Rate of convergence of the short cycle distribution in random regular graphs generated by pegging

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Submitted: Aug 30, 2008; Accepted: Mar 24, 2009; Published: Mar 31, 2009
Mathematics Subject Classification: 05C80

Abstract

The pegging algorithm is a method of generating large random regular graphs beginning with small ones. The ϵ-mixing time of the distribution of short cycle counts of these random regular graphs is the time at which the distribution reaches and maintains total variation distance at most ϵ from its limiting distribution. We show that this ϵ-mixing time is not $o(\epsilon^{-1})$. This demonstrates that the upper bound $O(\epsilon^{-1})$ proved recently by the authors is essentially tight.

1 Introduction

Different random graph models have been applied to analyse the behavior of real-world networks. The most classical and commonly studied one is the Erdős-Rényi model [1], which is the probability space of random graphs on n vertices with each edge appearing independently with some probability p. The properties of the random network (degree distribution, connectivity, diameter, etc.) vary when p is assigned different values. However, the Erdős-Rényi model cannot produce scale-free networks [2], whose degree distribution obeys the power law. The scale-free network caught a lot of attention because a diverse group of networks of interest are thought to be scale-free, such as the collaboration network and the World Wide Web. The preferential attachment model was first introduced by Yule [13] and then studied by many other authors [3, 7] in an attempt to simulate the properties of such scale-free networks.

A new type of peer-to-peer ad-hoc network called the SWAN network was introduced recently by Bourassa and Holt [4]. The underlying topology of the SWAN network is a random regular graph. In the SWAN network, clients arrive and leave randomly. To

*Research supported by the Canadian Research Chairs Program and NSERC

accommodate this, the network undergoes changes in structure using an operation called "clothespinning" (for arriving clients), and its reverse (for clients leaving), together with some other occasional adjustments to repair the network when these operations cause a problem, such as disconnection. Cooper, Dyer and Greenhill [6] defined a Markov chain on d-regular graphs with randomised size to model (a simplified version of) the SWAN network. The moves of the Markov chain are by clothespinning or the reverse. They obtained bounds on the mixing time of the chain. Along the way, they showed that, restricted to the times when the network has a given size, the stationary distribution is uniform. Thus, for this simplified version of the SWAN network, the limiting distribution of graphs coincides exactly with the model of random regular graphs which has already received the most attention from the theoretical viewpoint.

The related *pegging algorithm* to generate random d-regular graphs for constant d was first introduced by the authors in [10], where the clothespinning operation is called *pegging*. (The notion of pegging was also extended to odd degree graphs.) The pegging algorithm simply repeats pegging operations, without performing the reverse. This gives an extreme version of the SWAN network, in which no client ever leaves the network. By studying this extreme case we hope to gain knowledge of the properties of the random SWAN network in the case that it grows quickly, as opposed to the more steady-state scenario studied in [6]. Other models of random regular graphs generated algorithmically are discussed in [10].

Fix $d \geq 3$. For most models of random d-regular graphs, there are small numbers of short cycles and rarely any more complex structures, so the local structure is basically determined by the short cycle distribution. Although only describing local structure, the short cycle distribution has played a major role in the theory of contiguity of random regular graphs, which includes results on many global properties such as hamiltonicity (see [12]). In the random d-regular graph generated by pegging, the joint distribution of short cycle counts (up to some fixed length K) was proven to be asymptotically Poisson in [10]. Moreover, let $(\sigma_t)_{t\geq 0}$ be a sequence of distributions which converge to a distribution π. The *ϵ-mixing time* $\tau^*_\epsilon\big((\sigma_t)_{t\geq 0}\big)$ was defined in [10] to be the minimum $T \geq 0$ such that $d_{TV}(\sigma_t, \pi) \leq \epsilon$ for all $t \geq T$, where d_{TV} denotes total variation distance. For the joint distribution of short cycle counts mentioned above, the ϵ-mixing time was shown using coupling to be $O(\epsilon^{-1})$. It is often easy to find a coupling, but hard to find one that gives an optimal bound. Our goal in this paper is to show that the upper bound achieved by coupling in [10] is tight, in the sense that the ϵ-mixing time is not $o(\epsilon^{-1})$.

The proof focusses on the number of 3-cycles. During the pegging algorithm, the number of 3-cycles undergoes a random walk with transitions that are related to those of a Markov chain with limiting Poisson distribution. This was the technique used in the coupling argument in [10] to bound the total variation distance. The lower bound we obtain can be intuitively explained by "mistakes" made by this random walk that are of order $1/t$ after t steps. Actually, in a sense it is easy to show that such mistakes do occur occasionally, and the difficult part is to show that the mistakes do not usually cancel each other out.

For simplicity, we do not consider the case of odd d here. We expect that our method would show the same result in that case, but it would be more complicated to check the details.

2 Main result

We first recall the pegging algorithm to generate random regular graphs. In [10], the *pegging operation* was defined on a d-regular graph as follows for d even.

- Choose a set F of $d/2$ pairwise non-adjacent edges uniformly at random.
- Delete the edges in F.
- Add a new vertex u, together with d edges joining u to each endvertex of the edges in F.

The newly introduced vertex u is called the *peg vertex*, and we say that the edges deleted are *pegged*. Figure 1 illustrates the pegging operation with $d = 4$.

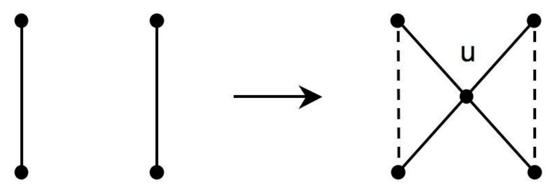

Figure 1: *Pegging operation when* $d = 4$

A similar operation for d odd was also defined in [10], but in the present paper we will consider only the case d even in detail. Thus, we henceforth assume that d is a fixed even integer, and at least 4.

The *pegging algorithm* starts from a nonempty d-regular graph G_0, for example, K_{d+1}, and repeatedly applies pegging operations. For $t > 0$, the random graph G_t is defined inductively to be the graph resulting when the pegging operation is applied to G_{t-1}. Clearly, G_t contains $n_t := n_0 + t$ vertices. We denote the resulting random graph process $(G_0, G_1, \ldots)$ by $\mathcal{P}(G_0, d)$.

For any fixed k, let $Y_{t,d,k}$ denote the number of k-cycles in $G_t \in \mathcal{P}(G_0, d)$ and let $\sigma_{t,d,k}$ denote the joint distribution of $Y_{t,d,3}, \ldots, Y_{t,d,k}$. Theorem 2.2 in [10] is essentially the following.

Theorem 2.1 *For any fixed* k, $Y_{t,d,3}, Y_{t,d,4}, \ldots, Y_{t,d,k}$ *are asymptotically independent Poisson random variables with means* $\mu_i = ((d-1)^i - (d-1)^2)/(2i)$, *for* $3 \le i \le k$, *and the* ϵ*-mixing time of* $(\sigma_{t,d,k})_{t\ge 0}$ *is* $O(1/\epsilon)$.

The main result of this paper is that the ϵ-mixing time $\tau^*_\epsilon\big((\sigma_{t,d,k})_{t\ge 0}\big)$ is not $o(1/\epsilon)$. In other words, there exists $c > 0$ such that $\tau^*_\epsilon\big((\sigma_{t,d,k})_{t\ge 0}\big) > c/\epsilon$ for arbitrarily small $\epsilon > 0$.

Theorem 2.2 *For fixed G_0 and $k \geq 3$, the ϵ-mixing time of the sequence of short cycle joint distributions in $\mathcal{P}(G_0)$ satisfies $\tau_\epsilon^*\big((\sigma_{t,d,k})_{t\geq 0}\big) \neq o\left(\epsilon^{-1}\right)$.*

Let $\mathbf{Po}(\mu_3, \ldots, \mu_k)$ denote the joint distribution of independent Poisson random variables with means μ_i for $3 \leq i \leq k$, where μ_i is as defined in Theorem 2.1. Note that Theorem 2.1 essentially states that there exists a constant $C > 0$ such that for all ϵ and $t \geq C/\epsilon$, $d_{TV}(\sigma_{t,d,k}, \mathbf{Po}(\mu_3, \ldots, \mu_k)) \leq \epsilon$. Putting $\epsilon = C/t$ and using the fact that $n_t = n_0 + t$ gives the following.

Corollary 2.1 *For any fixed integer $k \geq 3$, $d_{TV}(\sigma_{t,d,k}, \mathbf{Po}(\mu_3, \ldots, \mu_k)) = O(n_t^{-1})$.*

We note here that the difficulty in proving results about the random process $\mathcal{P}(G_0, d)$ lies in the lack of existence of a simple model by which probabilities of events can be calculated. Instead we are forced to find arguments that work with probabilities conditional upon the graph G_t existing at time t. The basic relevant observation is that the total number of ways to apply a pegging operation to G_t when $d = 4$ is

$$N_t = n_t(2n_t - 7) \tag{2.1}$$

since this is the number of pairs of nonadjacent edges.

3 Proof of the theorem

We begin with a simple technical lemma that will be used several times in the remaining part of the paper. The lemma holds for any $c > 0$ and p, though in our application we need only the case that $p < c$.

Lemma 3.1 *Let $c > 0$, p, a and ρ be constants with $p < c$. If $(a_n)_{n\geq 1}$ is a sequence of nonnegative real numbers with a_1 bounded, such that*

$$a_{n+1} = \big(1 - cn^{-1} + O(n^{-2})\big)\, a_n + \rho n^{-p} + \gamma(n)$$

for all $n \geq 1$, then

$$a_n = \begin{cases} \big(\rho/(c-p+1)\big)n^{-p+1} + O(n^{-p}) & \text{if } \gamma(n) = O(n^{-(p+1)}), \\ \big(\rho/(c-p+1)\big)n^{-p+1} + o(n^{-p+1}) & \text{if } \gamma(n) = o(n^{-p}). \end{cases}$$

Proof. When $\gamma(n) = O(n^{-(p+1)})$, we have

$$a_{n+1} = \exp\left(-\frac{c}{n} + O(n^{-2})\right) a_n + \frac{\rho}{n^p} + O(n^{-(p+1)}). \tag{3.1}$$

Iterating this gives

$$\begin{aligned}
a_n &= a_1 \exp\left(-\sum_{i=1}^{n-1} \frac{c}{i} + O(i^{-2})\right) + \sum_{i=1}^{n-1} \exp\left(-\sum_{j=i+1}^{n-1} \frac{c}{j} + O(j^{-2})\right)\left(\frac{\rho}{i^p} + O(i^{-(p+1)})\right) \\
&= a_1 \exp\left(-c\log n + O(1)\right) + \sum_{i=1}^{n-1} \exp\left(-c\log(n/i) + O(i^{-1})\right)\left(\frac{\rho}{i^p} + O(i^{-(p+1)})\right) \\
&= O(n^{-c}) + \sum_{i=1}^{n-1} \frac{\rho i^{c-p}}{n^c}\left(1 + O(i^{-1})\right) \\
&= \frac{\rho}{(c-p+1)} n^{-p+1} + O(n^{-p}).
\end{aligned}$$

When $\gamma(n) = o(n^{-p})$, by simply modifying the above computation we obtain

$$a_n = O(n^{-c}) + \sum_{i=1}^{n-1} \frac{\rho i^{c-p}}{n^c}(1 + o(1)) = \frac{\rho}{(c-p+1)} n^{-p+1} + o(n^{-p+1}).$$

Lemma 3.1 follows. ∎

Define $\Psi(i,r)$ to be the set of graphs with i vertices, minimum degree at least 2, and excess r, where the excess of a graph is the number of edges minus the number of vertices. Define $W_{t,i,r}$ to be the number of subgraphs of G_t in $\Psi(i,r)$. The following lemma was proven in [10] and is useful in this paper to bound the expected numbers of specific subgraphs.

Lemma 3.2 *[10, Lemma 3.3] For fixed $i > 0$ and $r \geq 0$,*

$$\mathbf{E}W_{t,i,r} = O(n_t^{-r}).$$

Let $[x]_j$ denote the j-th falling factorial of x.

Lemma 3.3 *For any fixed nonnegative integer j,*

$$\mathbf{E}([Y_{t,3}]_j) = 3^j + O(n_t^{-1}).$$

Proof. Multiplying an equation near the end of the proof of [10, Lemma 3.5] by $j!$ gives

$$\mathbf{E}([Y_{t+1,3}]_j) - \mathbf{E}([Y_{t,3}]_j) = \frac{9j}{n_t}\mathbf{E}([Y_{t,3}]_{j-1}) - \frac{3j}{n_t}\mathbf{E}([Y_{t,3}]_j) + O\big(n_t^{-2}\big(1 + \mathbf{E}(j[Y_{t,3}]_{j-1})\big)\big).$$

We apply induction on j, starting with $\mathbf{E}([Y_{t,3}]_0) = 1$. The error term is then simply $O(n_t^{-2})$. Hence for any $j \geq 1$,

$$\mathbf{E}([Y_{t+1,3}]_j) = \left(1 - \frac{3j}{n_t}\right)\mathbf{E}([Y_{t,3}]_j) + \frac{9j \cdot 3^{j-1}}{n_t} + O(n_t^{-2}).$$

Applying Lemma 3.1 with $c = 3j \geq 3$, $\rho = 9j \cdot 3^{j-1}$ and $p = 1$, we obtain the result claimed. ∎

For simplicity, we prove the main theorem for the case $d = 4$ in detail, and then at the end discuss the case of fixed $d > 4$. We drop the notation d from the subscript of $Y_{t,d,k}$ and $\sigma_{t,d,k}$ as convenience in this case. By considering just the events measurable in the σ-algebra generated by $Y_{t,3}$, we see immediately that

$$d_{TV}(\sigma_{t,3}, \pi_3) \leq d_{TV}(\sigma_{t,k}, \pi_k)$$

where π_k is the limit of $\sigma_{t,k}$. Hence, it suffices to show that the ϵ-mixing time for $\sigma_{t,3}$, which is the distribution of $Y_{t,3}$, is not $o(\epsilon^{-1})$. For convenience, in the rest of the paper we use the notation Y_t to denote $Y_{t,3}$.

Let C_4^* denote the graph consisting of a 4-cycle plus a chord (i.e. K_4 minus an edge), and let W_t denote the number of subgraphs of G_t that are isomorphic to C_4^*. Lemma 3.2 implies that a.a.s. $W_t = 0$. That is, a.a.s. all triangles are isolated, where an *isolated triangle* is a 3-cycle that shares no edges with any other 3-cycle. We also need more information on the distribution of the number of isolated triangles in the presence of one copy of C_4^*. In the following lemma, we show that this has the same asymptotic distribution as Y_t. This distribution is to be expected, since the creation of a copy of C_4^* will leave an asymptotically Poisson number of isolated triangles. Until the C_4^* disappears due to some pegging operation, this Poisson number of isolated triangles will undergo transitions with similar rules to Y_t and will therefore remain asymptotically Poisson. Instead of fleshing this argument out into a proof, it seems simpler to provide a complete argument using the method of moments, although this conceals the coincidence to a greater extent.

Lemma 3.4 *Conditional on $W_t = 1$, the random variable $Y_t - 2$ has a limiting distribution that is Poisson with mean* 3.

Proof. Let $U_{t,j}$ denote $[Y_t - 2]_j I\{W_t = 1\}$, i.e. the product of the j-th falling factorial of $Y_t - 2$ and the indicator random variable of the event that $W_t = 1$. Note that if we can show

$$\mathbf{E}(U_{t,j}) \to 3^j \, \mathbf{P}(W_t = 1), \tag{3.2}$$

then $\mathbf{E}([Y_t - 2]_j \mid W_t = 1) \to 3^j$. Lemma 3.4 then follows by the method of moments applied to the probability space obtained by conditioning on $W_t = 1$. So we only need to compute $\mathbf{P}(W_t = 1)$ and $\mathbf{E}(U_{t,j})$. We show that $\mathbf{P}(W_t = 1) = 27/(4n_t) + O(n_t^{-2})$, and show by induction on j that

$$\mathbf{E}(U_{t,j}) = \frac{27}{4n_t} 3^j + O(n_t^{-2}), \tag{3.3}$$

for any integer $j \geq 0$. This gives (3.2) as required.

Consider $\mathbf{P}(W_t = 1)$ first. Our way of estimating this quantity is by computing separately the expected numbers of copies of C_4^* that are created, or destroyed, in each step. There are two ways to create a C_4^*. One way is through the creation of a new triangle which shares an edge with an existing triangle, which we will call C. This requires two edges adjacent to different vertices of C (but not being edges of C) to be pegged. This is illustrated in Figure 2, where v is the peg vertex, and the two dashed edges e_1 and e_2 are pegged. Given C, if C is an isolated triangle, there are exactly 12 ways to choose such two edges. Otherwise, C is part of an existing C_4^* and the number of pegging operations using such a type of C is $O(W_t)$. Overall, the expected number of C_4^* created in this way is therefore $\big(12\hat{Y}_t + O(W_t)\big)/N_t$, where $\hat{Y}_t$ is the number of isolated triangles in G_t. The other way of creating a C_4^* from a triangle C is as illustrated in Figure 3, where e_1 is an edge in C, and e_2 is incident with some vertex of C, but not adjacent to e_1. Given C, there are 3 ways to choose e_1, and for each chosen e_1, there are 2 ways to choose e_2. Hence, there are 6 ways to choose the pair (e_1, e_2), and the expected number of C_4^* created in this way is $6Y_t/N_t$.

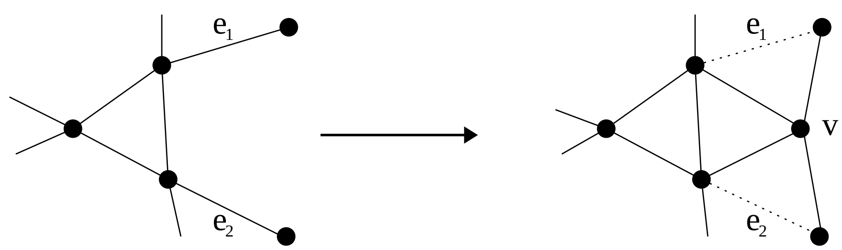

Figure 2: *pegging operation to create a C_4^*, first case*

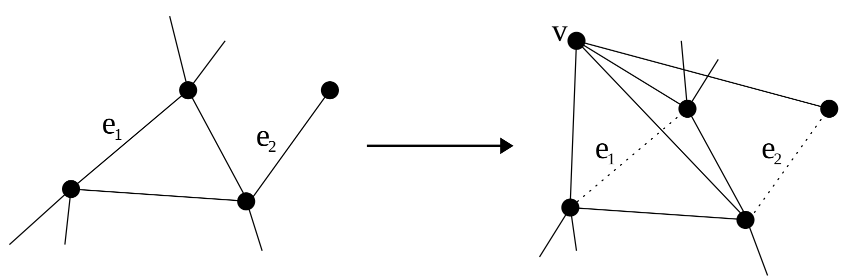

Figure 3: *pegging operation to create a C_4^*, second case*

Clearly $Y_t = \hat{Y}_t + O(W_t)$. So the expected number of C_4^* created in each step is $18\hat{Y}_t/N_t + O(W_t/N_t) = 9Y_t/n_t^2 + O(n_t^{-3}) + O(W_t n_t^{-2})$.

The expected number of C_4^* destroyed in each step is easily seen to be $5W_t(2n_t - 7)/N_t = 5W_t/n_t$. Thus

$$\mathbf{E}(W_{t+1} - W_t \mid W_t) = \frac{9Y_t}{n_t^2} - \frac{5W_t}{n_t} + O(W_t n_t^{-2} + n_t^{-3}).$$

Taking expected values and using the tower property of conditional expectation, this gives

$$\mathbf{E}W_{t+1} - \mathbf{E}W_t = \frac{9\mathbf{E}Y_t}{n_t^2} - \frac{5\mathbf{E}W_t}{n_t} + O(\mathbf{E}W_t n_t^{-2} + n_t^{-3}).$$

Since $\mathbf{E}Y_t = 3 + O(n_t^{-1})$, and $\mathbf{E}W_t = O(n_t^{-1})$, this yields

$$\mathbf{E}W_{t+1} = \left(1 - \frac{5}{n_t}\right)\mathbf{E}W_t + \frac{27}{n_t^2} + O(n_t^{-3}).$$

Applying Lemma 3.3 and Lemma 3.1 with $c = 5$, $p = 2$ and $\rho = 27$, we obtain that $\mathbf{E}W_t = 27/(4n_t) + O(n_t^{-2})$. Since $\mathbf{P}(W_t = i) \le \mathbf{E}([W_t]_i) = O(n_t^{-i})$ by Lemma 3.2,

$$\mathbf{P}(W_t = 1) = 27/(4n_t) + O(n_t^{-2}). \tag{3.4}$$

Next we compute $\mathbf{E}(U_{t,j})$ by induction on $j \ge 0$. The base case is $j = 0$, for which we begin by noting that $\mathbf{E}(U_{t,0}) = \mathbf{P}(W_t = 1) = 27/(4n_t) + O(n_t^{-2})$ as shown above. Now assume that $j \ge 1$ and that (3.3) holds for all smaller values of j. Given the graph G_t, the expected change in $U_{t,j}/j!$ when t changes to $t+1$ is, as explained below,

$$\begin{aligned}\mathbf{E}\left(\frac{U_{t+1,j}}{j!} - \frac{U_{t,j}}{j!}\;\middle|\; G_t\right) \;=\;& \left(\left(\frac{9 + O((1+Y_t+Y_{t,4})/n_t)}{n_t}\right)\frac{[Y_t-2]_{j-1}}{(j-1)!}\right) I\{W_t = 1\}\\ &+\left(\frac{9}{n_t^2} + O\left(n_t^{-3}\right)\right)\frac{(j+1)[Y_t]_{j+1}}{(j+1)!} I\{W_t = 0\}\\ &+f(j, G_t)\\ &-\left(\frac{(3j+5)[Y_t-2]_j/j!}{n_t} + O(n_t^{-2})\right) I\{W_t = 1\},\end{aligned} \tag{3.5}$$

where $f(j, G_t)$ denotes some assorted "error" terms described below. Note that, given $W_t = 1$, $[U_{t,1}]_j/j!$ is simply the number of subgraphs of G_t containing precisely j isolated triangles, so we may just compute the change in the number of such subgraphs in those cases where no copies of C_4^* are created or destroyed. The first term on the right in (3.5) is the positive contribution when $W_t = 1$ and the pegging step creates one new isolated triangle. Any set of $j-1$ isolated triangles, together with the new triangle, can potentially form a new set of j isolated triangles. A new triangle is created from pegging the two end-edges of a 3-path, the number of which in G_t is $4 \cdot 3 \cdot 3 \cdot n_t/2 + O(Y_t) = 18n_t + O(Y_t)$. Dividing this by N_t gives rise to the main term. The error term $O(1 + Y_t + Y_{t,4})$ accounts for choices of such edges which, when pegged, create two or more triangles (when both edges pegged are contained in a 4-cycle) or cause some existing triangle, including possibly the C_4^*, to be destroyed, or cause the new triangle or an existing one not to be isolated.

The second term on the right in (3.5) accounts for the contribution when $W_t = 0$ due to the creation of a C_4^*, when the set of j isolated triangles are all pre-existing. We have noted above that a new C_4^* can be created only from a triangle. So, when $W_t = 0$, a positive contribution to $U_{t+1,j} - U_{t,j}$ can arise from each set of $j+1$ isolated triangles, such that a new C_4^* comes from pegging near one of these triangles as in Figure 2 and 3. There are $[Y_t]_{j+1}/(j+1)!$ different $(j+1)$-sets of triangles, and for each $(j+1)$-set, there are $j+1$ ways to choose one particular triangle. There are 18 ways to peg two edges to create a C_4^* from any given triangle. This, together with $N_t = 2n_t^2(1 + O(n_t^{-1}))$, explains the significant part of this term and the first error term. There is also a correction required when the pegging that creates a C_4^* also "accidentally" destroys one or more of the other triangles in the $(j+1)$-set. This occurs only if the two triangles destroyed are near each other, so they create a small subgraph with more edges than vertices. This correction term is a sum of terms of the form $[Y_t]_{j'} W_{t,i',1}/n_t^2$ for a few different values of i' and j', whose expected value is $O(n_t^{-3})$.

The third term, $f(j, G_t)$, is a function that accounts for all other positive contributions, i.e. counts all other cases of newly created sets of j isolated triangles together with a copy of C_4^*. The situations included here are those in which

(a) $W_t = 1$ and $j' \geq 2$ new triangles are created, which only happens if both edges pegged are contained in a 4-cycle, contributing $O(I\{W_t = 1\}[Y_t]_{j-j'} Y_{t,4}/n_t^2)$, or

(b) $W_t = 1$, the copy of C_4^* is destroyed (leaving behind a new isolated triangle) and simultaneously another is created, contributing $O(I\{W_t = 1\}[Y_t]_{j-1}/n_t^2)$ or

(c) $W_t \geq 2$, and all but one of the copies of C_4^* are destroyed, possibly creating a number of isolated triangles and possibly destroying one. This contributes terms of the form $O(I\{W_t \geq 2\}[Y_t]_{j'}/n_t)$ for various $j' \leq j+1$, or

(d) $W_t = 0$, a C_4^* is created along with an isolated triangle, which is contained in the set of j isolated triangles. When this happens, there must be a triangle sharing a common edge with a 4-cycle, so that the triangle turns into C_4^* when two edges of the 4-cycle are pegged, whilst the other edge of the 4-cycle together with two new edges forms an isolated triangle. Figure 4 illustrates how this works. This case contributes $O(I\{W_t = 0\}[Y_t]_{j-1} W_{t,5,1}/n_t^2)$.

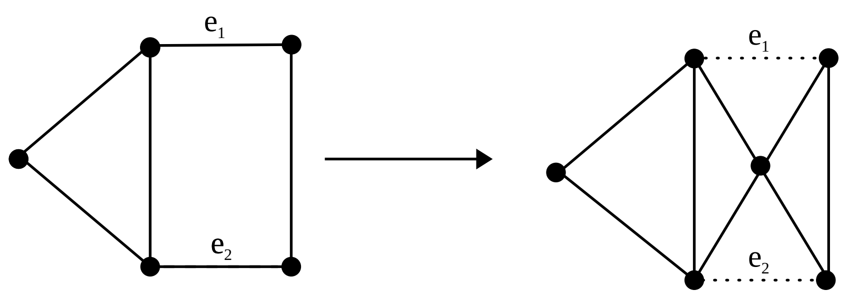

Figure 4: *pegging operation to create a C_4^* and a new triangle.*

We note here for later use that each of these cases involves a subgraph with excess at least 1, and at least 2 in the case (c). For instance $I\{W_t = 1\}[Y_t]_{j-j'} Y_{t,4} \leq W_t [Y_t]_{j-j'} Y_{t,4}$ counts subgraphs with $j - j'$ distinct triangles, a 4-cycle and a copy of C_4^*. Such subgraphs have at most $3(j-j') + 8$ vertices and excess at least 1. By Lemma 3.2, the expected number of such subgraphs is $O(n_t^{-1})$. Using this argument, we find that $\mathbf{E}(f(j, G_t)) = O(n_t^{-3})$.

The last term in (3.5) accounts for the negative contribution to $U_{t+1,j} - U_{t,j}$. Let F_i be the class of subgraphs consisting of i isolated triangles, for some fixed i. Then $U_{t,j}/j!$

counts the number of copies of subgraphs of G_t that are contained in F_j if $W_t = 1$, and is counted as 0 if $W_t \neq 1$. The negative contribution comes when an edge contained in some copy of a member of F_j is destroyed, or an edge contained in the C_4^* is destroyed. In the first case, each copy of an $f \in F_j$ in G_{t+1} that is destroyed contributes -1. The number of subgraphs of G_t that are in F_j is $[Y_t-2]_j/j!$, and for each copy there are $3j$ ways to choose an edge. Hence the expected contribution of this case is $-3j[Y_t-2]_j/(j!n_t)$. In the second case, the destruction of C_4^* kills the contribution of any copy of $f \in F_j$ to $U_{t+1,j}$, since W_{t+1} becomes 0. Hence the negative contribution is $-[Y_t-2]_j/j!$, the number subgraphs in F_j. There are 5 edges in C_4^*, hence the probability that the C_4^* is destroyed is $5/n_t$. So the expected negative contribution by destroying the C_4^* is $-5[Y_t-2]_j/(j!n_t)$.

Taking expectation of both sides of (3.5) and using the tower property of conditional expectation, we have

$$\begin{aligned}\mathbf{E}\left(\frac{U_{t+1,j}}{j!}\right) - \mathbf{E}\left(\frac{U_{t,j}}{j!}\right) &= \frac{9}{n_t}\mathbf{E}\left(\frac{U_{t,j-1}}{(j-1)!}\right) + \frac{9(j+1)}{n_t^2}\mathbf{E}\left(\frac{[Y_t]_{j+1}I\{W_t=0\}}{(j+1)!}\right) \\ &\quad -\frac{3j+5}{n_t}\mathbf{E}\left(\frac{U_{t,j}}{j!}\right) + O(n_t^{-3}).\end{aligned}$$

Note the error term $O(n_t^{-3})$ includes $\mathbf{E}(f(j,G_t))$ (as estimated above), as well as $\mathbf{E}((1+Y_t+Y_{t,4})[Y_t-2]_{j-2}I\{W_t=1\}/(j-2)!n_t^2)$, $\mathbf{E}([Y_t]_{j+1}I\{W_t=0\}/(j!n_t^3))$ and $\mathbf{E}(I\{W_t=1\}/n_t^2)$. This bound holds because $Y_t[Y_t-2]_{j-2}I\{W_t=1\}/(j-2)!$ counts subgraphs with $j-1$ triangles and a copy of C_4^*, $Y_{t,4}[Y_t-2]_{j-2}I\{W_t=1\}/(j-2)!$ counts subgraphs with one 4-cycle, $j-1$ triangles and a copy of C_4^*, and $[Y_t]_{j+1}I\{W_t=0\}/j!$ counts subgraphs with $j+1$ triangles, and hence by Lemma 3.2 $\mathbf{E}((1+Y_t+Y_{t,4})[Y_t-2]_{j-2}I\{W_t=1\}/(j-2)!) = O(n_t^{-1})$, $\mathbf{E}([Y_t]_{j+1}I\{W_t=0\}/j!) = O(1)$, and $\mathbf{E}(I\{W_t=1\}) = \mathbf{P}(W_t=1) = O(n_t^{-1})$.

Clearly for all fixed $j \geq 0$,

$$\mathbf{E}([Y_t]_jI\{W_t=0\}) = \mathbf{E}([Y_t]_j + O([Y_t]_jI\{W_t \geq 1\})) = \mathbf{E}([Y_t]_j) + O(\mathbf{E}([Y_t]_jW_t)). \tag{3.6}$$

Hence by Lemma 3.3 we have $\mathbf{E}([Y_t]_jI\{W_t=0\}) = 3^j + O(n_t^{-1})$. Together with $\mathbf{E}(U_{t,j-1}) = 27/(4n_t)3^{j-1} + O(n_t^{-2})$ by the induction hypothesis, we derive

$$\mathbf{E}(U_{t+1,j}/j!) = \left(1 - \frac{3j+5}{n_t}\right)\mathbf{E}(U_{t,j}/j!) + \frac{9}{n_t}\cdot\frac{27}{4n_t}\cdot\frac{3^{j-1}}{(j-1)!} + \frac{9}{n_t^2}\cdot\frac{3^{j+1}}{j!} + O(n_t^{-3}).$$

By Lemma 3.1 we obtain (3.3) as required. ∎

Proof of Theorem 2.2: As mentioned above, it is enough to show that the ϵ-mixing time for $\sigma_{t,3}$, i.e. the distribution of Y_t, is not $o(\epsilon^{-1})$.

A random walk $(X_t)_{t\geq 0}$ was defined in [10] as follows, and was used to derive the upper bound of the ϵ-mixing time by the coupling technique. Define $\mathbf{B}_{t,3} := \{i \in \mathbf{Z}_+ : (9+3i)/n_t \leq 1\}$, and the boundary of $\mathbf{B}_{t,3}$ to be $\partial\mathbf{B}_{t,3} := \{i \in \mathbf{B}_{t,3} : i+1 \notin \mathbf{B}_{t,3}\}$. The notation w.p. denotes "with probability."

For $X_t \in \mathbf{B}_{t,3} \setminus \partial\mathbf{B}_{t,3}$,

$$X_{t+1} = \begin{cases} X_t - 1 & \text{w.p. } 3X_t/n_t \\ X_t & \text{w.p. } 1 - 3X_t/n_t - 9/n_t \\ X_t + 1 & \text{w.p. } 9/n_t. \end{cases}$$

For $X_t \in \partial\mathbf{B}_{t,3}$,

$$X_{t+1} = \begin{cases} X_t - 1 & \text{w.p. } 3X_t/n_t \\ X_t & \text{w.p. } 1 - 3X_t/n_t. \end{cases}$$

For $X_t \notin \mathbf{B}_{t,3}$,

$$X_{t+1} = X_t \quad \text{w.p. } 1.$$

As was observed in [10], the Poisson distribution with mean 3, $\mathbf{Po}(3)$, is a stationary distribution of the Markov chain $(X_t)_{t\geq 0}$. Let the random walk X_t be defined as above and X_0 take the stationary distribution $\mathbf{Po}(3)$, so X_t has the same distribution for all $t \geq 0$. Let $(X_t)_{t\geq 0}$ walk independently of $(Y_t)_{t\geq 0}$ as generated by the graph process $(G_t)_{t\geq 0}$. We aim to estimate the total variation distance between Y_t and X_t.

Define $\delta_t = \mathbf{P}(X_t = 0) - \mathbf{P}(Y_t = 0)$. Then

$$d_{TV}(X_t, Y_t) \geq |\delta_t|.$$

From the definition of δ_t, we have

$$\begin{aligned} \delta_{t+1} \;=\; & \mathbf{P}(X_t = 0)\mathbf{P}(X_{t+1} = 0 \mid X_t = 0) - \mathbf{P}(Y_t = 0)\mathbf{P}(Y_{t+1} = 0 \mid Y_t = 0) \\ & + \mathbf{P}(X_t \neq 0)\mathbf{P}(X_{t+1} = 0 \mid X_t \neq 0) - \mathbf{P}(Y_t \neq 0)\mathbf{P}(Y_{t+1} = 0 \mid Y_t \neq 0). \end{aligned} \tag{3.7}$$

Without loss of generality, we may assume that $n_0 \geq 9$. Then from the transition probability of X_t we have

$$\mathbf{P}(X_{t+1} \neq 0 \mid X_t = 0) = \frac{9}{n_t} \quad \text{for all } t \geq 0. \tag{3.8}$$

Now we estimate $\mathbf{P}(Y_{t+1} \neq 0 \mid Y_t = 0)$. We consider the creation of a new triangle. Given an edge e of G_t, a new triangle is created containing e if and only if the two pegged edges e_1 and e_2 are both adjacent to e. Of course, in a view of the definition of pegging, they must be incident with different end-vertices of e. Since G_t is 4-regular, the number of ways to choose such e_1 and e_2 is precisely 9 conditional on $Y_t = 0$. It follows that the expected number of new triangles created is $9 \cdot 2n_t/N_t$. By (2.1),

$$\mathbf{E}(Y_{t+1} \mid Y_t = 0) = \frac{9 \cdot 2n_t}{n_t(2n_t - 7)} = \frac{9}{n_t} + \frac{63}{2n_t^2} + O(n_t^{-3}).$$

Conditional on $Y_t = 0$, there is no chord in any 4-cycle. Then it is impossible to create more than two triangles in a single step. Hence $\mathbf{P}(Y_{t+1} \geq 3 \mid Y_t = 0) = 0$. Hence we obtain

$$\mathbf{P}(Y_{t+1} = 1 \mid Y_t = 0) + 2\mathbf{P}(Y_{t+1} = 2 \mid Y_t = 0) = \frac{9}{n_t} + \frac{63}{2n_t^2} + O(n_t^{-3}). \tag{3.9}$$

To create two triangles in a single step, it is required to peg two non-adjacent edges both contained in a 4-cycle. For any 4-cycle, there are precisely two ways to choose two nonadjacent edges, so

$$\mathbf{P}(Y_{t+1} = 2 \mid Y_t = 0, Y_{t,4} = j) = \frac{2j}{N_t} = \frac{j(1+o(1))}{n_t^2},$$

and thus

$$\mathbf{P}(Y_{t+1} = 2 \mid Y_t = 0) \;=\; \sum_{j=0}^{\infty} \frac{j(1+o(1))}{n_t^2} \mathbf{P}(Y_{t,4} = j \mid Y_t = 0). \tag{3.10}$$

By Corollary 2.1, Y_t and $Y_{t,4}$ are asymptotically independent Poisson, with means 3 and 9 respectively, and the total variation distance between the joint distribution of $(Y_t, Y_{t,4})$ and its limit is at most $O(n_t^{-1})$. So $\mathbf{P}(Y_{t,4} = j \mid Y_t = 0) = e^{-9}9^j/j! + O(n_t^{-1})$. Hence

$$\sum_{j \le \log n_t} \frac{j(1+o(1))}{n_t^2} \mathbf{P}(Y_{t,4} = j \mid Y_t = 0) \;=\; \frac{9}{n_t^2} + o(n_t^{-2}). \tag{3.11}$$

It was shown in Theorem 2.1 of [10], that $\mathbf{E}Y_{t,4}^3 = O(1)$. By Corollary 2.1, the total variation distance between the distribution of Y_t and its limit $\mathbf{Po}(3)$ is $O(n_t^{-1})$. So $\mathbf{P}(Y_t = 0) = e^{-3} + O(n_t^{-1})$. Then by the Markov inequality,

$$\mathbf{P}(Y_{t,4} \ge j \mid Y_t = 0) = \mathbf{P}(Y_{t,4}^3 \ge j^3 \mid Y_t = 0) \le \frac{1}{j^3}\mathbf{E}(Y_{t,4}^3 \mid Y_t = 0) = O(1/j^3).$$

Thus

$$\sum_{j > \log n_t} \frac{j(1+o(1))}{n_t^2} \mathbf{P}(Y_{t,4} = j \mid Y_t = 0) = o(n_t^{-2}). \tag{3.12}$$

By (3.9)–(3.12),

$$\mathbf{P}(Y_{t+1} = 2 \mid Y_t = 0) \;=\; \frac{9}{n_t^2} + o(n_t^{-2}), \tag{3.13}$$

$$\mathbf{P}(Y_{t+1} = 1 \mid Y_t = 0) \;=\; \frac{9}{n_t} + \frac{27}{2n_t^2} + o(n_t^{-2}), \tag{3.14}$$

$$\mathbf{P}(Y_{t+1} \ne 0 \mid Y_t = 0) \;=\; \frac{9}{n_t} + \frac{45}{2n_t^2} + o(n_t^{-2}). \tag{3.15}$$

From (3.7), (3.8) and (3.15),

$$\begin{aligned}
\delta_{t+1} \;=\;& \mathbf{P}(X_t = 0)\left(1 - \frac{9}{n_t}\right) - (\mathbf{P}(X_t = 0) - \delta_t)\left(1 - \frac{9}{n_t} - \frac{45}{2n_t^2} + o(n_t^{-2})\right) \\
& + \mathbf{P}(X_t \ne 0)\mathbf{P}(X_{t+1} = 0 \mid X_t \ne 0) - (\mathbf{P}(X_t \ne 0) + \delta_t)\mathbf{P}(Y_{t+1} = 0 \mid Y_t \ne 0) \\
=\;& \delta_t\left(1 - \frac{9}{n_t} + O(n_t^{-2}) - \mathbf{P}(Y_{t+1} = 0 \mid Y_t \ne 0)\right) \\
& + \mathbf{P}(X_t \ne 0)\left(\mathbf{P}(X_{t+1} = 0 \mid X_t \ne 0) - \mathbf{P}(Y_{t+1} = 0 \mid Y_t \ne 0)\right) \\
& + \mathbf{P}(X_t = 0)\left(\frac{45}{2n_t^2} + o(n_t^{-2})\right).
\end{aligned} \tag{3.16}$$

It only remains to estimate $\mathbf{P}(X_{t+1} = 0 \mid X_t \neq 0)$ and $\mathbf{P}(Y_{t+1} = 0 \mid Y_t \neq 0)$. From the definition of the random walk of $(X_t)_{t\geq 0}$,

$$\begin{aligned}\mathbf{P}(X_{t+1} = 0 \mid X_t \neq 0) &= \frac{\mathbf{P}(X_t = 1)\mathbf{P}(X_{t+1} = 0 \mid X_t = 1)}{\mathbf{P}(X_t \neq 0)} \\ &= \frac{3}{n_t}\frac{\mathbf{P}(X_t = 1)}{\mathbf{P}(X_t \neq 0)}. \end{aligned} \tag{3.17}$$

The calculation of $\mathbf{P}(Y_{t+1} = 0 \mid Y_t \neq 0)$ is not so straightforward. Given any two distinct edges e_i and e_j, we can define a walk $e_i, e_{l_1}, e_{l_2}, \ldots, e_{l_k}, e_j$, such that every two consecutive edges appearing in the walk are adjacent. The *distance* of e_i and e_j is defined to be the length of the shortest walk between e_i and e_j. For instance, if e_i and e_j are adjacent, then their distance is 1. Conditional on $Y_t = 1$, i.e. the number of triangles in G_t being 1, if this triangle is destroyed without creating any new triangles, then one of the edges contained in the triangle must be pegged. Call it e_1. The other edge e_2 being pegged must be chosen from those whose distance from e_1 is at least 3. Let $\mathcal{R}$ be the rare event that at least one 4-cycle shares a common edge with this triangle, and $\overline{\mathcal{R}}$ be the complement of $\mathcal{R}$. There are 3 ways to choose e_1 and 21 edges within distance 2 from e_1, including e_1 itself, if $\overline{\mathcal{R}}$ occurs. Otherwise, there are in any case $O(1)$ edges within distance 2 from e_1. Hence

$$\mathbf{P}(Y_{t+1} = 0 \mid Y_t = 1) = \frac{3(2n_t - 21)}{N_t}\mathbf{P}(\overline{\mathcal{R}} \mid Y_t = 1) + \frac{3(2n_t - O(1))}{N_t}\mathbf{P}(\mathcal{R} \mid Y_t = 1).$$

Note that the occurrence of $\mathcal{R}$ implies that $W_{t,5,1} \geq 1$. So by Lemma 3.2,

$$\mathbf{P}(\mathcal{R} \mid Y_t = 1) \leq \frac{\mathbf{P}(\mathcal{R})}{\mathbf{P}(Y_t = 1)} = O(n_t^{-1}).$$

Noting that (2.1) implies $1/N_t = 1/(2n_t^2)(1 + 7/2n_t + O(n_t^{-2}))$,

$$\mathbf{P}(Y_{t+1} = 0 \mid Y_t = 1) = \frac{3}{n_t} - \frac{21}{n_t^2} + O(n_t^{-3}). \tag{3.18}$$

Given $Y_t = j$ for any $j \geq 3$, to destroy all j triangles in a single step, it is required either to peg an edge contained in j triangles, and hence a small subgraph with excess at least 2, or to peg two edges such that one edge is contained in at least one triangle, and the other edge contained in at least two triangles. The latter is a small subgraph with excess at least 1. Both cases imply that for $j \geq 3$,

$$\mathbf{P}(Y_{t+1} = 0 \mid Y_t = j) = O(n_t^{-3}). \tag{3.19}$$

Now we only need to compute $\mathbf{P}(Y_{t+1} = 0 \mid Y_t = 2)$. To destroy two triangles in a single step, either the two triangles are isolated and the algorithm pegs two edges which are contained in two triangles, or the two triangles share a common edge and the algorithm pegs the common edge, i.e. the chord of a C_4^*. Conditional on $Y_t = 2$, the number of C_4^*

can be either 0 or 1. Let W_t denote the number of C_4^* as before. If $W_t = 0$, the two triangles are isolated, and then two edges contained in different triangles are pegged, so $\mathbf{P}(Y_{t+1} = 0 \mid Y_t = 2, W_t = 0) = 9/N_t$. If $W_t = 1$, then the algorithm pegs the chord of the C_4^*. So $\mathbf{P}(Y_{t+1} = 0 \mid Y_t = 2, W_t = 1) = (2n_t - 7)/N_t$. Thus

$$\begin{aligned}\mathbf{P}(Y_{t+1} = 0 \mid Y_t = 2) &= \frac{9}{N_t}(1 - \mathbf{P}(W_t = 1 \mid Y_t = 2)) + \frac{2n_t - 7}{N_t}\mathbf{P}(W_t = 1 \mid Y_t = 2) \\ &= \frac{9}{N_t} + \frac{2n_t - 16}{N_t}\mathbf{P}(W_t = 1 \mid Y_t = 2). \qquad (3.20)\end{aligned}$$

By Lemma 3.4, $\mathbf{P}(Y_t = 2 \mid W_t = 1) = e^{-3} + o(1)$ and therefore using (3.4),

$$\mathbf{P}(W_t = 1 \mid Y_t = 2) = \frac{\mathbf{P}(Y_t = 2 \mid W_t = 1)\mathbf{P}(W_t = 1)}{\mathbf{P}(Y_t = 2)} = \frac{3 + o(1)}{2n_t} + O(n_t^{-2}).$$

Combining this with (3.20) and (2.1), we have

$$\mathbf{P}(Y_{t+1} = 0 \mid Y_t = 2) = \frac{6 + o(1)}{n_t^2} + O(n_t^{-3}). \qquad (3.21)$$

From (3.18), (3.19) and (3.21) we have

$$\mathbf{P}(Y_{t+1} = 0 \mid Y_t \neq 0) = \left(\frac{3}{n_t} - \frac{21}{n_t^2} + O(n_t^{-3})\right)\frac{\mathbf{P}(Y_t = 1)}{\mathbf{P}(Y_t \neq 0)} + \frac{6 + o(1)}{n_t^2}\frac{\mathbf{P}(Y_t = 2)}{\mathbf{P}(Y_t \neq 0)} + O(n_t^{-3}). \qquad (3.22)$$

By Corollary 2.1, $d_{TV}(X_t, Y_t) = O(n_t^{-1})$, and so (3.17) gives

$$\begin{aligned}&\mathbf{P}(X_{t+1} = 0 \mid X_t \neq 0) - \mathbf{P}(Y_{t+1} = 0 \mid Y_t \neq 0) \\ &\quad = \frac{3}{n_t}\left(\frac{\mathbf{P}(X_t = 1)}{\mathbf{P}(X_t \neq 0)} - \frac{\mathbf{P}(Y_t = 1)}{\mathbf{P}(Y_t \neq 0)}\right) + \frac{21}{n_t^2}\frac{\mathbf{P}(X_t = 1)}{\mathbf{P}(X_t \neq 0)} - \frac{6 + o(1)}{n_t^2}\frac{\mathbf{P}(X_t = 2)}{\mathbf{P}(X_t \neq 0)} + O(n_t^{-3}) \\ &\quad = \frac{3}{n_t}O(d_{TV}(X_t, Y_t)) + \frac{36e^{-3}}{(1 - e^{-3})n_t^2} + o(n_t^{-2}).\end{aligned}$$

Combining this with (3.16) and (3.22) gives

$$\delta_{t+1} \geq \delta_t(1 - \gamma(t)) + \frac{3(1 - e^{-3})}{n_t}O(d_{TV}(X_t, Y_t)) + \frac{117e^{-3}}{2n_t^2} + o(n_t^{-2}), \qquad (3.23)$$

where $\gamma(t) = 9/n_t + \mathbf{P}(Y_{t+1} = 0 \mid Y_t \neq 0) + O(n_t^{-2}) \geq 9/n_t + O(n_t^{-2})$. For a contradiction, assume that $d_{TV}(X_t, Y_t) = o(n_t^{-1})$. Then (3.23) gives

$$\delta_{t+1} \geq \delta_t(1 - \gamma(t)) + \frac{117e^{-3}}{2n_t^2} + w(n_t),$$

for some function $w(n_t)$ such that $w(n_t) = o(n_t^{-2})$.

Let $(a_t)_{t\geq 0}$ be defined as $a_0 = \delta_0$ and for all $t \geq 0$,

$$a_{t+1} = a_t(1-\gamma(t)) + \frac{117e^{-3}}{2n_t^2} + w(n_t).$$

Clearly $\delta_0 \geq a_0$. Assume $\delta_t \geq a_t$ for some $t \geq 0$. Then

$$\delta_{t+1} \geq \delta_t(1-\gamma(t)) + \frac{117e^{-3}}{2n_t^2} + w(n_t) \geq a_t(1-\gamma(t)) + \frac{117e^{-3}}{2n_t^2} + w(n_t) = a_{t+1}.$$

Hence $\delta_t \geq a_t$ for all $t \geq 0$. By Lemma 3.1, $a_t = \Theta(n_t^{-1})$. Hence $\delta_t = \Omega(n_t^{-1})$, which contradicts the assumption that $d_{TV}(X_t, Y_t) = o(n_t^{-1})$. So $d_{TV}(Y_t, \mathbf{Po}(3))$ is not $o(n_t^{-1})$. Clearly

$$d_{TV}(\mathbf{Y}_{t,k}, \mathbf{Po}(\mu_3, \dots, \mu_k)) \geq d_{TV}(Y_t, \mathbf{Po}(3)),$$

where $\mathbf{Po}(\mu_3, \dots, \mu_k)$ is the joint independent Poisson distribution with means $\mu_3, \dots, \mu_k$, and μ_i is as stated in Theorem 2.1, for all $3 \leq i \leq k$. So $d_{TV}(\mathbf{Y}_{t,k}, \mathbf{Po}(\mu_3, \dots, \mu_k))$ is not $o(n_t^{-1})$.

The analysis for even $d > 4$ is analogous but more complicated. The random walk $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are defined similarly, as follows. First, define $\mathbf{B}_{t,3} := \{i \in \mathbf{Z}_+ : ((d/2-1)(d-1)^2 + 3i)/n_t \leq 1\}$, and the boundary of $\mathbf{B}_{t,3}$ to be $\partial\mathbf{B}_{t,3} := \{i \in \mathbf{B}_{t,3} : i+1 \notin \mathbf{B}_{t,3}\}$. For $X_t \in \mathbf{B}_{t,3} \setminus \partial\mathbf{B}_{t,3}$,

$$X_{t+1} = \begin{cases} X_t - 1 & \text{w.p. } 3X_t/n_t \\ X_t & \text{w.p. } 1 - 3X_t/n_t - (d/2-1)(d-1)^2/n_t \\ X_t + 1 & \text{w.p. } (d/2-1)(d-1)^2/n_t. \end{cases}$$

For $X_t \in \partial\mathbf{B}_{t,3}$,

$$X_{t+1} = \begin{cases} X_t - 1 & \text{w.p. } 3X_t/n_t \\ X_t & \text{w.p. } 1 - 3X_t/n_t. \end{cases}$$

For $X_t \notin \mathbf{B}_{t,3}$,

$$X_{t+1} = X_t \quad \text{w.p. } 1.$$

It was shown in [10] that $\mathbf{Po}(\mu)$, the Poisson distribution with mean $\mu = ((d-1)^3 - (d-1)^2)/6$, is a stationary distribution of the Markov chain $(X_t)_{t\geq 0}$. The variable δ_t is defined the same as before. In order to bound δ_t, we need to compute

$$\mathbf{P}(Y_{t+1} \neq 0 \mid Y_t = 0), \quad \mathbf{P}(Y_{t+1} = 0 \mid Y_t \neq 0).$$

The calculation follows exactly the same path as in the case $d = 4$, though much more complicated. As an example, we explain the calculation of N_t, the number of possible pegging operations at step t. We also show as another example, the calculation of A_t, the number of pegging operations which create a triangle at step t, conditional on the number of triangles in G_t being 0.

Since G_t is d-regular, the number of edges in G_t is $m_t = dn_t/2$. At step $t+1$, the algorithm chooses $d/2$ non-adjacent edges. There are m_t ways to choose the first edge, and $m_t-(2d-1)i$ ways to choose the $(i+1)$-th edge for $1\le i\le d/2-1$, if we ignore the case that two or more of the previous i edges chosen are of distance 2, which will have a contribution of $O\left(m_t^{d/2-2}\right)$ to the total count. Hence,

$$\begin{aligned} N_t &= \frac{1}{(d/2)!}\left(\prod_{i=0}^{d/2-1}(m_t-(2d-1)i)+O\left(m_t^{d/2-2}\right)\right) \\ &= \frac{m_t^{d/2}}{(d/2)!}\left(1-\frac{d}{4}\left(\frac{d}{2}-1\right)(2d-1)m_t^{-1}+O(m_t^{-2})\right). \end{aligned}$$

Conditional on $Y_t=0$, if a triangle is created that contains an edge $e\in G_t$, the pegging algorithm pegs two edges e_1 and e_2 that are adjacent to e but at different end-vertices of e, together with $d/2-2$ other non-adjacent edges. There are m_t options for the choice of e, and for each fixed e, there are exactly $(d-1)^2$ ways to choose e_1 and e_2, since $Y_t=0$. Thus the number of ways to create a triangle when $Y_t=0$ is

$$\begin{aligned} A_t &= m_t(d-1)^2\frac{1}{(d/2-2)!}\left(\prod_{i=2}^{d/2-1}(m_t-(2d-1)i+1)+O\left(m_t^{d/2-2}\right)\right) \\ &= \frac{(d-1)^2m_t^{d/2-1}}{(d/2-2)!}\left(1-\left((2d-1)\left(\frac{d}{2}+1\right)\left(\frac{d}{4}-1\right)-\left(\frac{d}{2}-2\right)\right)m_t^{-1}+O(m_t^{-2})\right). \end{aligned}$$

Hence the expected number of triangles created, conditional on $Y_t=0$, is,

$$\frac{A_t}{N_t}=\frac{(d-1)^2\left(\frac{d}{2}-1\right)}{n_t}+\frac{(d-1)^2(d-2)}{dn_t^2}\left(\frac{5d}{2}-3\right)+O(n_t^{-3}).$$

We omit the calculational details of the probabilities of other events. Table 1 gives the significant terms in the probabilities of all events required to compute $\mathbf{P}(Y_{t+1}\neq 0\mid Y_t=0)$ and $\mathbf{P}(Y_{t+1}=0\mid Y_t\neq 0)$, as examined in detail in the special case when $d=4$. The values of the constants a_1, a_2, μ, k_1, k_2 in Table 1 are given in Table 2.

Hence

$$\begin{aligned} \delta_{t+1} &= \delta_t\left(1-\gamma(t)\right) \\ &\quad+\mathbf{P}(X_t\neq 0)\left(\frac{3}{n_t}O(d_{TV}(X_t,Y_t))-\frac{k_1}{n_t^2}\frac{\mathbf{P}(X_t=1)}{\mathbf{P}(X_t\neq 0)}-\frac{k_2}{n_t^2}\frac{\mathbf{P}(X_t=2)}{\mathbf{P}(X_t\neq 0)}+O(n_t^{-3})\right) \\ &\quad+\mathbf{P}(X_t=0)\frac{a_1-a_2}{n_t^2}+o(n_t^{-2}) \\ &= \delta_t\left(1-\gamma(t)\right)+\frac{O(d_{TV}(X_t,Y_t))}{n_t}+e^{-\mu}\left(a_1-a_2-\mu k_1-\frac{\mu^2}{2}k_2\right)n_t^{-2}+o(n_t^{-2}). \end{aligned}$$

where $\gamma(t)\ge (d-1)^2(d/2-1)/n_t+O(n_t^{-2})\ge 9/n_t+O(n_t^{-2})$. We only need to show that

$$a_1-a_2-\mu k_1-\frac{\mu^2}{2}k_2\neq 0.$$

$\mathbf{P}(Y_{t+1} = 2 \mid Y_t = 0)$	$a_2/n_t^2 + o(n_t^{-2})$
$\mathbf{P}(Y_{t+1} \neq 0 \mid Y_t = 0)$	$\frac{(d-1)^2\left(\frac{d}{2}-1\right)}{n_t} + (a_1 - a_2)n_t^{-2} + o(n_t^{-2})$
$\mathbf{P}(Y_{t+1} = 0 \mid Y_t = 1)$	$\frac{3}{n_t} + \frac{k_1}{n_t^2} + O(n_t^{-3})$
$\mathbf{P}(Y_{t+1} = 0 \mid Y_t = 2, W_t = 0)$	$\frac{9(d-2)}{d}n_t^{-2} + O(n_t^{-3})$
$\mathbf{P}(Y_{t+1} = 0 \mid Y_t = 2, W_t = 1)$	$\frac{1}{n_t} + O(n_t^{-2})$
$\mathbf{P}(W_t = 1)$	$\frac{3\mu}{4d}(d-2)^2(d-1)n_t^{-1} + O(n_t^{-2})$
$\mathbf{P}(Y_{t+1} = 0 \mid Y_t \neq 0)$	$\left(\frac{3}{n_t} + \frac{k_1}{n_t^2} + O(n_t^{-3})\right)\frac{\mathbf{P}(Y_t=1)}{\mathbf{P}(Y_t\neq 0)} + \frac{k_2}{n_t^2}\frac{\mathbf{P}(Y_t=2)}{\mathbf{P}(Y_t\neq 0)} + O(n_t^{-3})$

Table 1: Significant probabilities

By substituting the values of a_1, a_2, k_1 and k_2 in terms of d, and simplifying, we get

$$a_1 - a_2 - \mu k_1 - \frac{\mu^2}{2}k_2 = -\frac{(d-1)^2(d-2)(64 - 134d + 91d^2 - 25d^3 + 2d^4)}{8d},$$

which has no integral roots but 1 and 2, so

$$a_1 - a_2 - \mu k_1 - \frac{\mu^2}{2}k_2 \neq 0 \quad \text{for all even } d \geq 4.$$

Hence the ϵ-mixing time is not $o(\epsilon^{-1})$ for any even $d \geq 4$. ∎

4 Discussion

For any fixed $d \geq 3$, it is well known that the random d-regular graphs with the uniform distribution are d-connected and have diameter $O(\log n)$ a.a.s. (See [12] for terms and facts not referenced here.) These properties are of central interest where the graphs are used as communication networks. The first author determined the connectivity of random regular graphs in $\mathcal{P}(G_0, d)$ in [9], which supports the conjecture given in [10], that the probability space of d-regular graphs in the uniform model is contiguous with that of those generated by the pegging model. If the conjecture holds, it implies that the random regular graphs in $\mathcal{P}(G_0, d)$ are a.a.s. d-connected with diameter $O(\log n)$. In any case, the logarithmic diameter is common among random networks with average degree above 1. In the Erdős-Rényi model of random graphs, the components of the random graph a.a.s. all have diameter $O(\log n)$ if the edge probability p is at least c/n for some $c > 1$. Ferholz and Ramachandran [8] showed that the diameter of random sparse graphs with given degree sequences is a.a.s. $c(1 + o(1))\log n$, when the degree sequences satisfy some natural convergence conditions, and they determined the value of c. Bollobás and

a_1	$\frac{(d-1)^2(d-2)}{d}\left(\frac{5d}{2}-3\right)$
a_2	$\frac{2((d-1)^4-(d-1)^2)(d-2)+(d-1)^4(d-2)(d-4)(d-6)}{8d}$
c	$\frac{d}{4}\left(\frac{d}{2}-1\right)(2d-1)$
l	$3+3(d-2)+2(d-2)(d-1)$
μ	$\frac{(d-1)^3-(d-1)^2}{6}$
k_1	$\frac{6c}{d}-\frac{6}{d}\left(\left(\frac{d}{2}-1\right)l+\frac{1}{2}(2d-1)\left(\frac{d}{2}-1\right)\left(\frac{d}{2}-2\right)\right)$
k_2	$\frac{1}{d}\left(9(d-2)+\frac{3}{2\mu}(d-1)(d-2)^2\right)$

Table 2: Value of the constants appearing in Table 1

Riordan [5] proved that the random graphs generated by the preferential attachment model a.a.s. have diameter asymptotically $\log n/\log\log n$. We are currently studying the diameter of the graphs generated by the pegging process $\mathcal{P}(G_0)$.

Acknowledgement The authors wish to thank an anonymous referee for suggesting many small improvements in the manuscript.

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