle free* if the union of any two one-factors does not include the cycle C_k. Consequently, F is *S-cycle free* if the union of any two one-factors does not include cycles of lengths from the set S. In particular, if $S = \{4, 6, \ldots, k\}$, then F is called *$k^<$-cycle free*. It can be said that *F has a cycle of length k* if there are two one-factors in F, the union of which includes C_k.

The aim of this paper is to find, for each n and each even $k \geq 4$ such that $2n \neq k$, a k-cycle free one-factorization of K_{2n}. For $2n \neq p+1$, where p is a prime, or $2n \not\equiv 6, 12, 18 \pmod{24}$, the existence of $2n$-cycle free one-factorizations of K_{2n} is proven. Moreover, some infinite classes of $k^<$-cycle free one-factorizations are constructed.

2 Constructions and their properties

The following two facts are easily observed.

Claim 1 *For $n \geq 2$, if l is the minimum positive integer such that $\gcd(l, n) > 1$, then $\gcd(l, n) = l$. Moreover, for odd $n' \geq 3$, if l' is the minimum even positive integer such that $\gcd(l', n') > 1$, then $\gcd(l', n') = l'/2$.* □

Claim 2 *If $k > 2n \geq 4$, then any one-factorization of K_{2n} is k-cycle free.* □

The well-known canonical one-factorization GK_{2n} of K_{2n} has been published in Lucas' [13] and attributed to Walecki.

Construction A Let $V = \{\infty, 0, 1, \ldots, 2n-2\}$. Let GK_{2n} denote a one-factorization of K_{2n} which consists of one-factors $F_i = \{\{i-j, i+j\} : j = 1, 2, \ldots n-1\} \cup \{\infty, i\}$, for $i = 0, 1, \ldots 2n-2$, where labels are taken modulo $2n-1$.

It is well-known that GK_{2n} is perfect if and only if $2n-1$ is prime [11]. Dinitz et. al. [6] investigated lengths of cycles which may appear in the union of any two one-factors in GK_{2n}. Lemma 3 is a corollary to that result; a short proof is presented here in order to provide detailed constructions of cycles applied in further results.

Lemma 3 (cf. [6]) *For $n \geq 3$ and even k such that $4 \leq k \leq 2n$, the one-factorization GK_{2n} of K_{2n} contains a cycle of length k if and only if $k/2 \mid 2n-1$ or $k-1 \mid 2n-1$.*

Proof: Let $p = 2n - 1$. Assume that GK_{2n} contains a cycle C_k of length k which appears in the union H of two one-factors F_h and F_i, where $h < i$ and $h, i \in \{0, 1, \ldots, p-1\}$. Let $z = i - h$. Consider separately two cases.
Case I: C_k contains the vertex ∞. Then neighbors of ∞ in H are h and i. Consecutive vertices along the cycle C_k in H are: ∞, i, $h-z$, $i+2z$, $h-3z$, $i+4z$, $h-5z$, ..., $i+(k-2)z$, ∞, where k is the minimum even positive integer such that $i + (k-2)z \equiv h \pmod{p}$ (which is equivalent to $(k-1)z \equiv 0 \pmod{p}$). Since $0 < z < p$, $\gcd(k-1, p) = k-1$ follows by Claim 1.
Case II: C_k does not contain ∞. Let $h + x$ be a vertex of C_k. Then $x \neq 0$ and neighbors of $h + x$ in H are $h - x$ and $i + z - x$. Consecutive vertices along C_k in H are: $h + x$, $h - x$, $i + z + x$, $h - 2z - x$, $i + 3z + x$, $h - 4z - x$, ..., $h - (k-2)z - x$, $h + x$, where k is the minimum even positive integer such that $h - (k-2)z - x \equiv i + z - x \pmod{p}$. Similarly to the above, since $0 < z < p$, by the equivalence $kz \equiv 0 \pmod{p}$ and Claim 1, $\gcd(k, p) = k/2$.

To prove sufficiency, suppose first that $k \leq 2n$ and $k/2 \mid p$. Then $k \equiv 2 \pmod{4}$. In order to find a cycle of length k, take two one-factors F_0 and F_i, where $i = \frac{p}{k/2}$. Let l be the length of a cycle which does not contain ∞ in the union of F_0 and F_i. Then, repeating calculations of Case II, l is the minimum even positive integer such that $li \equiv 0 \pmod{p}$. Then $\frac{lp}{k/2} \equiv 0 \pmod{p}$ and next, since $k/2$ is odd, $l = k$. Similarly, for any even $k \leq 2n$ where $k - 1 \mid p$, two one-factors F_0 and F_j are taken, where $j = \frac{p}{k-1}$. If l is the length of a cycle which contains ∞, then as in Case I, l is the minimum even positive integer such that $(l-1)j \equiv 0 \pmod{p}$. Thus $l = k$. $\square$

The above Lemma 3 is equivalent to the following result.

Corollary 4 *For $n \geq 3$ and even $k \geq 4$, the one-factorization GK_{2n} of K_{2n} is k-cycle free if and only if $k/2 \nmid 2n-1$ and $k-1 \nmid 2n-1$.* $\square$

By Lemma 3, the one-factorization GK_{2n} has a trivial lower bound on the minimum length of cycles it contains.

Corollary 5 *Let r be the minimum prime factor of $2n-1$. If $r \geq 5$, then the one-factorization GK_{2n} of K_{2n} is $(r-1)^{<}$-cycle free.* $\square$

Lemma 3 immediately yields another property of GK_{2n}. Namely, for any order $2n$, GK_{2n} cannot be a non-perfect uniform one-factorization because GK_{2n} contains a cycle of length $2n$.

Another well-known one-factorization of the complete graph of order $2n$ for odd n is denoted by GA_{2n} [2].

Construction B Let n be odd. In what follows, labels of vertices are taken modulo n. Let $V = V_0 \cup V_1$, where $V_m = \{0_m, 1_m, \ldots, (n-1)_m\}$ for $m = 0, 1$. Let GA_{2n} be a one-factorization of K_{2n} with one-factors $F_0, F_1, \ldots, F_{2n-2}$. Let $F_i = \{\{(i-j)_m, (i+j)_m\} : \ j = 1, 2, \ldots (n-1)/2, \ m = 0, 1\} \cup \{i_0, i_1\}$, for $i = 0, 1, \ldots n-1$. Moreover, let $F_{n+i} = \{\{j_0, (j+i+1)_1\} : \ j = 0, 1, \ldots n-1\}$, for $i = 0, 1, \ldots n-2$.

It is well-known that GA_{2n} is perfect if and only if n is prime [1]. The following presents a stronger property of GA_{2n}.

Lemma 6 *For odd $n \geq 3$ and even k such that $4 \leq k \leq 2n$, the one-factorization GA_{2n} of K_{2n} contains a cycle of length k if and only if $k/2 \mid n$.*

Proof: Assume first that GA_{2n} contains a cycle C_k which is included in the union H of two one-factors F_h and F_i, where $h < i$ and $h, i \in \{0, 1, \ldots, 2n-2\}$. Consider separately three cases.
Case I: $h < i \leq n-1$. Note that, if in the construction of GK_{n+1} the vertex subset $V(K_{n+1}) \setminus \{\infty\}$ is replaced with V_m, for $m = 0, 1$, and moreover, GK_{n+1} is restricted to the vertices of V_m, then a near one-factorization of K_n into near one-factors $F_i^m = \{\{(i-j)_m, (i+j)_m\} : j = 1, 2, \ldots (n-1)/2\}$ is obtained, where $i = 0, 1, \ldots, n-1$. It is clear that $F_i^m \subset F_i$ (the one-factor of GA_{2n}) for every admissible i and m. A cycle C_k in H has all vertices either in V_0 or in V_1 or in both subsets together. In the previous two cases C_k corresponds to a cycle of the same length either in $F_h^0 \cup F_i^0$ or in $F_h^1 \cup F_i^1$. Then, by Case II in the proof of Lemma 3, $\gcd(k, n) = k/2$. In the latter case, $k \equiv 2 \pmod 4$ and C_k consists of two paths of length $k/2 - 1$ (one of them with all vertices in V_0 and the other one with all vertices in V_1) joint together by the edges $\{h_0, h_1\}$ (of F_h) and $\{i_0, i_1\}$ (of F_i). These two paths correspond to a path with endvertices h and i included in a cycle of length $k/2 + 1$ (which contains the vertex ∞), induced by one-factors with indices h and i in GK_{n+1}. Thus, by Case I in the proof of Lemma 3, $\gcd(k/2, n) = k/2$ holds.
Case II: $h < n \leq i$. Consider two subcases.
II.A: h_0 is not a vertex of the cycle C_k in H. Then also h_1 is not in C_k. Note that the length of C_k is divisible by 4. Let $(h+x)_0$ be a vertex of C_k for some $x \neq 0$. Then neighbors of $(h+x)_0$ in H are $(h-x)_0$ and $(h+x+i+1)_1$. Consecutive vertices along the cycle C_k in H are: $(h+x)_0$, $(h+x+i+1)_1$, $(h-x-i-1)_1$, $(h-x-2i-2)_0$, $(h+x+2i+2)_0$, $(h+x+3i+3)_1$, $(h-x-3i-3)_1$, $\ldots$, $(h-x-\frac{k(i+1)}{2})_0$, $(h+x)_0$, where k is the minimum even positive integer such that $h - x - \frac{k(i+1)}{2} \equiv h - x \pmod n$. Since $n < i+1 < 2n$, by the above equivalence $\frac{k}{2}(i+1) \equiv 0 \pmod n$ and Claim 1, $\gcd(k/2, n) = k/2$.
II.B: h_0 is a vertex of C_k. Then h_1 is in C_k as well. Note that $k \equiv 2 \pmod 4$. The neighbors of h_0 in H are h_1 and $(h+i+1)_1$. Consecutive vertices along the cycle C_k in H are: h_0, $(h+i+1)_1$, $(h-i-1)_1$, $(h-2i-2)_0$, $(h+2i+2)_0$, $(h+3i+3)_1$, $(h-3i-3)_1$, $\ldots$, $(h+\frac{k(i+1)}{2})_1$, h_0, where k is the minimum even positive integer such that $h + \frac{k(i+1)}{2} \equiv h \pmod n$. Analogously to the previous case, since $n < i+1 < 2n$, by Claim 1, $\gcd(k/2, n) = k/2$ is easily observed.
Case III: $n \leq h < i$. Then neighbors of y_0 in H are $(y+h+1)_1$ and $(y+i+1)_1$. Consecutive vertices along C_k in H are: y_0, $(y+i+1)_1$, $(y+i-h)_0$, $(y+2i-h+1)_1$, $(y+2i-2h)_0$, $\ldots$, $(y+\frac{ki-(k-2)h+2}{2})_1$, y_0, where $y + \frac{ki-(k-2)h+2}{2} \equiv y+h+1 \pmod n$. Similarly to the previous case, since $0 < i-h < n$, by $\frac{k}{2}(i-h) \equiv 0 \pmod n$ and Claim 1, $\gcd(k/2, n) = k/2$ holds.

To show sufficiency, suppose that $k \leq 2n$ and $k/2 \mid n$. To find a cycle of length k, take one-factors F_n and F_i such that $i = n + \frac{n}{k/2}$. Note that, if l is the length of a cycle in the union of F_n and F_i, then l is the minimum even positive integer such that $\frac{l}{2}(i-n) \equiv 0 \pmod{n}$ (cf. calculations of Case III above). Thus $\frac{l}{2}\frac{n}{k/2} \equiv 0 \pmod{n}$, whence $l = k$. □

Lemma 6 is equivalent to the following.

Corollary 7 *For odd $n \geq 3$ and even $k \geq 4$, the one-factorization GA_{2n} of K_{2n} is k-cycle free if and only if $k/2 \nmid n$.* □

Lemma 6 immediately provides a lower bound on the minimum length of cycles in GA_{2n}.

Corollary 8 *Let n be odd and $n \geq 3$. Let r be the minimum prime factor of n. Then the one-factorization GA_{2n} of K_{2n} is $(2r-2)^{<}$-cycle free.* □

Lemma 6 also yields an obvious corollary that GA_{2n} cannot be a non-perfect uniform one-factorization.

Presented below is an inductive construction for another family of one-factorizations of K_{2n}.

Construction C Let n be even. In what follows, labels of vertices are taken modulo n. Let $V = V_0 \cup V_1$, where $V_m = \{0_m, 1_m, \ldots, (n-1)_m\}$ for $m = 0, 1$. Let $\overline{F} = \{\overline{F}_0, \overline{F}_1, \ldots, \overline{F}_{n-2}\}$ be a k-cycle free one-factorization of K_n, where $\overline{V} = V(K_n) = \{0, 1, \ldots, n\}$. Two copies of $\overline{F}$ are taken by replacing $\overline{V}$ with V_0 and V_1, respectively. In this way, $n-1$ one-factors F_i of K_{2n} are obtained, $i = 0, 1, \ldots, n-2$. The nth one-factor is $F_{n-1} = \{\{j_0, j_1\} : j = 0, 1, \ldots n-1\}$. Remaining $n-1$ one-factors are built based on one-factors in $\overline{F}$; namely, if $\{v_0, u_0\}$ is the edge of one-factor $\overline{F}_h$, for some $h \in \{0, 1, \ldots, n-2\}$, then $\{v_0, u_1\}$ and $\{v_1, u_0\}$ are the edges of one-factor F_{n+h}.

The above method allows for the construction of k-cycle free one-factorizations of K_{2n}, where n is even and $k \not\equiv 4 \pmod{8}$.

Lemma 9 *For even $n \geq 4$ and even $k \geq 6$ such that $k \not\equiv 4 \pmod{8}$, if there is a k-cycle free one-factorization of K_n, then a k-cycle free one-factorization of K_{2n} exists.*

Proof: Assume that a k-cycle free one-factorization $\overline{F}$ of K_n is given. Let H be the union of two one-factors F_h and F_i in the one-factorization of K_{2n} obtained by applying Construction C, where $h < i$ and $h, i \in \{0, 1, \ldots, 2n-2\}$. If both $h, i < n-1$, then H does not contain C_k because all cycles in H are, in fact, copies of cycles in the given one-factorization $\overline{F}$ of K_n which is k-cycle free. If $i = n-1$ or $h = n-1$, one can see that every cycle in H has length 4. In what follows, assume that $i \geq n$. If $i - h = n$, it is evident that every cycle in H has length 4 as well. Otherwise $i - h \neq n$. Note that every cycle in H corresponds to a cycle in the union of one-factors $\overline{F}_h$ and $\overline{F}_{i-n}$ in K_n. Let C_l denote a cycle of length l in $\overline{F}_h \cup \overline{F}_{i-n}$ with consecutive vertices $v^1, v^2, v^3, \ldots, v^l$. Suppose that $h < n-1$. Note that C_l corresponds either to a cycle C'_l (if $l \equiv 0 \pmod{4}$) of length l or to a cycle C''_{2l} (if $l \equiv 2 \pmod{4}$) of length $2l$ in

H; consecutive vertices of C'_l are $v_0^1, v_0^2, v_1^3, v_1^4, v_0^5, v_0^6, \ldots, v_1^{l-1}, v_1^l$, while C'_{2l} has vertices $v_0^1, v_0^2, v_1^3, v_1^4, \ldots, v_0^{l-1}, v_0^l, v_1^{l+1}, v_1^{l+2}, \ldots, v_1^{2l-1}, v_1^{2l}$. In the latter case, by the assumption, $k \neq 2l$. Consider the last case $n \leq h$. Then C_l corresponds to a cycle C'''_l of the same length l in H with consecutive vertices $v_0^1, v_1^2, v_0^3, v_1^4, \ldots, v_0^{l-1}, v_1^l$. Hence, since K_n is k-cycle free, $k \neq l$ and the assertion holds. □

By the above Lemma 9, if $2n \equiv 0 \pmod 8$, then a one-factorization built by applying Construction C does not contain a cycle of length $2n$. Moreover, starting from $n = 4$ and applying the above inductive construction for consecutive powers of 2, a well-known class of uniform one-factorizations of complete graphs with all cycles of length 4 is easily obtained, cf. [4].

Construction C also enables the building of a $\{k/2, k\}$-cycle free one-factorization of K_{2n}, using a given $\{k/2, k\}$-cycle free one-factorization of K_n.

Lemma 10 *For even $n \geq 4$ and even $k \geq 12$ such that $k \equiv 4 \pmod 8$, if there is a $\{k/2, k\}$-cycle free one-factorization of K_n, then a $\{k/2, k\}$-cycle free one-factorization of K_{2n} exists.*

Proof: The assertion follows immediately from the proof of Lemma 9. Namely, by the assumption, a given one-factorization of K_n does not contain a cycle of length l such that $l \equiv 2 \pmod 4$. Hence, by the proof of Lemma 9, every cycle in a one-factorization of K_{2n}, obtained by applying Construction C, has either length 4 or has the same length as a corresponding cycle in a given one-factorization of K_n. □

The next infinite class of one-factorizations yields further examples of k-cycle free and $k^<$-cycle free one-factorizations of complete graphs.

Construction D Let $p \geq 3$ be a prime and $r = (p-1)/2$. Let n be an odd integer such that $n \geq p$ and $\gcd(n, r) = 1$. Let r^{-1} be the inverse of r in $\mathbb{Z}_n$. In what follows, labels of vertices are taken modulo n, while indices are taken modulo p. Consider a one-factorization of K_{pn+1} denoted by HK_{pn+1}. Let $V = V_0 \cup V_1 \cup \ldots \cup V_{p-1}$, where $V_m = \{\infty, 0_m, 1_m, \ldots, (n-1)_m\}$ for $m = 0, 1, \ldots, p-1$. Thus $V_0 \cap V_1 \cap \ldots V_{p-1} = \{\infty\}$. Let $F_{mn+i} = \{\{(i-j)_m, (i+j)_m\} : \ j = 1, 2, \ldots (n-1)/2\} \cup \{i_m, \infty\} \cup \{\{j_{m-s}, -(j + (i + m)r^{-1})_{m+s}\} : \ j = 0, 1, \ldots n-1, \ s = 1, 2, \ldots, r\}$ for $i = 0, 1, \ldots n-1$, $m = 0, 1, \ldots, p-1$.

Note that HK_{np+1} is an extension of GK_p: one-factorization induced by every V_i is the one-factorization GK_{n+1} of K_{n+1}. Moreover, if every set $V_i \setminus \infty$ is replaced by a single vertex u_i, and all edges with the same endvertices are contracted to a single edge, loops being removed, then the corresponding one-factorization GK_{p+1} of K_{p+1} would be obtained.

Presented below are investigations into possible lengths of cycles in HK_{pn+1}.

Lemma 11 *For odd prime p and for odd n such that $n \geq p$ and $\gcd(n, (p-1)/2) = 1$, and for even k such that $4 \leq k \leq pn+1$, the one-factorization HK_{pn+1} of K_{pn+1} contains a cycle of length k if and only if one of the following conditions holds:*

(1) $k \le n+1$ *and* $k-1 \mid n$,
(2) $k > n+1$ *and* $k-1 \mid np$,
(3) $k \le 2n$ *and* $k/2 \mid n$,
(4) $k > 2n$ *and* $k/2 \mid np$.

Proof: Assume that HK_{pn+1} contains a cycle of length k which appears in the union H of one-factors F_h and F_i, where $h < i$ and $h, i \in \{0, 1, \dots, pn-1\}$. Consider separately two cases.
Case I: $mn \le h < i < (m+1)n$ for some $m \in \{0, 1, \dots, p-1\}$. One-factorization induced by V_m is the one-factorization GK_{n+1} of K_{n+1} and therefore, by Lemma 3, either condition (1) or (3) is satisfied when $k \le n+1$ and all vertices of C_k come from V_m. Consider the case where all vertices of C_k are in $V \setminus V_m$. In fact, all vertices of C_k are in $V_{m-s} \cup V_{m+s}$ for some $s \in \{1, 2 \dots, r\}$. Then clearly $k \le 2n$. Let y_{m-s} be a vertex of C_k. Neighbors of the vertex y_{m-s} are $-(y+(h+m)r^{-1})_{m+s}$ and $-(y+(i+m)r^{-1})_{m+s}$. Consecutive vertices along the cycle C_k are: y_{m-s}, $-(y+(h+m)r^{-1})_{m+s}$, $(y+(h-i)r^{-1})_{m-s}$, $-(y+(2h-i+m)r^{-1})_{m+s}$, $(y+(2h-2i)r^{-1})_{m-s}$, $\dots$, $-(y+\frac{kh-(k-2)i+2m}{2}r^{-1})_{m+s}$, y_{m-s}, where k is the minimum even positive integer such that $-y-\frac{kh-(k-2)i+2m}{2}r^{-1} \equiv -y-(i+m)r^{-1} \pmod n$. Since $0 < i-h < n$, then $0 < (i-h)r^{-1} < n$ and, by the equivalence $\frac{k}{2}(i-h)r^{-1} \equiv 0 \pmod n$ and Claim 1, $\gcd(\frac{k}{2}, n) = \frac{k}{2}$ and then (3) holds.
Case II: $mn \le h < (m+1)n$ and $qn \le i < (q+1)n$ for some $m, q \in \{0, 1, \dots, p-1\}$, $m < q$. Let $z = q-m$. Consider two subcases.
II.A: ∞ is a vertex of C_k. Then $k \equiv p+1 \pmod{2p}$. Neighbors of ∞ in H are h_m and i_q. Note that indices of consecutive vertices in the cycle C_k appear in the order according to the labels of vertices in Case I of the proof of Lemma 3. Thus the first $p+1$ consecutive vertices along C_k in H are: ∞, $i_q = i_{m+z}$, $-(h+ri+m)r^{-1}_{m-z}$, $(h+(r-1)i+m-q)r^{-1}_{m+3z}$, $-(2h+(r-1)i+2m-q)r^{-1}_{m-3z}$, $(2h+(r-2)i+2m-2q)r^{-1}_{m+5z}$, $\dots$, $(rh+(r-r)i+rm-rq)r^{-1}_{m+pz} = (h-z)_m$. Note that $(h-z)_m \ne h_m$ because $0 < z < p \le n$. Thus the neighbor of $(h-z)_m$ in F_h is $(h+z)_m$. Moreover, $(i+2z)_{m+z} \ne i_{m+z}$. Then the next $2p$ consecutive vertices along C_k in H are: $(h+z)_m = (h+z)_{q-z}$, $-(rz+rh+i+q)r^{-1}_{q+z}$, $(rz+(r-1)h+i+q-m)r^{-1}_{q-3z}$, $-(rz+(r-1)h+2i+2q-m)r^{-1}_{q+3z}$, $(rz+(r-2)h+2i+2q-2m)r^{-1}_{q-5z}$, $\dots$, $(rz+(r-r)h+ri+rq-rm)r^{-1}_{q-pz} = (i+2z)_{m+z}$, $(i-2z)_{m+z}$, $-(-2rz+h+ri+m)r^{-1}_{m-z}$, $(-2rz+h+(r-1)i+m-q)r^{-1}_{m+3z}$, $-(-2rz+2h+(r-1)i+2m-q)r^{-1}_{m-3z}$, $(-2rz+2h+(r-2)i+2m-2q)r^{-1}_{m+5z}$, $\dots$, $(-2rz+rh+(r-r)i+rm-rq)r^{-1}_{m+pz} = (h-3z)_m$. Therefore, after the next $\frac{k-(3p+1)}{2p}$ segments, each of which contains $2p$ vertices, the kth vertex in C_k is $(h-\frac{k-1}{p}z)_m = h_m$. Since $0 < z < p \le n$, if k is the minimum even positive integer such that $\frac{k-1}{p}z \equiv 0 \pmod n$, then $k-1 > n$ and moreover, by Claim 1, $\gcd(\frac{k-1}{p}, n) = \frac{k-1}{p}$. Thus (2) is satisfied.
II.B: ∞ is not a vertex of C_k. Then $k \equiv 0 \pmod{2p}$. Let $(h+x)_m$ be a vertex of C_k. Then $x \ne 0$ and neighbors of $(h+x)_m$ in H are $(h-x)_m$ and $(-h+x-(i+q)r^{-1})_{q+z}$. First segment of $2p$ consecutive vertices along C_k is (cf. second segment of C_k in Subcase II.A): $(h+x)_m = (h+x)_{q-z}$, $-(rx+rh+i+q)r^{-1}_{q+z}$, $(rx+(r-1)h+i+q-m)r^{-1}_{q-3z}$,

$\ldots$, $(i+x+z)_q = (i+x+z)_{m+z}$, $(i-x-z)_{m+z}$, $-(-rx+h+ri+(r+1)m-rq)r^{-1}_{m-z}$, $(-rx+h+(r-1)i+(r+1)m-(r+1)q)r^{-1}_{m+3z}$, $\ldots$, $(h-x-2z)_m \neq (h-x)_m$. After the next $\frac{k}{2p}-1$ segments, each of which contains $2p$ vertices, we end up at $(h-x-\frac{k}{p}z)_m = (h-x)_m$. Since $0 < z < p \leq n$ and moreover, k is the minimum even positive integer such that $\frac{k}{p}z \equiv 0 \pmod n$, $k > 2n$ holds and, by Claim 1, $\gcd(\frac{k}{p}, n) = \frac{k}{2p}$. Hence (4) is satisfied.

To prove sufficiency, in order to find a cycle of length k, take the union of two one-factors F_0 and F_i. Let $i = \frac{n}{k-1}$ if $k \leq n+1$ and $k-1 \mid n$. Thus $1 \leq i < n$. Let l be the length of a cycle in the union of F_0 and F_i which contains ∞ and with all vertices in V_0. Then, by applying calculations of Case I in the proof of Lemma 3, l is the minimum even positive integer such that $(l-1)i \equiv 0 \pmod n$. Thus $\frac{l-1}{k-1}n \equiv 0 \pmod n$ and therefore $l = k$. Similarly, let $i = \frac{nr}{k/2} \pmod n$ if $k \leq 2n$ and $k/2 \mid n$. Hence, if l is the length of a cycle in $F_0 \cup F_i$ with all vertices in $V_{p-1} \cup V_1$, by calculations as in Case I above, l is the minimum even positive integer such that $\frac{l}{2}ir^{-1} \equiv 0 \pmod n$. Hence $l = k$. Analogously, let $i = n\frac{np}{k-1}$ $(\geq n)$ if $k > n+1 \geq p+1$ and $k-1 \mid np$. If l is the length of a cycle in $F_0 \cup F_i$ which contains ∞, by calculations as in Subcase II.A above, l is the minimum even positive integer such that $\frac{l-1}{p}z \equiv 0 \pmod n$, where $z = i/n = \frac{np}{k-1} < n$. Then $\frac{l-1}{p}\frac{np}{k-1} \equiv 0 \pmod n$, whence $k = l$. In the last case, if $k > 2n \geq 2p$ and $k/2 \mid np$, then $i = n\frac{np}{k/2} > n$. Note that $k \equiv 2 \pmod 4$. If l is the length of a cycle in the union $F_0 \cup F_i$ which does not contain ∞, then l is the minimum even positive integer such that $\frac{l}{p}z \equiv 0 \pmod n$, where $z = i/n = \frac{np}{k/2} < n$, cf. Subcase II.B. Hence $\frac{l}{p}\frac{np}{k/2} \equiv 0 \pmod n$ and, since $k/2$ is odd, $k = l$ holds. □

Lemma 11 is equivalent to the following result.

Corollary 12 *For odd prime p and for odd n such that $n \geq p$ and $\gcd(n, (p-1)/2) = 1$, and for even $k \geq 4$, the one-factorization HK_{pn+1} of K_{pn+1} is k-cycle free if and only if all of the following conditions hold:*
(1) $k-1 \nmid n$ if $k \leq n+1$,
(2) $k-1 \nmid np$ if $k > n+1$,
(3) $k/2 \nmid n$ if $k \leq 2n$,
(4) $k/2 \nmid np$ if $k > 2n$. □

Lemma 11 yields a trivial lower bound on the minimum length of cycles in HK_{pn+1}.

Corollary 13 *Let p be an odd prime and n be odd such that $n \geq p$ and $\gcd(n, (p-1)/2) = 1$. Let r be the minimum prime factor of n. If $r \geq 5$, then the one-factorization HK_{pn+1} of K_{pn+1} is $(r-1)^{<}$-cycle free.* □

It is clear that HK_{pn+1} cannot be uniform. Taking two one-factors F_0 and F_1, its union H has a cycle of length $n+1$ with all vertices in V_0, while one-factors F_0 and F_n make a Hamiltonian cycle in K_{pn+1}.

The next inductive construction, similar to HK_{pn+1}, produces a one-factorization of K_{pn+1} for odd n and odd prime p, which does not have cycles of even lengths k, where $k \not\equiv 0, p+1 \pmod{2p}$ or $k = p+1$.

Construction E Let $p \geq 3$ be a prime and $r = (p-1)/2$. Let n be an odd integer such that $n \geq p$ and $\gcd(n,r) = 1$. Let r^{-1} be the inverse of r in $\mathbb{Z}_n$. In what follows, labels of vertices are taken modulo n, while indices are taken modulo p. Let $V = V_0 \cup V_1 \cup \ldots \cup V_{p-1}$, where $V_m = \{\infty, 0_m, 1_m, \ldots, (n-1)_m\}$ for $m = 0,1,\ldots,p-1$. Let $\tilde{F}$ be a k-cycle free one-factorization of K_{n+1}, where $\tilde{V} = V(K_{n+1}) = \{\infty, 0, 1, \ldots, n-1\}$. Let $\tilde{F_i}$ be a one-factor in $\tilde{F}$, $i = 0,1,\ldots n-1$. To construct one-factor F_{mn+i} of K_{pn+1}, for $m = 0,1,\ldots,p-1$ and $i = 0,1,\ldots,n-1$, copies of all edges of $\tilde{F_i}$ are taken by replacing $\tilde{V}$ with V_m, and moreover, the set of edges $\{\{j_{m-s}, -(j+(i+m)r^{-1})_{m+s}\} : \ j = 0,1,\ldots n-1, \ s = 1,2,\ldots,r\}$ is added.

Lemma 14 *For odd prime p and for odd $n \geq p$ such that $\gcd(n,(p-1)/2) = 1$, and for even $k \geq 4$, where $k \not\equiv 0, p+1 \pmod{2p}$ or $k = p+1$, and moreover, $k/2 \nmid n$, if there is a k-cycle free one-factorization of K_{n+1}, then a k-cycle free one-factorization of K_{pn+1} exists.*

Proof: Assume that a k-cycle free one-factorization $\tilde{F}$ of K_{n+1} is given. Let H be the union of two one-factors F_h and F_i in the one-factorization obtained by applying Construction E, where $h < i$ and $h, i \in \{0,1,\ldots,pn-1\}$.

Suppose that h and i satisfy $mn \leq h < i < (m+1)n$ for some $m \in \{0,1,\ldots,p-1\}$. Then H does not contain a cycle of length k with all vertices in V_m because one-factorization induced by V_m is the given k-cycle free one-factorization $\tilde{F}$ of K_{n+1}. Moreover, let C_l be a cycle of H with all vertices in $V \setminus V_m$ and let y_{m-s} be a vertex of C_l, for some $s \in \{1, 2\ldots, r\}$. Note that C_l is exactly the same cycle as in Case I of the proof of Lemma 11 and, since $\gcd(k/2, n) < k/2$ by the assumption, $l \neq k$ is satisfied.

It remains to consider the case when $mn \leq h < (m+1)n$ and $qn \leq i < (q+1)n$ for some $m, q \in \{0,1,\ldots,p-1\}$, $m < q$. Let $z = q - m$. If ∞ is a vertex of a cycle C_l in H, then $l \equiv p+1 \pmod{2p}$, cf. Subcase II.A in the proof of Lemma 11. Moreover, neighbors of ∞ in H are h_m and i_q and the first $p+1$ consecutive vertices along the cycle C_l in H (by Subcase II.A in the proof of Lemma 11) are: ∞, i_q, …, $(h-z)_m \neq h_m$. Hence $l \neq p+1$. If ∞ is not a vertex of C_l in H, then $l \equiv 0 \pmod{2p}$, cf. Subcase II.B in the proof of Lemma 11. Thus $l \neq k$. □

To prove main results one more construction, slightly different from Construction E, is needed.

Construction F Let $p \geq 3$ be a prime and $r = (p-1)/2$. Let n be an odd integer such that $n \geq p$ and $\gcd(n,r) = 1$. Let r^{-1} be the inverse of r in $\mathbb{Z}_n$. In what follows, labels of vertices are taken modulo n and moreover, indices are taken modulo p. Let $r = (p-1)/2$. Let $V = V_0 \cup V_1 \cup \ldots \cup V_{p-1}$, where $V_m = \{\infty, 0_m, 1_m, \ldots, (n-1)_m\}$ for $m = 0,1,\ldots,p-1$. Let $\tilde{F}$ be a k-cycle free one-factorization of K_{n+1}, where $\tilde{V} = V(K_{n+1}) = \{\infty, 0, 1, \ldots, n-1\}$. Let $\tilde{F_i}$ be a one-factor in $\tilde{F}$, $i = 0,1,\ldots n-1$. To construct one-factor F_{mn+i} of K_{pn+1}, for $m = 0,1,\ldots,p-1$ and $i = 0,1,\ldots,n-1$, copies of all edges of $\tilde{F_i}$ are taken by replacing $\tilde{V}$ with V_m, and the set of edges $\{\{j_{m-s}, -(j+ir^{-1})_{m+s}\} :$ $j = 0,1,\ldots n-1, \ s = 1,2,\ldots,r\}$ is added.

Lemma 15 *For odd prime p and for odd $n \geq 3$ such that $\gcd(n, (p-1)/2) = 1$, and for even $k \geq 4$ where $k \neq 2p$, $k \neq p+1$ and moreover, $k/2 \nmid n$, if there is a k-cycle free one-factorization of K_{n+1}, then a k-cycle free one-factorization of K_{pn+1} exists.*

Proof: Assume that a k-cycle free one-factorization $\tilde{F}$ of K_n is given. Let H be the union of two one-factors F_h and F_i in the one-factorization constructed according to Construction F, where $h < i$ and $h, i \in \{0, 1, \ldots, pn-1\}$.

Suppose that h and i satisfy $mn \leq h < i < (m+1)n$ for some $m \in \{0, 1, \ldots, p-1\}$. Then clearly H does not contain a cycle of length k with all vertices in V_m because one-factorization induced by V_m is the given one-factorization $\tilde{F}$ of K_{n+1}, which is k-cycle free. Let C_l be a cycle of H with all vertices in $V \setminus V_m$. In fact, all vertices of C_l are in $V_{m-s} \cup V_{m+s}$ for some $s \in \{1, 2 \ldots, r\}$. Clearly $l \leq 2n$. Let y_{m-s} be a vertex of C_l. Neighbors of the vertex y_{m-s} in H are $-(y + hr^{-1})_{m+s}$ and $-(y + ir^{-1})_{m+s}$. Consecutive vertices along the cycle C_l are: y_{m-s}, $-(y + hr^{-1})_{m+s}$, $(y + (h-i)r^{-1})_{m-s}$, $-(y + (2h-i)r^{-1})_{m+s}$, $(y + (2h-2i)r^{-1})_{m-s}$, $\ldots$, $-(y + \frac{lh-(l-2)i}{2} r^{-1})_{m+s}$, y_{m-s}, where l is the minimum even positive integer such that $-y - \frac{lh-(l-2)i}{2} r^{-1} \equiv -y - ir^{-1} \pmod{n}$. Since $0 < (i-h)r^{-1} < n$, by the equivalence $\frac{l}{2}(i-h)r^{-1} \equiv 0 \pmod{n}$ and Claim 1, $\gcd(\frac{l}{2}, n) = l/2$ holds. Thus $l \neq k$.

It remains to consider the case when $mn \leq h < (m+1)n$ and $qn \leq i < (q+1)n$ for some $m, q \in \{0, 1, \ldots, p-1\}$, $m < q$. Let $z = q - m$. Assume that ∞ is a vertex of C_l in H. Neighbors of ∞ in H are h_m and i_q. Note that $p+1$ consecutive vertices along C_l in H are: ∞, $i_q = i_{m+z}$, $-(h+ri)r^{-1}_{m-z}$, $(h+(r-1)i)r^{-1}_{m+3z}$, $-(2h+(r-1)i)r^{-1}_{m-3z}$, $(2h+(r-2)i)r^{-1}_{m+5z}$, $\ldots$, $(rh + (r-r)i)r^{-1}_{m+pz} = h_m$. Hence $l = p+1 \neq k$. Consider the case when ∞ is not a vertex of C_l in H. Let $(h+x)_m$ be a vertex of C_l for some $x \neq 0$. Then neighbors of $(h+x)_m$ in H are $(h-x)_m$ and $-(h+x+ir^{-1})_{q+z} = -(h+x+ir^{-1})_{m+2z}$. Therefore, $2p$ consecutive vertices along C_l are: $(h+x)_m$, $-(rx+rh+i)r^{-1}_{m+2z}$, $(rx+(r-1)h+i)r^{-1}_{m-2z}$, $\ldots$, $(rx + (r-r)h + ri)r^{-1}_{m-(p-1)z} = (i+x)_{m+z}$, $(i-x)_{m+z}$, $-(-rx+h+ri)r^{-1}_{m-z}$, $(-rx+h+(r-1)i)r^{-1}_{m+3z}$, $\ldots$, $(-rx+rh+(r-r)i)r^{-1}_{m+pz} = (h-x)_m$. Thus $l = 2p$ and, by the assumption, $l \neq k$. □

Note that a one-factorization made by Construction F does not contain a cycle of length $np+1$. Moreover, if $n = p$ and GK_{n+1} is taken as a one-factorization $\tilde{F}$ of K_{n+1}, then one-factorization produced in this way is a known uniform one-factorization of K_{p^2+1} with cycles of lengths $p+1, 2p, 2p, \ldots 2p$. Applying Construction F more than once for just-obtained uniform one-factorization easily produces a series of uniform one-factorizations for all orders of the form $p^x + 1$, $x \geq 2$, where every one-factor has one cycle of length $p+1$ and $(p^{x-1} - 1)/2$ cycles of length $2p$, cf. [4].

3 Main results

The constructions presented in the previous section are used to prove general results on k-cycle free one-factorizations.

Theorem 16 *For each n and each even $k \geq 4$ such that $k \neq 2n$, the complete graph K_{2n} has a k-cycle free one-factorization.*

Proof: Let $k = 2^{\lambda_0} p_1^{\lambda_1} p_2^{\lambda_2} \ldots p_w^{\lambda_w}$ be the prime factorization of k into non-trivial factors, $\lambda_j \geq 1$ for each p_j and $p_1 < p_2 < \ldots < p_w$. Since k is even, $\lambda_0 \geq 1$. If $k > 2n$, by Claim 2 the assertion is true. Thus, the result is trivial for $n = 4$. In what follows, let $k < 2n$. For the induction, assume that a k-free one-factorization of K_{2m} exists for every m such that $2 \leq m < n$ and $2m \neq k$. Consider separately two cases:
Case I: $k/2 \nmid n$. Thus $k \neq n$. For odd n, by Corollary 7, the one-factorization GA_{2n} is k-cycle free. Assume that n is even. If $\lambda_0 \neq 2$, then to find a required one-factorization of K_{2n} apply Lemma 9. Consider the case $\lambda_0 = 2$. Note that $k > 4$ because otherwise $k/2 = 2|n$. Let $x = \max\{y : \gcd(2^y, n) = 2^y\}$. Hence immediately $k \neq n/2^y$ for every $y \leq x$. Let $n' = n/2^x$. Note that both $k/2 \nmid n'$ and $k/4 \nmid n'$. Thus, the one-factorization $GA_{2n'}$ of $K_{2n'}$, by Corollary 7, is $\{k/2, k\}$-cycle free. In the next steps apply x times Construction C to get, by Lemma 10, one-factorizations of $K_{4n'}$, of $K_{8n'}, \ldots$, of K_{2n}, respectively, which are $\{k/2, k\}$-cycle free.
Case II: $k/2 \mid n$. Hence, for every $j = 1, 2, \ldots, w$, $p_j \mid n$ and clearly $p_j \nmid 2n-1$. Thus $\gcd(k/2, 2n-1) = 1$. If $\gcd(k-1, 2n-1) < k-1$, by Corollary 4 the one-factorization GK_{2n} is k-cycle free. Consider the opposite case $\gcd(k-1, 2n-1) = k-1$. Let f be the minimum nontrivial factor of $2n-1$ and $e = \frac{2n-1}{f}$. Thus $e \geq f \geq 3$ and $\gcd(e, (f-1)/2) = 1$. Moreover, since $\gcd(k/2, ef) = 1$, $\gcd(k/2, e) = 1$ and $f \nmid k/2$ immediately follow, and then $k \not\equiv 0 \pmod{2f}$. The aim is to show that $e \neq k-1$. Suppose to the contrary that $e = k-1$. Then $2n-1 = ef = (k-1)f$ and, since $n = z\frac{k}{2}$ for some integer z, $k(f-z) = f-1$. Thus, k is a divisor of $f-1$, whence $f \geq k+1 = e+2$, which contradicts the fact that f is the minimum factor of $2n-1$. By the inductive assumption there is a k-cycle free one-factorization of K_{e+1}. If f is not a factor of $k-1$ (it means $k \not\equiv f+1 \pmod{2f}$) or $f = k-1$, then to find a required one-factorization of K_{ef+1} apply Lemma 14 (with $p := f$). Otherwise $f|k-1$ and $f < k-1$. In this case, to find a k-cycle free one-factorization of K_{ef+1}, apply Lemma 15 (with $p := f$). $\square$

The existence of 4-cycle free one-factorizations of complete graphs has already been stated in [9].

For an infinite class of even orders $2n$ of complete graphs, $2n$-cycle free one-factorizations may be constructed. Note that all one-factorizations GK_{2n}, GA_{2n}, as well as HK_{2n}, are not useful for this purpose since, as was noted earlier, they contain Hamiltonian cycles.

Theorem 17 *Let $2n \neq p+1$, where p is a prime, or $2n \not\equiv 6, 12, 18 \pmod{24}$. Then the complete graph K_{2n} has a $2n$-cycle free one-factorization.*

Proof: Let $2n \neq p+1$ for every prime p. Let f be the minimum prime factor of $2n-1$ and $e = \frac{2n-1}{f}$. Then $e \geq f \geq 3$ and to construct a $2n$-cycle free one-factorization of K_{ef+1} apply, by Lemma 15, Construction F.

If $2n \equiv 2, 4 \pmod{6}$, then it is easily observed than any Steiner one-factorization of order $2n$ (cf. [12]) is $2n$-cycle free; in fact, the union of any two one-factors contains the cycle C_4. If $2n \equiv 0 \pmod 8$, then n is even and, by Claim 2, any one-factorization of K_n is $2n$-cycle free. Hence, by Lemma 9, Construction C produces a required one-factorization. $\square$

At present, the existence problem of k-cycle free one-factorizations when $k = 2n$ has been only partially solved. In contrast to perfect one-factorizations, orders of the form $2n = p + 1$, for p being prime, appear to be the most difficult regarding constructions of $2n$-cycle free one-factorizations of K_{2n}. However, the existence of n-cycle free one-factorization of K_n when $n \equiv 2 \pmod 4$, by Lemma 10, immediately implies the existence of $2n$-cycle free one-factorization of K_{2n}. Moreover, known examples of non-perfect uniform one-factorizations of K_{2n} (cf. [5]), as well as the $2n$-cycle free one-factorizations for $2n = 18$ given in the Appendix, cover all unsolved cases for orders less than 102.

The more general question concerns $k^<$-cycle free one-factorizations of the complete graph. This appears to be much more difficult. One obvious argument is that perfect one-factorizations of K_{2n} are simply $(2\lfloor n/2 \rfloor)^<$-cycle free one-factorizations. Even for $k = 6$, all constructions presented in this paper are not sufficient to obtain a complete classification, i.e. the case $2n = 28$ remains unsolved. However, for every order $2n \equiv 2 \pmod 4$, a $6^<$-cycle free one-factorization of K_{2n} may be constructed.

Theorem 18 *For every odd $n \geq 5$, there exists one-factorization of K_{2n} which is $6^<$-cycle free.*

Proof: Let q be the minimum prime factor of n. If $q \geq 5$, then the one-factorization GA_{2n}, by Corollary 8, is $8^<$-cycle free. Therefore, assume that $q = 3$. Clearly, $3 \nmid 2n - 1$. If 5 is not a factor of $2n - 1$, then the one-factorization GK_{2n}, by Corollary 5, is $6^<$-cycle free. It remains to consider the case when $5 \mid 2n-1$. Let $2n-1 = r_1 r_2 \ldots r_v$ be the prime factorization of $2n - 1$ into non-trivial factors, where $5 = r_1 \leq r_2 \leq \ldots \leq r_v$ and $v \geq 2$. Note that for $r_v \geq 7$ there exists a $6^<$-cycle free one-factorization $\hat{F}$ of K_{r_v+1}, namely, by Corollary 5, as $\hat{F}$ the one-factorization GK_{v+1} may be substituted. Otherwise $r_v = 5$ and $2n - 1 = 5^x$ for some $x \geq 2$. Let $\hat{F}$ be the one-factorization GA_{5^2+1} of K_{5^2+1} which is clearly perfect. In the next steps apply $v - 1$ times ($v - 2$ times if $r_v = 5$) the inductive Construction E, taking as p's consecutive prime factors of $2n - 1$ in the non-increasing order. In this way, by Lemma 14, a series of $6^<$-cycle free one-factorizations is constructed, ending up at the order $2n$. $\square$

Although it is not possible to construct $k^<$-cycle free one-factorizations for all orders $2n \geq k \geq 6$, infinite families of orders may be provided, for which such one-factorizations do exist. Evidently, by Corollary 5, the one-factorization GK_{2n} is $k^<$-cycle free for every order $2n$ such that the prime factorization of $2n - 1$ does not contain a factor less than k. Let $n \geq 3$ and let r be the minimum prime factor of $2n - 1$. Moreover, let $l = \max\{r_1 - 1, 2r_2 - 2\}$, where r_1 is the minimum prime factor of $\frac{2n-1}{r}$ and r_2 the minimum prime factor of n. If $r \geq 5$, then there exists an $l^<$-cycle free one-factorization of K_{2n} (which follows directly from Corollaries 8 and 13).

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Appendix

One-factors of 18-cycle free one-factorization of K_{18}, $V(K_{18}) = \{0, 1, \ldots, 17\}$:

0-1,2-3,4-5,6-7,8-17,9-10,11-12,13-14,15-16; 0-2,1-3,4-7,5-8,6-15,9-11,10-12,13-16,14-17;
0-3,1-2,4-13,5-6,7-8,9-12,10-11,14-15,16-17; 0-4,1-5,2-6,3-8,7-16,9-13,10-14,11-15,12-17;
0-5,1-4,2-11,3-7,6-8,9-14,10-13,12-16,15-17; 0-6,1-7,2-8,3-4,5-14,9-15,10-16,11-17,12-13;
0-7,1-6,2-5,3-12,4-8,9-16,10-15,11-14,13-17; 0-8,1-10,2-4,3-6,5-7,9-17,11-13,12-15,14-16;
0-9,1-8,2-7,3-5,4-6,10-17,11-16,12-14,13-15; 0-10,1-9,2-12,3-11,4-14,5-16,6-17,7-15,8-13;
0-11,1-17,2-16,3-9,4-15,5-12,6-13,7-14,8-10; 0-12,1-11,2-10,3-15,4-9,5-13,6-14,7-17,8-16;
0-13,1-14,2-15,3-17,4-16,5-10,6-11,7-12,8-9; 0-14,1-13,2-17,3-16,4-10,5-9,6-12,7-11,8-15;
0-15,1-12,2-9,3-10,4-11,5-17,6-16,7-13,8-14; 0-16,1-15,2-13,3-14,4-17,5-11,6-10,7-9,8-12;
0-17,1-16,2-14,3-13,4-12,5-15,6-9,7-10,8-11.