The number of elements in the mutation class of a quiver of type D_n

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Abstract

We show that the number of quivers in the mutation class of a quiver of Dynkin type D_n is given by $\sum_{d|n} \phi(n/d)\binom{2d}{d}/(2n)$ for $n \geq 5$. To obtain this formula, we give a correspondence between the quivers in the mutation class and certain rooted trees.

Introduction

Quiver mutation is an important ingredient in the definition of cluster algebras [FZ1]. It is an operation on quivers, which induces an equivalence relation on the set of quivers. The mutation class $\mathcal{M}$ of a quiver Q consists of all quivers mutation equivalent to Q. If Q is a Dynkin quiver, then $\mathcal{M}$ is finite. In [T] an excplicit formula for $|\mathcal{M}|$ is given for Dynkin type A_n. Here we give an explicit formula for the number of quivers in the mutation class of a quiver of Dynkin type D_n. The formula is given by

$$d(n) = \begin{cases} \sum_{d|n} \phi(n/d)\binom{2d}{d}/(2n) & \text{if } n \geq 5, \\ 6 & \text{if } n = 4, \end{cases}$$

where ϕ is the Euler function.

The proof for this formula consists of two parts. The first part shows that the mutation class of type D_n is in 1–1 correspondence with the triangulations (with tagged edges) of a punctured n-gon, up to rotation and inversion of tags. This is a generalization of the method used in [T] to count the number of elements in the mutation class of quivers of Dynkin type A_n. Here we are strongly using the ideas in [FST] and [S].

In the second part we count the number of (equivalence classes of) triangulations of a punctured n-gon, by describing an explicit correspondence to a certain class of rooted trees. A tree in this class is constructed by taking a family of full binary trees $T_1, \ldots, T_s$ such that the total number of leaves is n, and then adding a node S and an edge from this node to the root of T_i for each i, such that S becomes a root (Figure 21 displays all such trees for $n = 5$).

When these rooted trees are considered up to rotation at the root, they are in 1–1 correspondence with the above mentioned equivalence classes of triangulations of the punctured n-gon. To count these rooted trees we use a simple adaption of a known formula found in [I] and [St, exercise 7.112 b].

We also point out a mutation operation on these rooted trees, corresponding to the other mutation operations involved (on triangulations and on quivers).

Our formula and the bijection to triangulations of the punctured n-gon were presented at the ICRA in Torun, August 2007 [T2].

After completing our work, we learnt about the paper [GLZ]. They also generalize the methods in [T] to prove the bijection from the mutation class of D_n to triangulations of the punctured n-gon. However, their method of counting triangulations is very different from ours. They use the classification of quivers of mutation type D_n, recently given in [V]. The authors of [GLZ] end up with a very different formula than ours. In particular, their formula is not explicit, and it seems they get a different output than we get, e.g. for $n = 6$.

We are grateful to Hugh Thomas for several useful discussions and for the idea of making use of binary trees as an alternative to rooted planar trees. We would also like to thank Dagfinn Vatne for useful discussions.

1 Quiver mutation

Let Q be a quiver with no multiple arrows, no loops and no oriented cycles of length two. Mutation of Q at the vertex k gives a quiver Q' obtained from Q in the following way.

1. Add a vertex k^*.

2. If there is a path $i \to k \to j$, then if there is an arrow from j to i, remove this arrow. If there is no arrow from j to i, add an arrow from i to j.

3. For any vertex i replace all arrows from i to k with arrows from k^* to i, and replace all arrows from k to i with arrows from i to k^*.

4. Remove the vertex k.

It is easy to see that mutating Q twice at k gives Q. We say that two quivers Q and Q' are mutation equivalent if Q' can be obtained from Q by a finite number of mutations. The mutation class of Q consists of all quivers mutation equivalent to Q. Figure 1 gives all quivers in the mutation class of D_4, up to isomorphism.

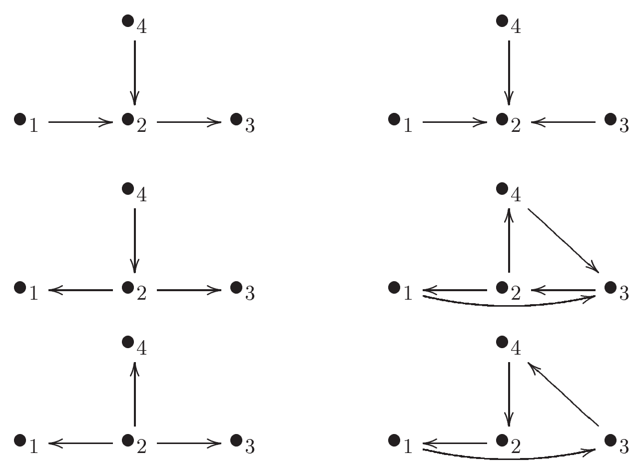

Figure 1: The mutation class of D_4.

It is know from [FZ3] that the mutation class of a Dynkin quiver Q is finite. An explicit formula for the number of equivalence classes in the mutation class of any quiver of type A_n was given in [T].

The Catalan number $C(i)$ can be defined as the number of triangulations of an $i+2$-gon with $i-1$ diagonals. It is given by

$$C(i) = \frac{1}{i+1}\binom{2i}{i}.$$

The number of equivalence classes in the mutation class of any quiver of type A_n is then given by the formula [T]

$$a(n) = C(n+1)/(n+3) + C((n+1)/2)/2 + (2/3)C(n/3)$$

where the second term is omitted if $(n+1)/2$ is not an integer and the third term is omitted if $n/3$ is not an integer. This formula counts the triangulations of the disk with n diagonals [B].

2 Cluster-tilted algebras

The cluster category was defined independently in [BMRRT] for the general case and in [CCS] for the A_n case. Let $\mathcal{D}^b(\operatorname{mod} H)$ be the bounded derived category of the finitely generated modules over a finite dimensional hereditary algebra H over a field K. In [BMRRT] the cluster category was defined as the orbit category $\mathcal{C} = \mathcal{D}^b(\mathrm{mod}\, H)/\tau^{-1}[1]$, where τ is the Auslander-Reiten translation and [1] the suspension functor. The cluster-tilted algebras are the algebras of the form $\Gamma = \mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$, where T is a cluster-tilting object in $\mathcal{C}$ (see [BMR1]). In this paper we will mostly consider the case where the underlying graph of the quiver of H is of Dynkin type D.

If $\Gamma = \mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ for a cluster-tilting object T in $\mathcal{C}$, and $\mathcal{C}$ is the cluster category of a path algebra of type D_n, then we say that Γ is of type D_n.

Let Q be a quiver of a cluster-tilted algebra Γ. From [BMR2] it is known that if Q' is obtained from Q by a finite number of mutations, then there is a cluster-tilted algebra Γ' with quiver Q'. Moreover, Γ is of finite representation type if and only if Γ' is of finite representation type [BMR1]. We also have that Γ is of type D_n if and only if Γ' is of type D_n. It is well known that we can obtain all orientations of a Dynkin quiver by reflections, and hence all orientations of a Dynkin quiver are mutation equivalent. From [BMR3, BIRS] we know that a cluster-tilted algebra is up to isomorphism uniquely determined by its quiver (see also [CCS2]).

It follows from this that the number of non-isomorphic cluster-tilted algebras of type D_n is equal to the number of equivalence classes in the mutation class of any quiver with underlying graph D_n.

3 Category of diagonals of a regular $n+3$-gon

In [CCS] Caldero, Chapoton and Schiffler considered regular polygons with $n+3$ vertices and triangulations of such polygons. A diagonal is a straight line between two non-adjacent vertices on the border of the polygon, and a triangulation is a maximal set of diagonals which do not cross. A triangulation of an $(n+3)$-gon consists of exactly n diagonals. In [CCS] the category of diagonals of such polygons was defined, and it was shown to be equivalent to the cluster category, as defined in Section 2, in the A_n case. It was also shown that a cluster-tilting object in the cluster category $\mathcal{C}$ corresponds to a triangulation of the regular $(n+3)$-gon in the A_n case. In [T] it was shown that there is a bijection between isomorphism classes of cluster-tilted algebras of type A_n (or equivalently isomorphism classes of quivers in the mutation class of any quiver with underlying graph A_n) and triangulations of the disk with n diagonals (i.e. triangulations of the regular $(n+3)$-gon up to rotation).

For any triangulation of the regular $(n+3)$-gon we can define a quiver with n vertices in the following way. The vertices are the midpoints of the diagonals. There is an arrow between i and j if the corresponding diagonals bound a common triangle. The orientation is $i \to j$ if the diagonal corresponding to j can be obtained from the diagonal corresponding to i by rotating anticlockwise about their common vertex. It is also known from [CCS] that all quivers obtained in this way are quivers of cluster-tilted algebras of type A_n. This means that we can define a function γ_n from the mutation class of A_n to the set of all triangulations of the regular $(n+3)$-gon. There is an induced function $\widetilde{\gamma}_n$

from the mutation class of A_n to the set of all triangulations of the disk with n diagonals. It was shown in [T] that $\widetilde{\gamma}_n$ is a bijection.

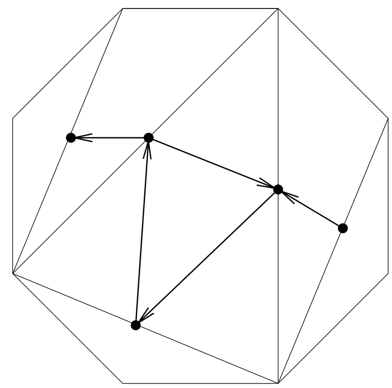

Figure 2: A triangulation Δ of the regular 8-gon and the corresponding quiver $\gamma_5(\Delta)$ of type A_5.

4 Category of diagonals of a punctured regular n-gon

In this paper we will consider the D_n case and we will first recall some results and notions from [S] and [FST].

Let $\mathcal{P}_n$ be a regular polygon with n vertices and one puncture in the center. Diagonals (or edges) will be homotopy classes of paths between two vertices on the border of the polygon. We follow the definitions from [S].

Let $\delta_{a,b}$ be an oriented path between two vertices $a \neq b$ on the border of $\mathcal{P}_n$ in counterclockwise direction, such that $\delta_{a,b}$ does not run through the same point twice. Also let $\delta_{a,a}$ be the path that runs from a to a, i.e. around the polygon exactly one time. We define $|\delta_{a,b}|$ to be the number of vertices on the path $\delta_{a,b}$, including a and b.

An edge is a triple (a, α, b) where a and b are vertices on the border of the polygon and α is an oriented path from a to b lying in the interior of $\mathcal{P}_n$ and that is homotopic to $\delta_{a,b}$. Furthermore, the path should not cross itself and $|\delta_{a,b}| \geq 3$. Two edges are equivalent if they start in the same vertex, end in the same vertex and are homotopic.

Let E be the set of equivalence classes of edges, and denote by $M_{a,b}$ the equivalence class of edges in E going from a to b. In [S] the set of tagged edges is defined as follows.

$$\{M^{\epsilon}_{a,b} | M_{a,b} \in E,\ \epsilon \in \{-1, 1\} \text{ with } \epsilon = 1 \text{ if } a \neq b\}$$

From now on tagged edges will be called *diagonals*. Diagonals starting and ending in the same vertex a will be represented as lines between the puncture and the vertex a. Diagonals with $\epsilon = -1$ will be drawn with a tag on it. In some cases we will draw them as loops.

The crossing number $e(M^{\epsilon}_{a,b}, N^{\epsilon'}_{c,d})$ is the minimal number of intersection of representations of $M^{\epsilon}_{a,b}$ and $N^{\epsilon'}_{c,d}$ in the interior of the punctured polygon. When $a = b$ and $c = d$, we let the crossing number be 1 if $a \neq c$ and $\epsilon \neq \epsilon'$ and 0 otherwise. If $e(M^{\epsilon}_{a,b}, N^{\epsilon'}_{c,d}) = 0$, we say that $M^{\epsilon}_{a,b}$ and $N^{\epsilon'}_{c,d}$ do not cross.

Now we can define a triangulation of the punctured n-gon, which is a maximal set of non-crossing diagonals. Any such set will have n elements [S]. See some examples of triangulations of the punctures 6-gon in Figure 3.

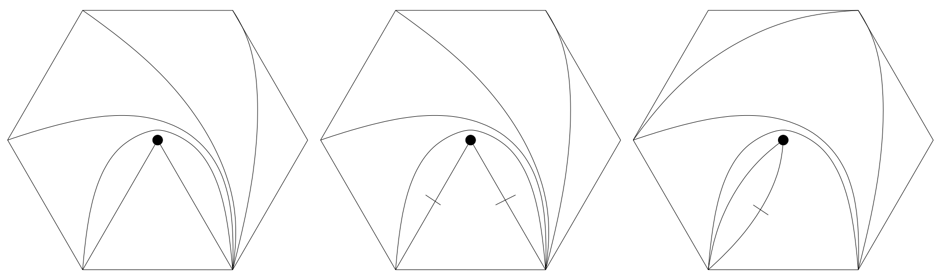

Figure 3: Examples of triangulations of the punctured 6-gon.

[S] defines a category which is equivalent to the cluster catecory in the D_n case in the following way. The objects are direct sums of diagonals (tagged edges), and the morphism space from α to β is spanned by sequences of elementary moves modulo the mesh-relations. The equivalence between this category $\mathcal{C}$ and the cluster category in the D_n case was proved in [S]. Furthermore we have the following important results:

- $\dim \operatorname{Ext}^1_{\mathcal{C}}(\alpha, \beta)$ is equal to the crossing number of α and β.
- A cluster-tilting object corresponds to a triangulation.
- The Auslander-Reiten translation of a diagonal from a to b is given by clockwise rotation of the diagonal if $a \neq b$. If $a = b$ the AR-translation is given by clockwise rotation and inverting the tag.

Let $\mathcal{T}_n$ be the set of all triangulations of $\mathcal{P}_n$, and let Δ be an element in $\mathcal{T}_n$. We can assign to Δ a quiver in the following way (see [FST]). Just as in the A_n case, the vertices are the midpoints of the diagonals. There is an arrow between i and j if the corresponding diagonals bound a common triangle. The orientation is $i \to j$ if the diagonal corresponding to j can be obtained from the diagonal corresponding to i by rotating anticlockwise about their common vertex. In the case when there are two diagonals α and α' between the puncture and the same vertex on the border, both adjacent to a diagonal β and a border edge δ, we consider the triangle with edges α, β and δ separately from the triangle with edges α', β and δ, when thinking of α and α' as loops around the puncture. If we end up with an oriented cycle of length 2, delete both arrows in the cycle. See some examples in Figure 4.

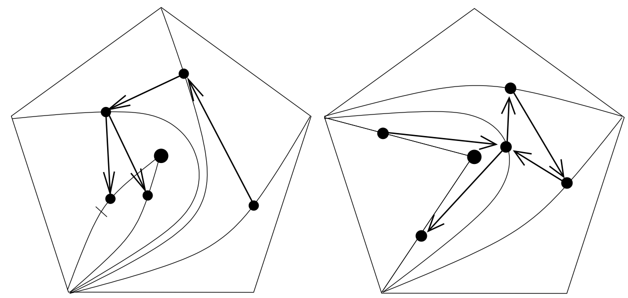

Figure 4: Some examples of triangulations and corresponding quiver.

Let $\mathcal{M}_n$ be the mutation class of D_n, i.e. all quivers obtained by repeated mutations from D_n, up to isomorphisms of quivers. We can define a function $\epsilon_n : \mathcal{T}_n \to \mathcal{M}_n$, where we set $\epsilon_n(\Delta) = Q_\Delta$ for any triangulation in $\mathcal{T}_n$. It is known that Q_Δ is a quiver of Dynkin type D_n and that all quiver of type D can be obtained this way, hence ϵ is surjective.

We can define a mutation operation on a triangulation. If α is a diagonal in a triangulation, then mutation at α is defined as replacing α with another diagonal such that we obtain a new triangulation. This can be done in one and only one way. It is known that mutation of quivers commutes with mutation of triangulations under ϵ (see [S, FST]).

5 Bijection between the mutation class of a quiver of type D_n and triangulations up to rotation and inverting tags

Here we adapt the methods and ideas of [T] to obtain a bijection between the mutation class of a quiver of type D_n and the set of triangulations of a punctured n-gon up to rotations and inversion of tags. See also [GLZ].

We say that a diagonal from a to b is *close to the border* if $|\delta(a,b)| = 3$. For a quiver Q_Δ corresponding to a triangulation Δ, we will always denote by v_α the vertex in Q_Δ corresponding to the diagonal α. From now on we let $n \geq 5$. Let us denote by S_n the triangulation of $\mathcal{P}_n$ shown in Figure 5. Note that this triangulation and the triangulation S_n with all tags inverted are the only triangulations that correspond to the quiver consisting of the oriented cycle of length n, Q_n.

Lemma 5.1 *Let Δ be a triangulation of $\mathcal{P}_n$, with $\Delta \neq S_n$. Then there exists a diagonal in Δ which is close to the border.*

Proof: Let Δ be a triangulation of $\mathcal{P}_n$. If Δ is not S_n, then there is at least one diagonal α which connects two vertices on the border. See Figure 6.

Consider the non-punctured surface B determined by this diagonal. If α is not close to the border, there exist a diagonal that divides the surface B into two smaller surfaces. By induction, there exists a diagonal close to the border. ■

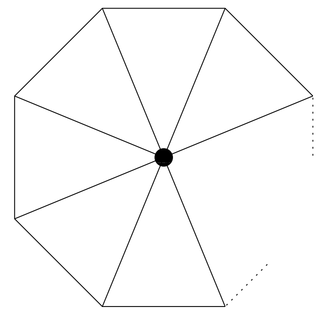

Figure 5: Triangulation R_n corresponding to the quiver consisting of the oriented cycle of length n.

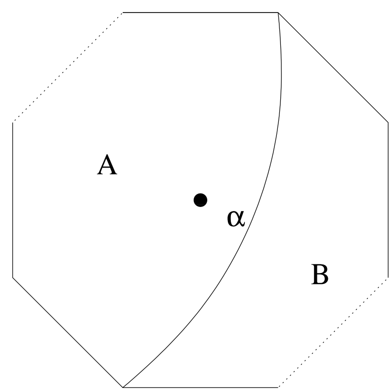

Figure 6: The diagonal α divides the polygon into a punctured and a non-punctured surface.

Lemma 5.2 *If a diagonal α of a triangulation Δ is close to the border, then the corresponding vertex v_α in $\epsilon_n(\Delta) = Q_\Delta$ is either a source, a sink or lies on an oriented cycle of length 3.*

Proof: Suppose α is a diagonal close to the border. We have to consider the eight cases shown in Figure 7. In the first picture in Figure 7, α corresponds to a source since no other vertex except v_β can be adjacent to v_α, or else the corresponding diagonal would cross β. In the second picture α corresponds to a sink. In picture three, four, five and six, there are arrows between v_α, v_β and $v_{\beta'}$, and in the last two pictures, there are arrows between v_α, v_β and v_γ, so v_α lies on an oriented cycle of length 3. ■

Let Δ be a triangulation of $\mathcal{P}_n$ and let α be a diagonal close to the border. We define a triangulation Δ/α of $\mathcal{P}_n$ obtained from Δ by letting α be a border edge and leaving all the other diagonals unchanged. We write Δ/α for the new triangulation obtained and we say that we factor out α. See Figure 8. Note that this operation is well-defined for each case in Figure 7.

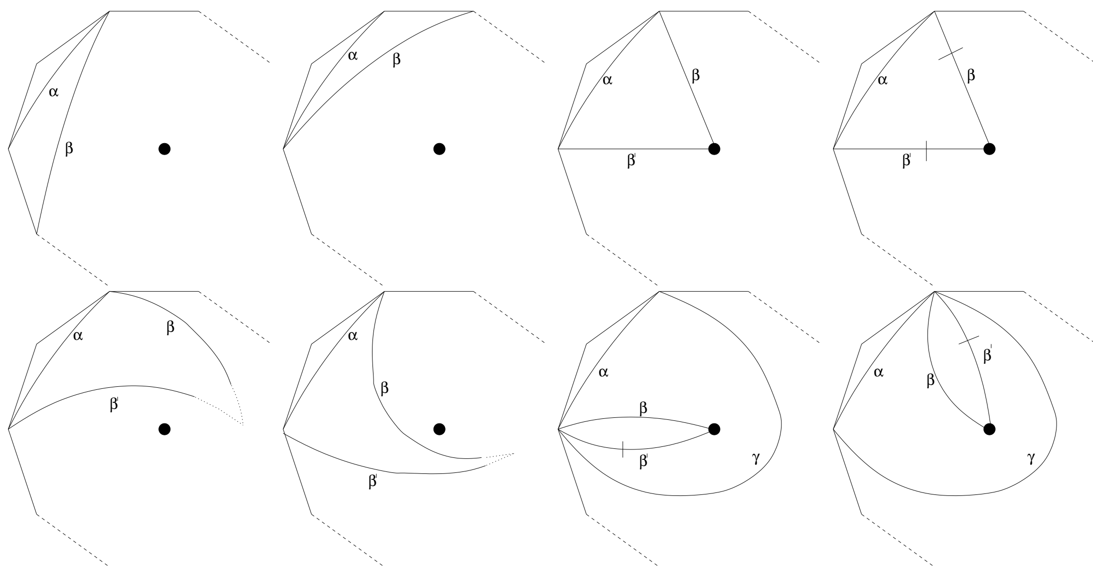

Figure 7: See the proof of Lemma 5.2

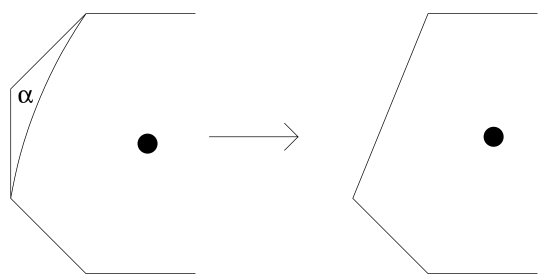

Figure 8: Factoring out a diagonal close to the border.

Lemma 5.3 *Let Δ be a triangulation of $\mathcal{P}_n$, with $\Delta \neq S_n$ and let $\epsilon_n(\Delta) = Q_\Delta$ be the corresponding quiver. If α is a diagonal close to the border in Δ, then the quiver Q_Δ/v_α obtained from Q_Δ by factoring out the vertex v_α is connected and of type D_{n-1}. Furthermore, we have that $\epsilon_{n-1}(\Delta/\alpha) = Q_\Delta/v_\alpha$, when α is close to the border.*

Proof: By Lemma 5.2 we have that Q_Δ/v_α is connected. It is also straightforward to verify that $\epsilon_{n-1}(\Delta/\alpha) = Q_\Delta/v_\alpha$ for each case, and hence Q_Δ/v_α is of type D_{n-1} since Δ/α is a triangulation of $\mathcal{P}_{n-1}$. ■

Now we describe what happens when we factor out a vertex corresponding to a diagonal not close to the border. We need to consider two cases. We first deal with the case when α is a diagonal not going between the puncture and the border.

Lemma 5.4 *Let Δ be a triangulation and $\epsilon_n(\Delta) = Q_\Delta$. If we factor out a vertex in Q_Δ corresponding to a diagonal that is not close to the border and that is not a diagonal between the puncture and the border, then the resulting quiver is disconnected.*

Proof: Let α be a diagonal not close to the border and not between the puncture and the border. Then the diagonal divides $\mathcal{P}_n$ into two surfaces A and B. See Figure 6. Let β be a diagonal in A and β' a diagonal in B. If β and β' would determine a common triangle, the third diagonal would cross α, hence there is no arrow between the subquiver determined by A and the subquiver determined by B, except those passing through v_α. It follows that factoring out v_α disconnects the quiver. ■

Let Δ be a triangulation of $\mathcal{P}_n$ and let α be a diagonal between the puncture and a vertex b_i on the border of the polygon. We want to understand the effect of factoring out v_α (see Figure 9). In $\mathcal{P}_n$, create a new vertex c between b_{i-1} and b_i and a new vertex d between b_i and b_{i+1}, such that we obtain a $(n+2)$-polygon. Let all diagonals that started in b_i now start in d and all diagonals ending in b_i now end in c. Remove the diagonal α and identify the puncture with the vertex b_i. If there were two diagonals between the puncture and b_i, remove both and draw a diagonal from c to d. Leave all the other diagonals unchanged. We will see that this is a triangulation of the non-punctured $(n+2)$-polygon in the next lemma.

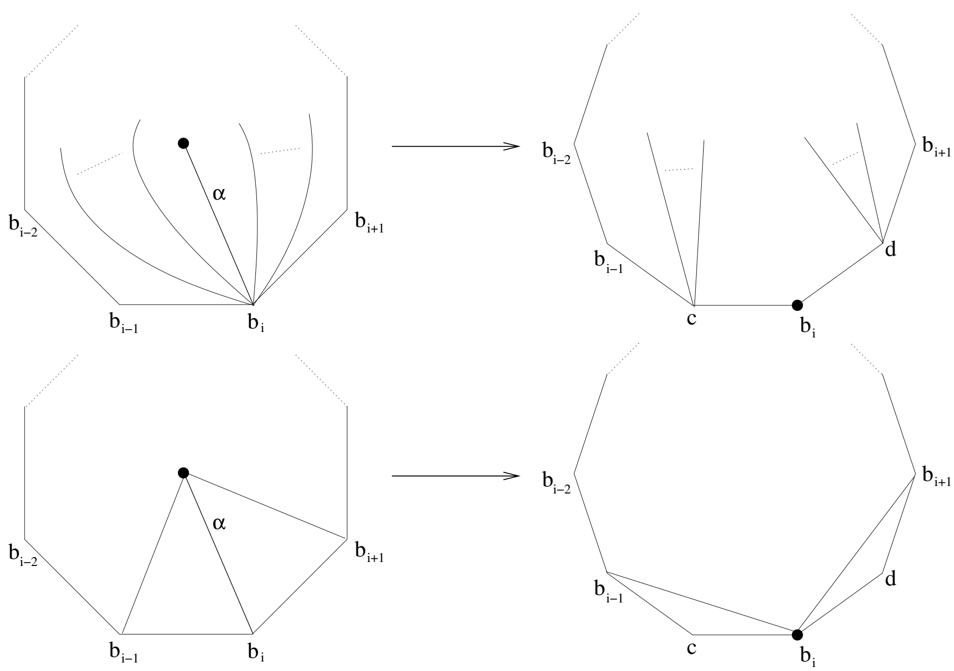

Figure 9: Factoring out a diagonal from the puncture to the border.

Recall that γ_n is the function from the set of all triangulations of the regular $(n+3)$-gon to the mutation class of A_n, defined in Section 2. We have the following.

Lemma 5.5 *Let Δ be a triangulation and $\epsilon_n(\Delta) = Q_\Delta$. If α is a diagonal between the puncture and the border, then the quiver Q_Δ/v_α obtained from Q_Δ by factoring out v_α is connected and of type A_{n-1}. Furthermore, we have that $\gamma_{n+2}(\Delta/\alpha) = Q_\Delta/v_\alpha$ when α is a diagonal between the puncture and a vertex on the border.*

Proof: It is clear that Δ/α has $n-1$ diagonals and that no diagonals cross. This means that the new triangulation is a triangulation of the $(n+2)$ polygon without a puncture. We want to show that all triangles are preserved by factoring out a diagonal as described above and hence we will have that $\gamma_{n+2}(\Delta/\alpha) = Q_\Delta/v_\alpha$, and that Q_Δ/v_α is of type A_{n-1}.

First suppose that there is only one diagonal from the puncture to the vertex b_i (see Figure 9). Then it is easy to see that all triangles are preserved. Next, suppose there are two diagonals α and β from the puncture to b_i. In this case we add a new diagonal β' between b_{i-1} and b_{i+1} and remove α and β. Then the diagonals bounding a common triangle with β before factoring out α will bound a common triangle with β' after factoring out α. ■

Summarizing, we get the following Proposition.

Proposition 5.6 *Let Δ be a triangulation and let $\epsilon_n(\Delta) = Q_\Delta$ be the corresponding quiver. Then $\epsilon_{n-1}(\Delta/\alpha) = Q_\Delta/v_\alpha$ is of type D_{n-1} if and only if the corresponding diagonal α is close to the border.*

Proof: From Lemma 5.3, we have that if α is close to the border, then Q_Δ/v_α is of type D_{n-1}. If α is not close to the border, we have by Lemma 5.4 and Lemma 5.5 that Q_Δ/v_α is either disconnected or of type A_{n-1}. ■

If Δ is a triangulation of $\mathcal{P}_n$, we want to add a diagonal α and a vertex on the polygon such that α is a diagonal close to the border and such that $\Delta \cup \alpha$ is a triangulation of $\mathcal{P}_{n+1}$. Consider any border edge m on $\mathcal{P}_n$. We consider the eight different cases for the triangle containing m, as shown in Figure 10. We can define the extension at m for each case. See Figure 7 for the corresponding extensions.

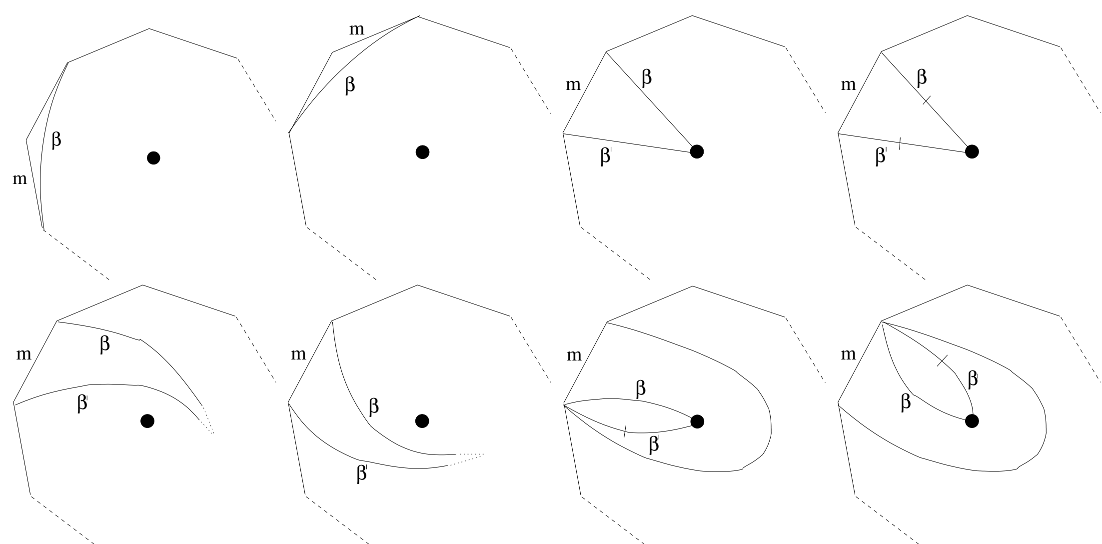

Figure 10: Extension at m.

For a given diagonal β, there are at most three ways to extend the polygon with a diagonal α such that α is adjacent to β. These extensions give non-isomorphic quivers, except when the triangulation is S_n.

Combining Lemma 5.1 and Lemma 5.3, we get that for a quiver Q which is not Q_n, there always exist a vertex v such that Q' obtained from Q by factoring out v is connected and a quiver of a cluster-tilted algebra of type D. Furthermore, such a vertex must correspond to a diagonal close to the border in any triangulation Δ such that $\epsilon_n(\Delta) = Q_\Delta$.

For a triangulation Δ of $\mathcal{P}_n$, let us denote by $\Delta(i)$ the triangulation obtained from Δ by rotating i steps in the clockwise direction. Also denote by Δ^{-1} the triangulation obtained from Δ by inverting all tags. We define an equivalence relation on $\mathcal{T}_n$ where we let $\Delta \sim \Delta(i)$ for all i and $\Delta^{-1} \sim \Delta$. We define a new function $\widetilde{\epsilon}_n : (\mathcal{T}_n/\sim) \to \mathcal{M}_n$ induced from ϵ_n. This is well-defined, and since ϵ_n is a surjection, we also have that $\widetilde{\epsilon}_n$ is a surjection. We actually have the following.

Theorem 5.7 *The function $\widetilde{\epsilon}_n : (\mathcal{T}_n/\sim) \to \mathcal{M}_n$ is bijective for all $n \geq 5$.*

Proof: We already know that $\widetilde{\epsilon}_n$ is surjective.

Suppose $\widetilde{\epsilon}_n(\Delta) = \widetilde{\epsilon}_n(\Delta')$. We want to show that $\Delta = \Delta'$ in $(\mathcal{T}_n/\sim)$ using induction.

It is straightforward to check that $\widetilde{\epsilon}_5 : (\mathcal{T}_5/\sim) \to \mathcal{M}_5$ is injective. Suppose $\widetilde{\epsilon}_{n-1} : (\mathcal{T}_{n-1}/\sim) \to \mathcal{M}_{n-1}$ is injective. Let α be a diagonal close to the border in Δ, with image v_α in Q, where Q is a representative for $\widetilde{\epsilon}_n(\Delta)$. Then the diagonal α' in Δ' corresponding to v_α in Q is also close to the border by Proposition 5.6. We have $\widetilde{\epsilon}_{n-1}(\Delta/\alpha) = \widetilde{\epsilon}_{n-1}(\Delta'/\alpha') = Q/v_\alpha$, and hence by hypothesis, $\Delta/\alpha = \Delta'/\alpha'$ in $(\mathcal{T}_n/\sim)$.

We can obtain Δ and Δ' from $\Delta/\alpha = \Delta'/\alpha'$ by extending the polygon at some border edge. Fix a diagonal β in Δ such that v_α and v_β are adjacent. This can be done since Q is connected. Let β' be the diagonal in Δ' corresponding to v_β. By the above there are at most three ways to extend Δ/α such that the new diagonal is adjacent to β. It is clear that these extensions will be mapped by $\widetilde{\epsilon}_n$ to non-isomorphic quivers. Also there are at most three ways to extend Δ'/α' such that the new diagonal is adjacent to β', and all these extensions are mapped to non-isomorphic quivers, thus $\Delta = \Delta'$ in $(\mathcal{T}_n/\sim)$. ∎

Corollary 5.8 *The number $d(n)$ of elements in the mutation class of any quiver of type D_n is equal to the number of triangulations of the punctured regular n polygon up to rotations and inverting all tags.*

6 Equivalences on the cluster category in the D_n case

Since the Auslander-Reiten translation τ is an equivalence, it is clear that if T is a cluster-tilting object in $\mathcal{C}$, then the cluster-tilted algebras $\mathrm{End}_\mathcal{C}(T)^{\mathrm{op}}$ and $\mathrm{End}_\mathcal{C}(\tau T)^{\mathrm{op}}$ are isomorphic. We know that τ corresponds to rotation of diagonals. In [T] it was proven that if T and T' are cluster-tilting objects in $\mathcal{C}$, then the cluster-tilted algebras $\mathrm{End}_\mathcal{C}(T)^{\mathrm{op}}$ and $\mathrm{End}_\mathcal{C}(T')^{\mathrm{op}}$ are isomorphic if and only if $T' = \tau^i T$ for an $i \in \mathbb{Z}$ in the A_n case.

Let α be a diagonal (indecomposable object in $\mathcal{C}$). If α is a diagonal between the puncture and the border, let α^{-1} denote the diagonal α with inverted tag. We define

$$\mu\alpha = \begin{cases} \alpha^{-1} & \text{if } \alpha \text{ is a diagonal between the puncture and the border,} \\ \alpha & \text{otherwise.} \end{cases}$$

If α is not a diagonal between the puncture and the border, then clearly $\tau^n\alpha = \alpha$. Now, let α be a diagonal between the puncture and the border. Suppose n is even. Then it is clear from combinatorial reasons that $\tau^n\alpha = \alpha$ and that $\tau^i\alpha \neq \alpha^{-1}$ for any i. If n is odd, then $\tau^n\alpha = \alpha^{-1}$ and hence $\tau^n = \mu$. See Figure 11 for an example of an AR-quiver in the D_5 case.

Theorem 6.1 *Let T and T' be cluster-tilting objects in $\mathcal{C}$. Then the cluster-tilted algebras* $\mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ *and* $\mathrm{End}_{\mathcal{C}}(T')^{\mathrm{op}}$ *are isomorphic if and only if $T' = \mu^i\tau^jT$ $i, j \in \mathbb{Z}$.*

Proof: Let Δ be a triangulation corresponding to T and Δ' a triangulation corresponding to T'. If $T' \not\simeq \tau^iT$ for any i, then Δ' is not obtained from Δ by a rotation. If $T' \not\simeq \mu T$, then $\Delta \neq \Delta^{-1}$. It then follows from Theorem 5.7 that $\mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ is not isomorphic to $\mathrm{End}_{\mathcal{C}}(T')^{\mathrm{op}}$. ■

It is clear that μ is an equivalence on the cluster category, since $\mu^2 = \mathrm{id}$.

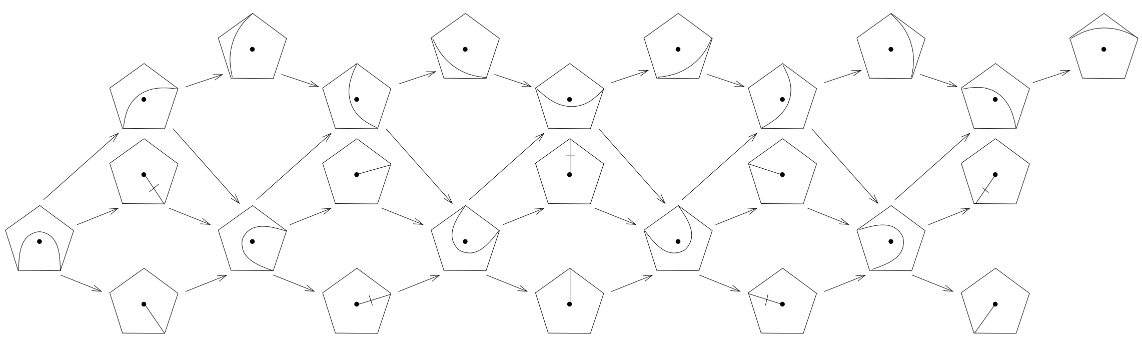

Figure 11: AR quiver for the cluster category in the D_5 case.

7 The number of triangulations of punctured polygons

In this section we want to find an explicit formula for the number of triangulations of punctured polygons up to rotation and tags. Let $\mathcal{B}_n$ be the set of equivalence classes of trees such that

- any full subtree not including the root is binary and every inner node has either two or no children,

- there are exactly n leaves and
- two trees are equivalent if one can be obtained from the other by rotating at the root.

As before, let $\mathcal{T}_5/\sim$ be the set of triangulations of the punctured n-gon, where rotations and inverting tags gives equivalent triangulations. In this section we will draw certain tagged edges as loops. If there are two diagonals between the puncture and the same vertex, we will draw one diagonal as a loop. See Figure 12.

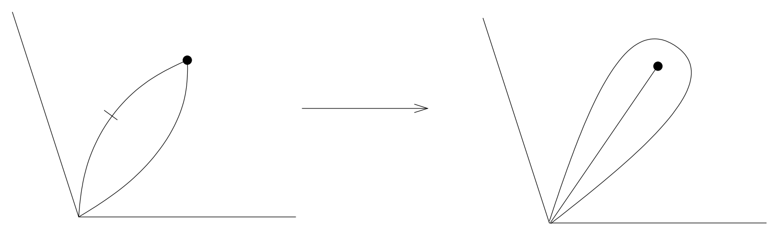

Figure 12: Drawing tagged edges as loops

We define a function $\sigma : \mathcal{T}_n/\sim \to \mathcal{B}_n$ by assigning to a triangulation a tree. Let Δ be a triangulation. We let $\sigma(\Delta)$ be the tree obtained in the following way. Draw an edge between two triangles E and E' if they are adjacent and their common diagonal is not a diagonal between the puncture and the border. Note that a loop in this case is not an edge between the puncture and the border. When a triangle E contains one or two border edges, also draw one or two edges from the vertex to the outside of the polygon, crossing the border edges. These will be the leaf edges. Then identify the vertices adjacent to the puncture to be the root in the tree. See Figure 13 for some examples.

It is clear that σ is a well-defined function. Our aim is to show that σ is a bijection.

Let the tree R_n be the tree consisting of exactly n edges from the root, as shown in Figure 14. Note that this is the unique tree which is the image of the triangulation S_n.

Now we want to define a function $\lambda : \mathcal{B}_n \to \mathcal{T}_n/\sim$ and we will see that this is the inverse of σ.

Given a tree T with n leaves, we will here describe $\lambda(T)$. We know that an inner edge of a tree (an edge not going to a leaf) corresponds to a diagonal α not going between the puncture and the border.

Suppose α is an inner edge of T. Let T' be the full subtree of T with root ending in α. If T' has n leaves, we draw a segment of a polygon consisting of n border edges. See Figure 15. Suppose the subtree to the left of the root in T' has $r \geq 2$ leaves. Then we draw a diagonal β from v_1 to v_{r+1}. If $r+1 \neq n$ we draw a diagonal δ from v_{r+1} to v_{n+1}. We can continue like this with β and δ until we made a complete triangulation of the segment of the polygon, by induction.

Now, suppose T has k edges from the root, namely $t_1, t_2, ..., t_k$. Suppose the full subtree with root ending in t_i has d_i leaves. Then $\sum_i d_i = n$. Draw a punctured polygon with n

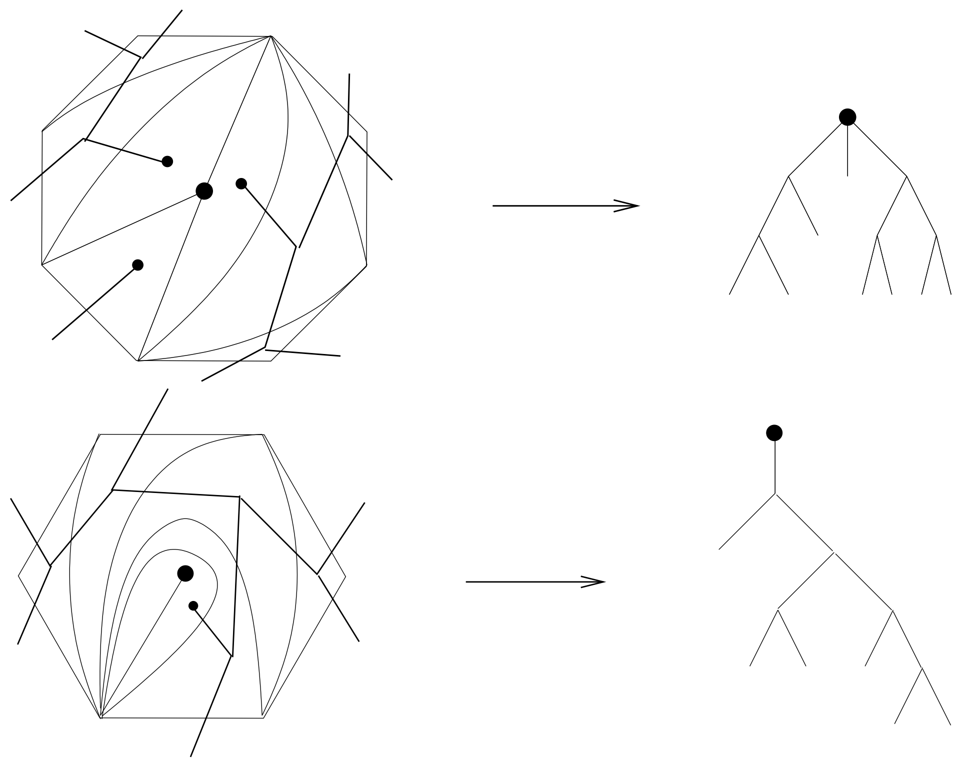

Figure 13: Triangulation and corresponding tree.

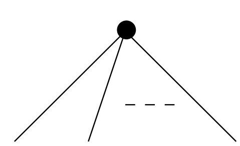

Figure 14: The tree R_n consisting of exactly n edges from the root.

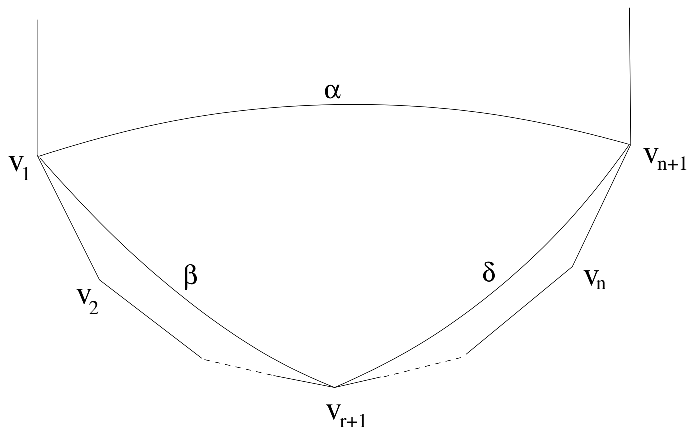

Figure 15

border edges and draw k diagonals between the puncture and vertices on the border such that each segment has d_i border edges in anticlockwise direction.

For each segment defined by t_i, apply the procedure described above to obtain a triangulation of the segment. See Figure 16.

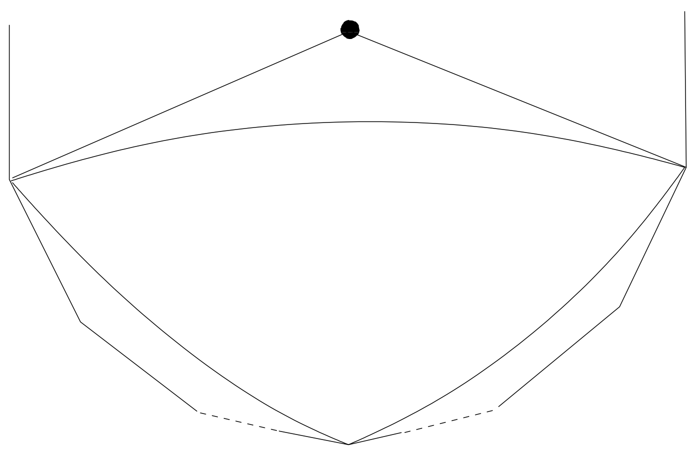

Figure 16

It is clear from the construction that λ is the inverse of σ, so we have the following.

Theorem 7.1 *$\sigma : \mathcal{T}_n \to \mathcal{B}_n$ is a bijection.*

The number of rooted planar trees with $n+1$ nodes where rotating at the root gives equivalent trees, is given by the formula

$$\sum_{d|n} \phi(n/d) \binom{2d}{d} / (2n)$$

n	$d(n)$
3	4
4	6
5	26
6	80
7	246

n	$d(n)$
8	810
9	2704
10	9252
11	32066
12	112720

Table 1: Some values of $d(n)$.

where ϕ is the Euler function (see [I] and the references given there and exercise 7.112 b in [St]).

The number of planar trees with $n+1$ nodes and the number of planar binary trees with $n+1$ leaves are both given by the n'th Catalan number. It follows that the number of elements in $\mathcal{B}_n$ is given by the above formula.

Corollary 7.2 *The number $d(n)$ of elements in the mutation class of any quiver of type D_n is given by:*

$$d(n) = \begin{cases} \sum_{d|n} \phi(n/d)\binom{2d}{d}/(2n) & \text{if } n \geq 5, \\ 6 & \text{if } n = 4, \end{cases}$$

where ϕ is the Euler function.

We proved this for $n \geq 5$ and for $n = 4$ the number is 6. See Figure 1 for all quivers in the mutation class of D_4. See Table 1 for some values of $d(n)$.

8 Mutation of trees

We want to define a mutation operation on the elements in $\mathcal{B}$, and we want this to commute with mutation of triangulations. Mutating a triangulation at a given diagonal is defined as removing this diagonal and replacing it with another one to obtain a new triangulation. This can be done in one and only one way.

Let Δ be a triangulation in $\mathcal{T}_n$ and let $\sigma(\Delta) = T$ be the corresponding tree. An inner edge of T corresponds to a diagonal in Δ not going from the puncture to the border, since the edges crosses these diagonals when we construct T from Δ. However, when we construct T from Δ, no edges in T crosses a diagonal between the puncture and the border. To define mutation on T corresponding to mutating at a diagonal α between the puncture and the border, we instead define mutation at two adjacent edges from the root in T, namely the two edges from the root in T separated by α.

1. Let v_1 be an edge from the root in T. The mutation of T at v_1 is a new tree obtained in the following way. Remove the edge v_1. Identify the root of the full subtree of T ending in v_1 with the root in T. See the first picture in Figure 17.

2. Let x and y be two adjacent edges from the root of T. The mutation of T at x and y is a new tree obtained in the following way. Disconnect the full subtree of T containing x and y. Add an edge v_1 from the root and connect the subtree to the end of v_1. See the second picture in Figure 17.

3. Let v be an inner edge not going from the root or to a leaf. The mutation of T at v is a new tree obtained in the following way. Suppose v is an edge from the nodes r to t, going down in the tree. Let x be the other edge starting in r, and let y and z be the two edges starting in t. See the third and fourth picture in Figure 17. Suppose x goes to the left from r and v goes to the right, as in the third picture. Disconnect the full subtree with t as a root. Remove the edge v and identify r with t. Disconnect the full subtree T' containing x and y. Create a new vertex v' starting in r and identify the root of T' with the node ending in v'. See the third picture in Figure 17. If x goes to the right from r and v goes to the left, we define mutation at v in a similar way as shown in the fourth picture.

We claim that mutation of a tree as defined above commutes with mutation of triangulations. We leave the details of the proof to the reader.

Proposition 8.1 *Mutation of trees commutes with mutations of triangulations and quivers.*

Sketch of proof:
For mutation of type 3, we mutate at a diagonal not going between the puncture and the border, so we are in the situation shown in Figure 18. We see that mutation of trees commutes with mutation of triangulations.

For mutation of type 1 and 2 we have to consider the three cases shown in Figure 19. In these cases we also see that mutation as defined above commutes with mutation of triangulations. ■

Figure 20 and 21 shows the mutations of type 1 and 2 for both triangulations and trees in the D_5 case. Note that mutation of type 2 adds an edge from the root, or equivalently replaces a diagonal not between the puncture and the border with a diagonal between the puncture and the border. Mutation of type 1 is the opposite operation. This defines a tree of mutations as shown in Figure 20 and 21, where going down in the tree corresponds to mutatation of type 1 and going up in the tree corresponds to mutation of type 2. If we drew arrows for mutations of type 3, the arrows would go to trees (or triangulations) in the same level in the tree of mutations. It is easy to see that this holds in general for any n.

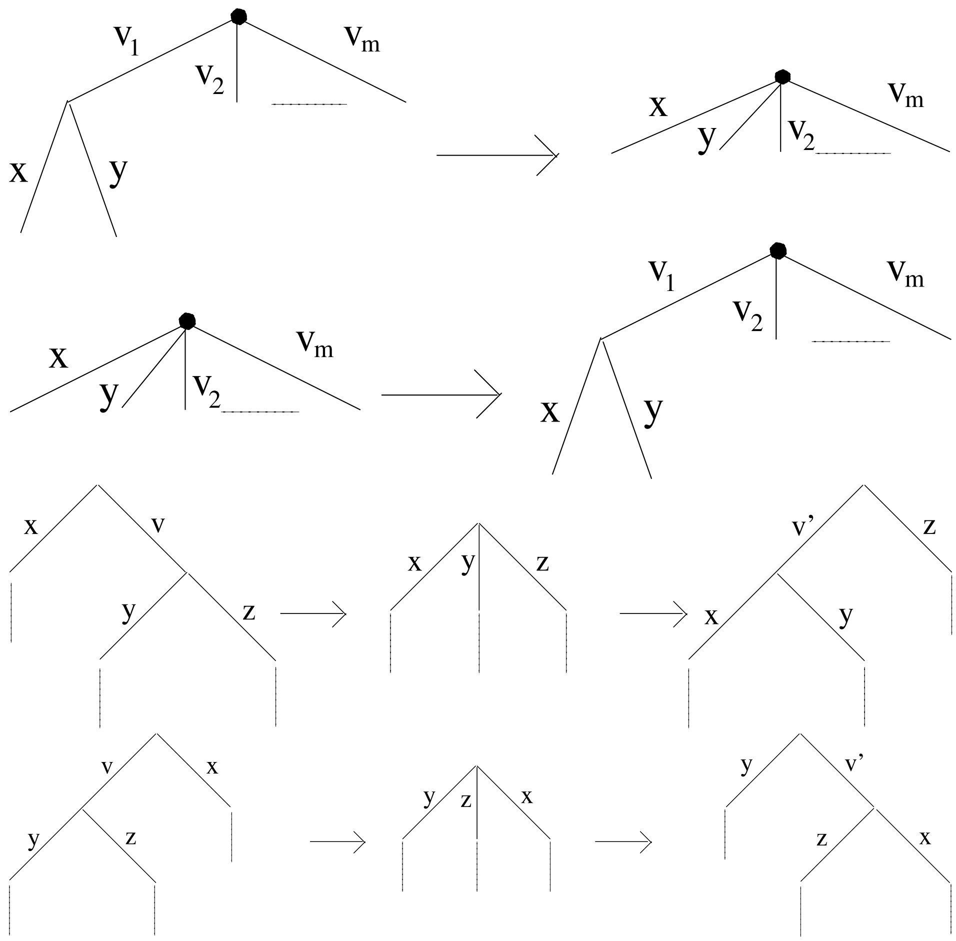

Figure 17: Mutation of a tree.

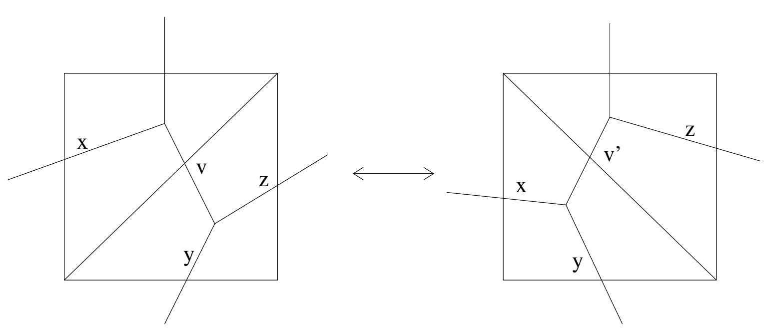

Figure 18: Mutation of triangulations and trees commute. See proof of Proposition 8.1.

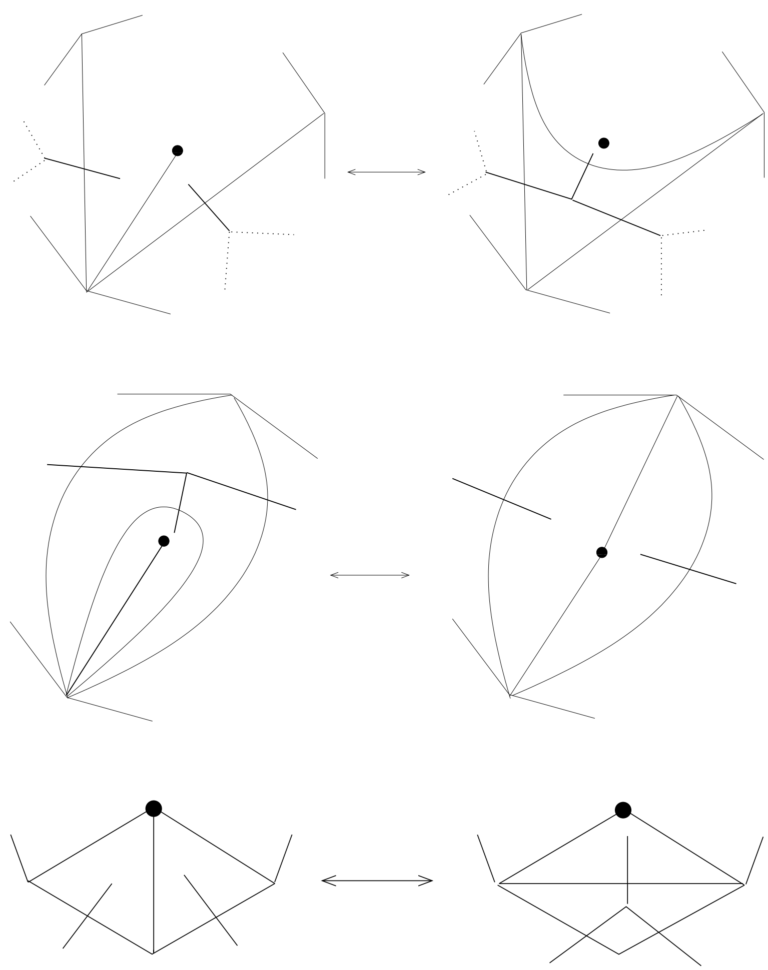

Figure 19: Mutation of triangulations and trees commutes. See proof of Proposition 8.1.

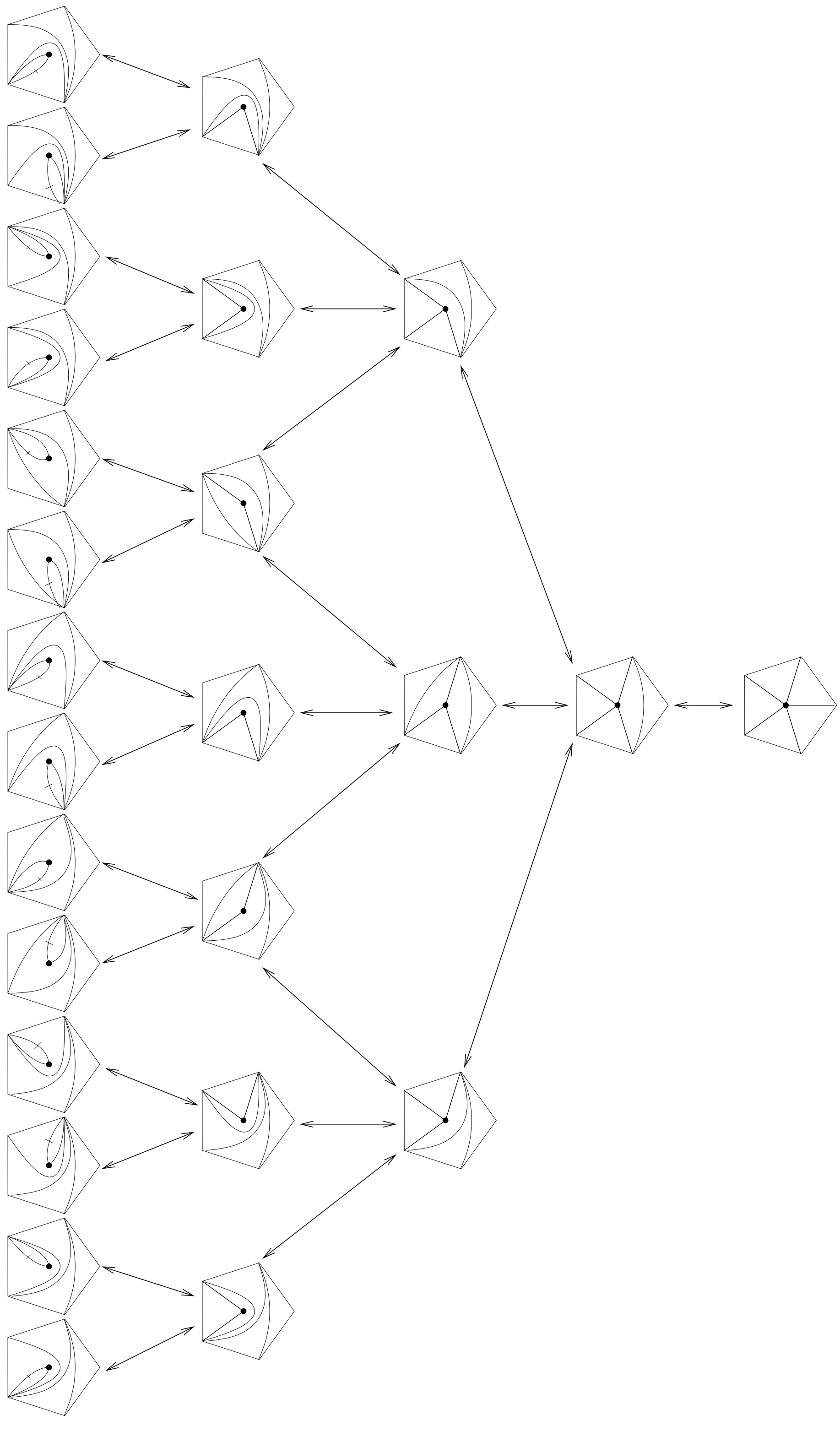

Figure 20: All triangulations of type D_5.

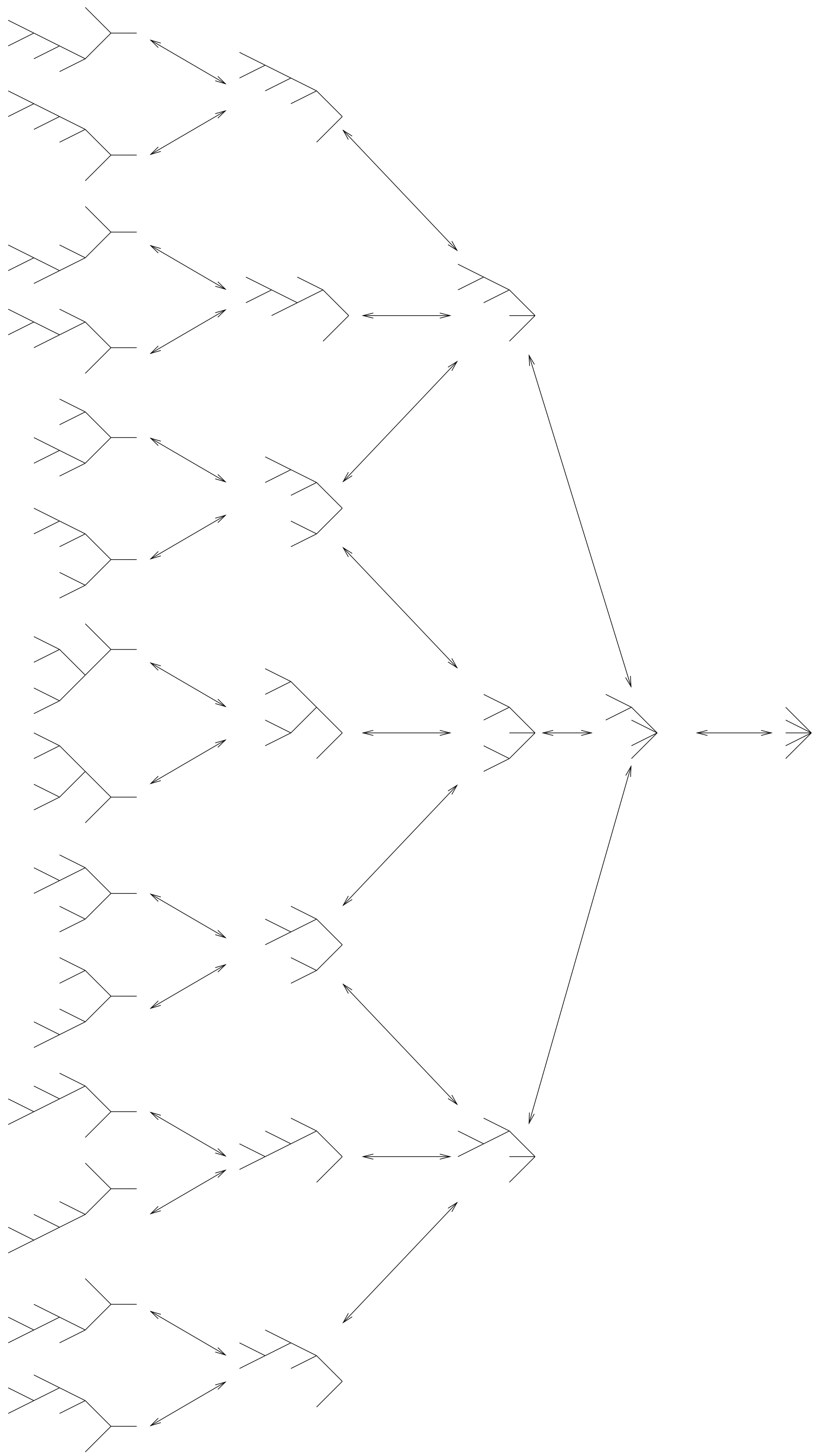

Figure 21: All trees of type D_5.

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