# Spectral saturation: inverting the spectral Turán theorem

Vladimir Nikiforov

Department of Mathematical Sciences
University of Memphis, Memphis TN, USA

vnikifrv@memphis.edu

Submitted: Nov 28, 2007; Accepted: Feb 26, 2009; Published: Mar 13, 2009
Mathematics Subject Classifications: 05C50, 05C35

**Abstract**

Let $\mu(G)$ be the largest eigenvalue of a graph $G$ and $T_r(n)$ be the $r$-partite Turán graph of order $n$.

We prove that if $G$ is a graph of order $n$ with $\mu(G) > \mu(T_r(n))$, then $G$ contains various large supergraphs of the complete graph of order $r+1$, e.g., the complete $r$-partite graph with all parts of size $\log n$ with an edge added to the first part.

We also give corresponding stability results.

**Keywords:** *complete $r$-partite graph; stability, spectral Turán's theorem; largest eigenvalue of a graph.*

## 1 Introduction

This note is part of an ongoing project aiming to build extremal graph theory on spectral basis, see, e.g., [3], [13, 18].

Let $\mu(G)$ be the largest adjacency eigenvalue of a graph $G$ and $T_r(n)$ be the $r$-partite Turán graph of order $n$. The spectral Turán theorem [15] implies that if $G$ is a graph of order $n$ with $\mu(G) > \mu(T_r(n))$, then $G$ contains a $K_{r+1}$, the complete graph of order $r+1$.

On the other hand, it is known (e.g., [2], [4], [9], [12]) that if $e(G) > e(T_r(n))$, then $G$ contains large supergraphs of $K_{r+1}$. It turns out that essentially the same results also follow from the inequality $\mu(G) > \mu(T_r(n))$.

Recall first a family of graphs, studied initially by Erdős [7] and recently in [2]: an *$r$-joint* of size $t$ is the union of $t$ distinct $r$-cliques sharing an edge. Write $js_r(G)$ for the maximum size of an $r$-joint in a graph $G$. Erdős [7], Theorem 3$'$, showed that *if $G$ is a graph of sufficiently large order $n$ satisfying $e(G) > e(T_r(n))$, then $js_{r+1}(G) > n^{r-1}/(10(r+1))^{6(r+1)}$.*

Here is a explicit spectral analogue of this result.

**Theorem 1** *Let $r \geq 2$, $n > r^{15}$, and $G$ be a graph of order $n$. If $\mu(G) > \mu(T_r(n))$, then $js_{r+1}(G) > n^{r-1}/r^{2r+4}$.*

Erdős [4] introduced yet another graph related to Turán's theorem: let $K_r^+(s_1, \ldots, s_r)$ be the complete $r$-partite graph with parts of sizes $s_1 \geq 2, s_2, \ldots, s_r$, with an edge added to the first part. The extremal results about this graph given in [4] and [9] were recently extended in [12] to:

*Let $r \geq 2$, $2/\ln n \leq c \leq r^{-(r+7)(r+1)}$, and $G$ be a graph of order $n$. If $G$ has $t_r(n)+1$ edges, then $G$ contains a $K_r^+\left(\lfloor c\ln n\rfloor, \ldots, \lfloor c\ln n\rfloor, \lceil n^{1-\sqrt{c}}\rceil\right)$.*

Here we give a similar spectral extremal result.

**Theorem 2** *Let $r \geq 2$, $2/\ln n \leq c \leq r^{-(2r+9)(r+1)}$, and $G$ be a graph of order $n$. If $\mu(G) > \mu(T_r(n))$, then $G$ contains a $K_r^+\left(\lfloor c\ln n\rfloor, \ldots, \lfloor c\ln n\rfloor, \lceil n^{1-\sqrt{c}}\rceil\right)$.*

As an easy consequence of Theorem 2 we obtain

**Theorem 3** *Let $r \geq 2$, $c = r^{-(2r+9)(r+1)}$, $n \geq e^{2/c}$, and $G$ be a graph of order $n$. If $\mu(G) > \mu(T_r(n))$, then $G$ contains a $K_r^+(\lfloor c\ln n\rfloor, \ldots, \lfloor c\ln n\rfloor)$.*

Theorems 1, 2, and 3 have corresponding stability results.

**Theorem 4** *Let $r \geq 2$, $0 < b < 2^{-10}r^{-6}$, $n \geq r^{20}$, and $G$ be a graph of order $n$. If $\mu(G) > (1 - 1/r - b)n$, then $G$ satsisfies one of the conditions:*

*(a) $js_{r+1}(G) > n^{r-1}/r^{2r+5}$;*

*(b) $G$ contains an induced $r$-partite subgraph $G_0$ of order at least $\left(1-4b^{1/3}\right)n$ with minimum degree $\delta(G_0) > \left(1 - 1/r - 7b^{1/3}\right)n$.*

**Theorem 5** *Let $r \geq 2$, $2/\ln n \leq c \leq r^{-(2r+9)(r+1)}/2$, $0 < b < 2^{-10}r^{-6}$, and $G$ be a graph of order $n$. If $\mu(G) > (1 - 1/r - b)n$, then $G$ satsisfies one of the conditions:*

*(a) $G$ contains a $K_r^+\left(\lfloor c\ln n\rfloor, \ldots, \lfloor c\ln n\rfloor, \lceil n^{1-2\sqrt{c}}\rceil\right)$;*

*(b) $G$ contains an induced $r$-partite subgraph $G_0$ of order at least $\left(1-4b^{1/3}\right)n$ with minimum degree $\delta(G_0) > \left(1 - 1/r - 7b^{1/3}\right)n$.*

**Theorem 6** *Let $r \geq 2$, $c = r^{-(2r+9)(r+1)}/2$, $0 < b < 2^{-10}r^{-6}$, $n \geq e^{2/c}$, and $G$ be a graph of order $n$. If $\mu(G) > (1 - 1/r - b)n$, then one of the following conditions holds:*

*(a) $G$ contains a $K_r^+(\lfloor c\ln n\rfloor, \ldots, \lfloor c\ln n\rfloor)$;*

*(b) $G$ contains an induced $r$-partite subgraph $G_0$ of order at least $\left(1-4b^{1/3}\right)n$ with minimum degree $\delta(G_0) > \left(1 - 1/r - 7b^{1/3}\right)n$.*

**Remarks**

- Obviously Theorems 1, 2, and 3 are tight since $T_r(n)$ contains no $(r+1)$-cliques.

- Theorems 2, 3, 5, and 6 are essentially best possible since for every $\varepsilon > 0$, choosing randomly a graph $G$ of order $n$ with $e(G) = \lceil (1-\varepsilon) n^2/2 \rceil$ edges we see that $\mu(G) > (1-\varepsilon) n$, but $G$ contains no $K_2(c \ln n, c \ln n)$ for some $c > 0$, independent of $n$.

- In Theorem 1, it is not known what is the best possible value of $js_{r+1}(G)$, given $G$ is a graph of order $n$ and $\mu(G) > \mu(T_r(n))$.

- Theorem 1 implies in turn spectral versions of other known results, like Theorem 3.8 in [8]:

  *Every graph $G$ of order $n$ with $\mu(G) > \mu(T_r(n))$ contains $cn$ distinct $(r+1)$-cliques sharing an $r$-clique, where $c > 0$ is independent of $n$.*

- It is not difficult to show that if $G$ is a graph of order $n$, then the inequality $e(G) > e(T_r(n))$ implies the inequality $\mu(G) > \mu(T_r(n))$. Therefore, Theorems 1-6 imply the corresponding nonspectral extremal results of [12] with narrower ranges of the parameters.

- The relations between $c$ and $n$ in Theorems 2 and 5 need some explanation. First, for fixed $c$, they show how large must be $n$ so that the vertex classes of the required $K_r^+(s, \ldots s, t)$ are nonempty. But also $c$ may depend on $n$, e.g., letting $c = 1/\ln\ln n$, the conclusion is meaningful for sufficiently large $n$.

- Note that, in Theorems 2 and 5, if the conclusion holds for some $c$, it holds also for $0 < c' < c$, provided $n$ is sufficiently large, i.e., as $n$ grows, we can find a larger and more lopsided $K_r^+(s, \ldots s, t)$;

- The stability conditions *(b)* in Theorems 4, 5, and 6 are stronger than the conditions in the stability theorems of [6], [19] and [11]. Indeed, in all these theorems, condition *(b)* implies that $G_0$ is an induced, almost balanced, and almost complete $r$-partite graph containing almost all the vertices of $G$;

- The exponents $1 - \sqrt{c}$ and $1 - 2\sqrt{c}$ in Theorems 2 and 5 are far from the best ones, but are simple.

The next section contains notation and results needed to prove the theorems. The proofs are presented in Section 3.

# 2 Preliminary results

Our notation follows [1]. Given a graph $G$, we write:

- $V(G)$ for the vertex set of $G$ and $|G|$ for $|V(G)|$;
- $E(G)$ for the edge set of $G$ and $e(G)$ for $|E(G)|$;
- $d(u)$ for the degree of a vertex $u$;
- $\delta(G)$ for the minimum degree of $G$;
- $k_r(G)$ for the number of $r$-cliques of $G$;
- $K_r(s_1, \ldots, s_r)$ for the complete $r$-partite graph with parts of sizes $s_1, \ldots, s_r$.

The following facts play crucial roles in our proofs.

**Fact 7 ([15], Theorem 1)** *Every graph $G$ of order $n$ with $\mu(G) > \mu(T_r(n))$ contains a $K_{r+1}$.* □

**Fact 8 ([16], Theorem 5)** *Let $0 < \alpha \leq 1/4$, $0 < \beta \leq 1/2$, $1/2 - \alpha/4 \leq \gamma < 1$, $K \geq 0$, $n \geq (42K+4)/\alpha^2\beta$, and $G$ be a graph of order $n$. If*

$$\mu(G) > \gamma n - K/n \quad \text{and} \quad \delta(G) \leq (\gamma - \alpha) n,$$

*then $G$ contains an induced subgraph $H$ satisfying $|H| \geq (1-\beta) n$ and one of the conditions:*

*(a) $\mu(H) > \gamma(1 + \beta\alpha/2)|H|$;*
*(b) $\mu(H) > \gamma|H|$ and $\delta(H) > (\gamma - \alpha)|H|$.* □

**Fact 9 ([2], Lemma 6)** *Let $r \geq 2$ and $G$ be graph a of order $n$. If $G$ contains a $K_{r+1}$ and $\delta(G) > (1 - 1/r - 1/r^4) n$, then $js_{r+1}(G) > n^{r-1}/r^{r+3}$.* □

**Fact 10 ([3], Theorem 2)** *If $r \geq 2$ and $G$ is a graph of order $n$, then*

$$k_r(G) \geq \left(\frac{\mu(G)}{n} - 1 + \frac{1}{r}\right) \frac{r(r-1)}{r+1} \left(\frac{n}{r}\right)^{r+1}.$$

□

**Fact 11 ([3], Theorem 4)** *Let $r \geq 2$, $0 \leq b \leq 2^{-10}r^{-6}$, and $G$ be a graph of order $n$. If $G$ contains no $K_{r+1}$ and $\mu(G) \geq (1 - 1/r - b) n$, then $G$ contains an induced $r$-partite graph $G_0$ satisfying $|G_0| \geq \left(1 - 3b^{1/3}\right) n$ and $\delta(G_0) > \left(1 - 1/r - 6b^{1/3}\right) n$.* □

**Fact 12 ([12], Theorem 6)** *Let $r \geq 2$, $2/\ln n \leq c \leq r^{-(r+8)r}$, and $G$ be a graph of order $n$. If $G$ contains a $K_{r+1}$ and $\delta(G) > (1 - 1/r - 1/r^4) n$, then $G$ contains a $K_r^+\left(\lfloor c \ln n \rfloor, \ldots, \lfloor c \ln n \rfloor, \left\lceil n^{1-cr^3} \right\rceil\right)$.* □

**Fact 13 ([10], Theorem 1)** *Let $r \geq 2$, $c^r \ln n \geq 1$, and $G$ be a graph of order $n$. If $k_r(G) \geq cn^r$, then $G$ contains a $K_r(s, \ldots s, t)$ with $s = \lfloor c^r \ln n \rfloor$ and $t > n^{1-c^{r-1}}$.* □

**Fact 14** *The number of edges of $T_r(n)$ satisfies $2e(T_r(n)) \geq (1 - 1/r) n^2 - r/4$.* □

## 3 Proofs

Below we prove Theorems 1, 2, 4, and 5. We omit the proofs of Theorems 3 and 6 since they are easy consequences of Theorems 2 and 5.

All proofs have similar simple structure and follow from the facts listed above.

### Proof of Theorem 1

Let $G$ be a graph of order $n$ with $\mu(G) > \mu(T_r(n))$; thus, by Fact 7, $G$ contains a $K_{r+1}$. If

$$\delta(G) > \left(1 - r^{-1} - r^{-4}\right) n, \tag{1}$$

then, by Fact 9, $js_{r+1}(G) > n^{r-1}/r^{r+3}$, completing the proof.

Thus, we shall assume that (1) fails. Then, letting

$$\alpha = 1/r^4, \quad \beta = 1/2, \quad \gamma = 1 - 1/r, \quad K = r/4, \tag{2}$$

we see that

$$\delta(G) \leq (\gamma - \alpha) n \tag{3}$$

and also, in view of Fact 14,

$$\mu(G) > \mu(T_r(n)) \geq 2e(T_r(n))/n \geq (1 - 1/r) n - r/4n = \gamma n - K/n. \tag{4}$$

Given (2), (3) and (4), Fact 8 implies that, for $n \geq r^{15}$, $G$ contains an induced subgraph $H$ satisfying $|H| \geq n/2$ and one of the conditions:

*(i)* $\mu(H) > (1 - 1/r + 1/(4r^4))|H|$;

*(ii)* $\mu(H) > (1 - 1/r)|H|$ and $\delta(H) > (1 - 1/r - 1/r^4)|H|$.

If condition *(i)* holds, Fact 10 gives

$$k_{r+1}(H) > \left(\frac{\mu(H)}{|H|} - 1 - \frac{1}{r}\right)\frac{r(r-1)}{r+1}\left(\frac{|H|}{r}\right)^{r+1} > \frac{r(r-1)}{4r^4(r+1)}\left(\frac{|H|}{r}\right)^{r+1},$$

and so,

$$\begin{aligned} js_{r+1}(G) &\geq js_{r+1}(H) \geq \binom{r+1}{2}\frac{k_{r+1}(H)}{e(H)} > r(r+1)\frac{k_{r+1}(H)}{|H|^2} \\ &> \frac{r(r+1)r(r-1)}{4r^4(r+1)r^{r+1}}|H|^{r-1} \geq \frac{1}{4r^{r+3}}|H|^{r-1} \geq \frac{1}{2^{r+1}r^{r+3}}n^{r-1} \geq \frac{1}{r^{2r+4}}n^{r-1}, \end{aligned}$$

completing the proof.

If condition *(ii)* holds, then $H$ contains a $K_{r+1}$; thus, $js_{r+1}(H) > |H|^{r-1}/r^{r+3}$ by Fact 9. To complete the proof, notice that

$$js_{r+1}(G) > js_{r+1}(H) > \frac{|H|^{r-1}}{r^{r+3}} \geq \frac{1}{2^{r-1}r^{r+3}}n^{r-1} > \frac{1}{r^{2r+4}}n^{r-1}.$$

□

**Proof of Theorem 2**

Let $G$ be a graph of order $n$ with $\mu(G) > \mu(T_r(n))$; thus, by Fact 7, $G$ contains a $K_{r+1}$. If

$$\delta(G) > \left(1 - 1/r - 1/r^4\right) n, \tag{5}$$

then, by Fact 12, $G$ contains a $K_r^+\left(\lfloor c\ln n\rfloor, \ldots, \lfloor c\ln n\rfloor, \left\lceil n^{1-cr^3}\right\rceil\right)$, completing the proof, in view of $cr^3 < \sqrt{c}$.

Thus, we shall assume that (5) fails. Then, letting

$$\alpha = 1/r^4, \quad \beta = 1/2, \quad \gamma = 1 - 1/r, \quad K = r/4, \tag{6}$$

we see that

$$\delta(G) \leq (\gamma - \alpha) n \tag{7}$$

and also, in view of Fact 14,

$$\mu(G) > \mu(T_r(n)) \geq 2e(T_r(n))/n \geq (1 - 1/r) n - r/4n = \gamma n - K/n. \tag{8}$$

Given (6), (7) and (8), Fact 8 implies that, for $n > r^{15}$, $G$ contains an induced subgraph $H$ satisfying $|H| \geq n/2$ and one of the conditions:

*(i)* $\mu(H) > (1 - 1/r + 1/(4r^4))|H|$;
*(ii)* $\mu(H) > (1 - 1/r)|H|$ and $\delta(H) > (1 - 1/r - 1/r^4)|H|$.

If condition *(i)* holds, Fact 10 gives

$$\begin{aligned} k_{r+1}(H) &> \left(\frac{\mu(H)}{|H|} - 1 - \frac{1}{r}\right)\frac{r(r-1)}{r+1}\left(\frac{|H|}{r}\right)^{r+1} > \frac{r(r-1)}{4r^4(r+1)}\left(\frac{|H|}{r}\right)^{r+1} \\ &> \frac{1}{2^{r+3}r^{r+4}(r+1)}n^{r+1} > \frac{1}{r^{2r+9}}n^{r+1} \geq c^{1/(r+1)}n^{r+1}. \end{aligned}$$

Thus, by Fact 13, $G$ contains a $K_{r+1}(s, \ldots, s, t)$ with $s = \lfloor c\ln n\rfloor$ and $t > n^{1-c^{r/(r+1)}} > n^{1-\sqrt{c}}$. Then, obviously, $G$ contains a $K_r^+\left(\lfloor c\ln n\rfloor, \ldots, \lfloor c\ln n\rfloor, \lceil n^{1-\sqrt{c}}\rceil\right)$, completing the proof.

If condition *(ii)* holds, then $H$ contains a $K_{r+1}$; thus, by Fact 12, $H$ contains a

$$K_r^+\left(\lfloor 2c\ln|H|\rfloor, \ldots, \lfloor 2c\ln|H|\rfloor, \left\lceil |H|^{1-2cr^3}\right\rceil\right).$$

To complete the proof, note that $2c\ln|H| \geq 2c\ln\frac{n}{2} > c\ln n$ and

$$|H|^{1-2cr^3} \geq \left(\frac{n}{2}\right)^{1-2cr^3} \geq \frac{1}{2}n^{1-2cr^3} > n^{1-\sqrt{c}}.$$

□

**Proof of Theorem 4** Let $G$ be a graph of order $n$ with $\mu(G) > (1 - 1/r - b)n$. If $G$ contains no $K_{r+1}$, then condition *(b)* follows from Fact 11; thus we assume that $G$ contains a $K_{r+1}$. If

$$\delta(G) > \left(1 - 1/r - 1/r^4\right) n, \tag{9}$$

then Fact 9 implies condition *(a)*.

Thus, we shall assume that (9) fails. Then, letting

$$\alpha = 1/r^4 - b, \quad \beta = 4b/\alpha, \quad \gamma = 1 - 1/r - b, \quad K = 0, \tag{10}$$

we easily see that

$$\beta = \frac{4b}{1/r^4 - b} \leq \frac{1}{2}, \qquad \delta(G) \leq (\gamma - \alpha) n, \tag{11}$$

and

$$\mu(G) > (1 - 1/r - b) n = \gamma n. \tag{12}$$

Given (10), (11) and (12), Theorem 8 implies that, for $n \geq r^{20}$, $G$ contains an induced subgraph $H$ satisfying $|H| \geq (1-\beta) n$ and one of the conditions:

*(i)* $\mu(H) > (1 - 1/r) |H|$;

*(ii)* $\mu(H) > (1 - 1/r - b) |H|$ and $\delta(H) > (1 - 1/r - 1/r^4) |H|$.

If condition *(i)* holds, by Theorem 1 we have

$$\begin{aligned} js_{r+1}(G) &\geq js_{r+1}(H) \geq \frac{|H|^{r-1}}{r^{2r+4}} \geq (1-\beta)^{r-1} \frac{n^{r-1}}{r^{2r+4}} = \left(1 - \frac{4b}{1/r^4 - b}\right)^{r-1} \frac{n^{r-1}}{r^{2r+4}} \\ &> \left(1 - \frac{1}{r^2}\right)^{r-1} \frac{n^{r-1}}{r^{2r+4}} \geq \left(1 - \frac{r-1}{r^2}\right) \frac{n^{r-1}}{r^{2r+4}} > \frac{n^{r-1}}{r^{2r+5}}, \end{aligned}$$

implying condition *(a)* and completing the proof.

Suppose now that condition *(ii)* holds. If $H$ contains a $K_{r+1}$, by Fact 9, we see that

$$js_{r+1}(G) \geq js_{r+1}(H) \geq \frac{|H|^{r-1}}{r^{r+3}} \geq (1-\beta)^{r-1} \frac{n^{r-1}}{r^{r+3}} > \frac{n^{r-1}}{2^{r-1} r^{r+3}} > \frac{n^{r-1}}{r^{2r+5}},$$

implying condition *(a)*.

If $H$ contains no $K_{r+1}$, by Fact 11, $H$ contains an induced $r$-partite subgraph $H_0$ satisfying $|H_0| > \left(1 - 3b^{1/3}\right) |H|$ and $\delta(H_0) > \left(1 - 6b^{1/3}\right) |H|$. Now from

$$\beta = \frac{4b}{1/r^4 - b} \leq \frac{4b}{1/r^4 - 1/(2^{10} r^6)} \leq 8r^4 b < b^{1/3},$$

we deduce that

$$|H_0| \geq \left(1 - 3b^{1/3}\right) |H| \geq \left(1 - 3b^{1/3}\right)(1-\beta) n > \left(1 - 4b^{1/3}\right) n$$

and

$$\delta(H_0) \geq \left(1 - 6b^{1/3}\right) |H| \geq \left(1 - 6b^{1/3}\right)(1-\beta) n > \left(1 - 7b^{1/3}\right) n.$$

Thus condition *(b)* holds, completing the proof. □

**Proof of Theorem 5** Let $G$ be a graph of order $n$ with $\mu(G) > (1 - 1/r - b) n$. If $G$ contains no $K_{r+1}$, then condition *(b)* follows from Fact 11; thus we assume that $G$ contains a $K_{r+1}$. If

$$\delta(G) > \left(1 - 1/r - 1/r^4\right) n, \tag{13}$$

then Fact 12 implies condition *(a)*.

Thus, we shall assume that (13) fails. Then, letting

$$\alpha = 1/r^4 - b, \quad \beta = 4b/\alpha, \quad \gamma = 1 - 1/r - b, \quad K = 0, \tag{14}$$

we easily see that

$$\beta = \frac{4b}{1/r^4 - b} \leq \frac{1}{2}, \qquad \delta(G) \leq (\gamma - \alpha) n, \tag{15}$$

and

$$\mu(G) > (1 - 1/r - b) n = \gamma n. \tag{16}$$

Given (14), (15) and (16), Theorem 8 implies that, for $n \geq r^{20}$, $G$ contains an induced subgraph $H$ satisfying $|H| \geq (1 - \beta) n$ and one of the conditions:

*(i)* $\mu(H) > (1 - 1/r) |H|$;

*(ii)* $\mu(H) > (1 - 1/r - b) |H|$ and $\delta(H) > (1 - 1/r - 1/r^4) |H|$.

If condition *(i)* holds, Theorem 2 implies that $H$ contains a

$$K_r^+ \left( \lfloor 2c \ln |H| \rfloor, \ldots, \lfloor 2c \ln |H| \rfloor, \left\lceil |H|^{1-2cr^3} \right\rceil \right).$$

Now condition *(a)* follows in view of $2c \ln |H| \geq 2c \ln \frac{n}{2} > c \ln n$ and

$$|H|^{1-2cr^3} \geq \left(\frac{n}{2}\right)^{1-2cr^3} \geq \frac{1}{2} n^{1-2cr^3} > n^{1-\sqrt{c}},$$

completing the proof.

Suppose now that condition *(ii)* holds. If $H$ contains a $K_{r+1}$, by Fact 12, $H$ contains a

$$K_r^+ \left( \lfloor 2c \ln |H| \rfloor, \ldots, \lfloor 2c \ln |H| \rfloor, \left\lceil |H|^{1-2cr^3} \right\rceil \right).$$

This implies condition *(a)* in view of $2c \ln |H| \geq 2c \ln \frac{n}{2} > c \ln n$ and

$$|H|^{1-2cr^3} \geq \left(\frac{n}{2}\right)^{1-2cr^3} \geq \frac{1}{2} n^{1-2cr^3} > n^{1-\sqrt{c}}.$$

If $H$ contains no $K_{r+1}$, the proof is completed as the proof of Theorem 4. $\square$

**Acknowledgement.** Thanks are due to the referee for careful reading and useful remarks.

# References

[1] B. Bollobás, *Modern Graph Theory,* Graduate Texts in Mathematics, 184, Springer-Verlag, New York (1998).

[2] B. Bollobás, V. Nikiforov, Joints in graphs, to appear in *Discrete Math.*

[3] B. Bollobás, V. Nikiforov, Cliques and the Spectral Radius, *J. Combin. Theory Ser. B.* **97** (2007), 859-865.

[4] P. Erdős, On the structure of linear graphs, *Israel J. Math.* **1** (1963), 156–160.

[5] P. Erdős, Some recent results on extremal problems in graph theory (results) in *Theory of Graphs (Internat. Sympos., Rome, 1966)*, pp. 117–130, Gordon and Breach, New York; Dunod, Paris.

[6] P. Erdős, On some new inequalities concerning extremal properties of graphs, in: *Theory of Graphs (Proc. Colloq., Tihany, 1966),* pp. 77–81, Academic Press, New York, 1968.

[7] P. Erdős, On the number of complete subgraphs and circuits contained in graphs, *Časopis Pěst. Mat.* **94** (1969), 290–296.

[8] P. Erdős, R.J. Faudree, C.C. Rousseau, Extremal problems and generalized degrees, *Discrete Math.* **127** (1994), 139-152.

[9] P. Erdős, M. Simonovits, On a valence problem in extremal graph theory, *Discrete Math.* **5** (1973), p. 323-334.

[10] V. Nikiforov, Graphs with many $r$-cliques have large complete $r$-partite subgraphs, *Bull. of London Math. Soc.* **40** (2008), 23-25. Update available at *http://arxiv.org/math.CO/0703554*

[11] V. Nikiforov, Stability for large forbidden graphs, to appear in *J. Graph Theory.* Preprint available at *http://arxiv.org/abs/0707.2563*

[12] V. Nikiforov, Turán's theorem inverted, submitted for publication. Preprint available at *http://arxiv.org/abs/0707.3439*

[13] V. Nikiforov, Some inequalities for the largest eigenvalue of a graph, *Combin. Probab. Comp.* **11** (2002), 179-189.

[14] V. Nikiforov, The smallest eigenvalue of $K_r$-free graphs, *Discrete Math.* **306** (2006), 612-616.

[15] V. Nikiforov, Bounds on graph eigenvalues II, *Linear Algebra Appl.* **427** (2007) 183-189.

[16] V. Nikiforov, A spectral condition for odd cycles, *Linear Algebra Appl.* **428** (2008), 1492-1498. Update available at *http://arxiv.org/abs/0707.4499*

[17] V. Nikiforov, More spectral bounds on the clique and independence numbers, to appear in *J. Combin. Theory Ser. B.* Preprint available at *http://arxiv.org/abs/0706.0548*

[18] V. Nikiforov, A spectral Erdős-Stone-Bollobás theorem, to appear in *Combin. Probab. Comput.* Preprint available at *http://arxiv.org/abs/0707.2259*

[19] M. Simonovits, A method for solving extremal problems in graph theory, stability problems, in: *Theory of Graphs (Proc. Colloq., Tihany, 1966),* pp. 279–319, Academic Press, New York, 1968.