A Rainbow k-Matching in the Complete Graph with r Colors

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Abstract

An *r-edge-coloring* of a graph is an assignment of r colors to the edges of the graph. An *exactly r-edge-coloring* of a graph is an r-edge-coloring of the graph that uses all r colors. A matching of an edge-colored graph is called *rainbow matching*, if no two edges have the same color in the matching. In this paper, we prove that an exactly r-edge-colored complete graph of order n has a rainbow matching of size $k(\geq 2)$ if $r \geq max\{\binom{2k-3}{2}+2, \binom{k-2}{2}+(k-2)(n-k+2)+2\}$, $k \geq 2$, and $n \geq 2k+1$. The bound on r is best possible.

Keyword(s): edge-coloring, matching, complete graph, anti-Ramsey, rainbow, heterochromatic, totally multicolored

1 Introduction

We consider finite, undirected, simple graphs G with the vertex set $V(G)$ and the edge set $E(G)$. An *r-edge-coloring* of a graph G is a mapping $color : E(G) \to C$, where C is

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a set of r colors. An *exactly r-edge-coloring* of a graph is an r-edge-coloring of the graph such that all r colors is used, namely, every color appears in the r-edge-colored graph. A subgraph H of an edge-colored graph is said to be *rainbow* (or *heterochromatic*, or *totally multicolored*) if no two edges of H have the same color, that is, if $color(e) \neq color(f)$ for any two distinct edges e and f of H. A matching of size k is called a *k-matching*. Let P_k and C_k are the path and the cycle of order k, respectively.

We begin with a brief introduction of the background concerning anti-Ramsey numbers. Let $h_p(n)$ be the minimum number of colors r such that every exactly r-edge-colored complete graph K_n contains a rainbow K_p. The pioneering paper [2] by Erdős, Simonovits and Sós proved the existence of a number $n_0(p)$ such that $h_p(n) = t_{p-1}(n)+2$ for $n > n_0(p)$, where $t_{p-1}(n)$ is the Turán number. Montellano-Ballesteros and Neumann-Lara [5] proved that for all integers n and p such that $3 \leq p < n$, the corresponding anti-Ramsey function is such. Along a slightly different line, Eroh [3, 4] studied rainbow Ramsey numbers for matchings, which is a certain generalization of the Ramsey and anti-Ramsey numbers. For two graphs G_1 and G_2, let $RM(G_1, G_2)$ be the minimum integer n such that any edge-coloring of K_n contains either a monochromatic G_1 or a rainbow G_2. In [3], the case where G_1 is a star and G_2 is a matching is discussed. Also, in [4], the case where each G_i with $i = 1, 2$ is a k_i-matching is treated. There, in particular, it is conjectured that $RM(G_1, G_2) = k_2(k_1 - 1) + 2$, and a proof in the case where $k_2 \leq \frac{3}{2}(k_1 - 1)$ is given.

In this paper, we study anti-Ramsey numbers for k-matchings. Given an exactly r-edge-colored complete graph of order n, is there a rainbow k-matching? Since the case $k = 0$ and $k = 1$ is trivial, we assume $k \geq 2$. For example, if $n = 7$ and $r \geq 2$ then we can find easily a rainbow 2-matching, but we may not find a rainbow 3-matching. Generally, if $n \geq 2k + 1$ then the following colorings do not allow a rainbow k-matching to exist. (See Figure 1.)

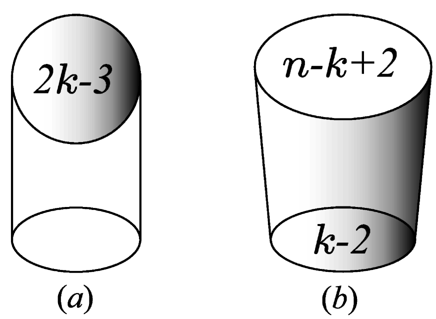

Figure 1: Colorings without rainbow k-matchings.

In the coloring (a) of Figure 1, a complete subgraph K_{2k-3} of G is rainbow and the other edges are colored with exactly one color, namely, monochromatic. In the coloring (b), a complete subgraph $K_{n-(k-2)}$ of G is monochromatic and the other edges are rainbow. In each coloring, it is clear that there is no rainbow k-matching. However, if there are more colors than in these colorings, is there a rainbow k-matching ? Schiermeyer [6] solved this problem affirmatively for $k \geq 2$ and $n \geq 3k + 3$. In this paper, we solve this problem for $n \geq 2k+1$.

Theorem 1.1. *An exactly r-edge-colored complete graph of order n has a rainbow k-matching, if $r \geq max\{\binom{2k-3}{2}+2, \binom{k-2}{2}+(k-2)(n-k+2)+2\}$, $k \geq 2$, and $n \geq 2k+1$.*

If $n = 2k$ then there exists an exactly r-edge-coloring with $r = \binom{2k-3}{2}+2$ for $k \geq 3$ or $r = \binom{2k-3}{2}+3$ for $k = 2$ such that there is no rainbow k-matching. (See Figure 2.)

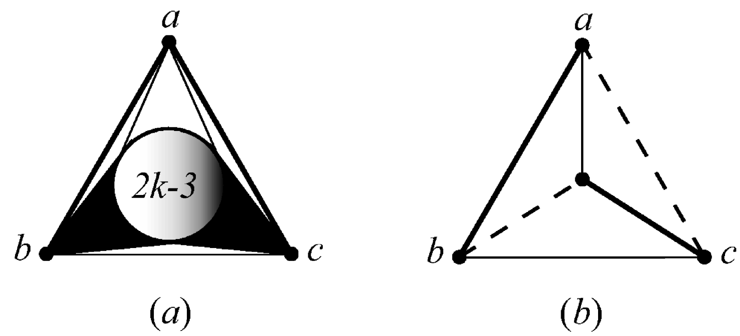

Figure 2: $(\binom{2k-3}{2}+2$ or $+3)$-Colorings without rainbow k-matchings.

In the coloring (a) of Figure 2, a complete subgraph K_{2k-3} of G is rainbow and the other edges are colored with exactly two colors red and blue, so that ab, ac and the edges between $\{b,c\}$ and $G-\{a,b,c\}$ are red, and bc and the edges between a and $G-\{a,b,c\}$ are blue. Thus, the number of colors is $\binom{2k-3}{2}+2$, but there is no rainbow 1-factor. In the coloring (b) of Figure 2, K_4 is colored with three colors. Then, $k=2$, $\binom{2k-3}{2}+3=3$, and $\binom{k-2}{2}+(k-2)(n-k+2)+2=2$, but any 1-factor is monochromatic. We propose the following conjecture.

Conjecture 1.2. *An exactly r-edge-colored complete graph of order $2k(\geq 6)$ has a rainbow 1-factor, if $r \geq \max\{\binom{2k-3}{2}+3, \binom{k-2}{2}+k^2-2\}$.*

We have proved that this conjecture holds for $3 \leq k \leq 4$ in our preprint (we can send the proof upon request), but for $k \geq 5$ this is still open.

The concept of rainbow matchings is linked with the relationship between the maximum number of edges and the edge independence number in graphs. In 1959, Erdős and Gallai [1] proved the following theorem.

Theorem 1.3 ([1]). *Let G be a graph of order $n \geq 2k+1$ with edge independence number at most k. Then $|E(G)| \leq \max\{\binom{2k+1}{2}, \binom{k}{2}+k(n-k)\}$.*

In fact, Theorem 1.1 nearly implies Theorem 1.3, that is, the following corollary is obtained by Theorem 1.1.

Corollary 1.4. *If $n \geq 2k+5$, then the assertion of Theorem 1.3 follows from Theorem 1.1.*

Proof. Color the edges of the complete graph K_n of order $n \geq 2k+5 = 2(k+2)+1$, so that, a spanning subgraph H isomorphic to G is rainbow and the other edges are colored with one new color. Then the number of colors r is $|E(H)|+1 = |E(G)|+1$. Since the edge independence number of H is at most k, H has no rainbow $(k+1)$-matching. Thus, K_n has no rainbow $(k+2)$-matching. Hence, by Theorem 1.1, $r \leq max\{\binom{2(k+2)-3}{2} + 1, \binom{(k+2)-2}{2} + ((k+2)-2)(n-(k+2)+2)+1\} = max\{\binom{2k+1}{2}+1, \binom{k}{2}+k(n-k)+1\}$. Therefore, $|E(G)| = r-1 \leq max\{\binom{2k+1}{2}, \binom{k}{2}+k(n-k)\}$. ■

In the next section, we give the proof of Theorem 1.1. In the rest of this section, we introduce some notation for the proof of the theorem. For a graph G and a vertex subset M of $V(G)$, let $G[M]$ denote the induced subgraph by M. For an element x of a set S, we denote $S-\{x\}$ by $S-x$. For a matching M and edges $e_1,\dots,e_k,f_1,\dots,f_l$, we denote $(M-\{e_1,\dots,e_k\})\cup\{f_1,\dots,f_l\}$ by $M-e_1-\cdots-e_k+f_1+\cdots+f_l$. We often denote an edge $e=\{x,y\}$ by xy or yx. For an edge-colored graph G and an edge set $E \subseteq E(G)$, we define $color(E) = \{color(e) \mid e \in E\}$.

2 Proof of Theorem 1.1

Proof. Let G be an exactly r-edge-colored complete graph of order $n \geq 2k+1$ with no rainbow k-matchings. We may assume that r is chosen as large as possible under the above assumption. To prove the theorem, it suffices to show that

$$r < max\{\binom{2k-3}{2}+2, \binom{k-2}{2}+(k-2)(n-k+2)+2\}.$$

We begin with the following basic Claim.

Claim 1. *G has a rainbow $(k-1)$-matching.*

Proof. We may assume that G is not rainbow, because the complete graph of order at least $2k$ has a k-matching. Hence, there are two edges e,f such that $color(e) = color(f)$. Change the color of e into the $(r+1)$-th new color. Then, by the maximality of r, there is a rainbow k-matching M_k. Therefore, $M_k - e$ is a desired rainbow matching of G, because $|M_k - e| \geq k-1$. ■

Let $M = \{e_1, e_2, \dots, e_{k-1}\}$ be a rainbow $(k-1)$-matching of G. Let x_i and y_i be the end vertices of e_i, namely $e_i = x_i y_i$. Remove these vertices x_i and y_i, and let H be the resulting graph, namely $H = G - \bigcup_{1\leq i\leq k-1}\{x_i,y_i\}$. Since $n \geq 2k+1$, we have $|V(H)| \geq 3$. Hence $E(H) \neq \emptyset$.

Claim 2. *$color(E(H)) \subseteq color(M)$.*

Proof. If $color(E(H)) \not\subseteq color(M)$, then we have a rainbow k-matching $M+e$ of G where e is an edge of H with $color(e) \in color(E(H)) - color(M)$, which is a contradiction. ■

Without loss of generality, we may assume $color(E(H)) = \{color(e_1), color(e_2), \ldots, color(e_p)\}$ for some positive integer $p \le k-1$. Since $E(H) \neq \emptyset$, note that $1 \le p$. Let $M_1 = \{e_1, e_2, \ldots, e_p\}$ and $M_2 = M - M_1$. (See Figure 3.)

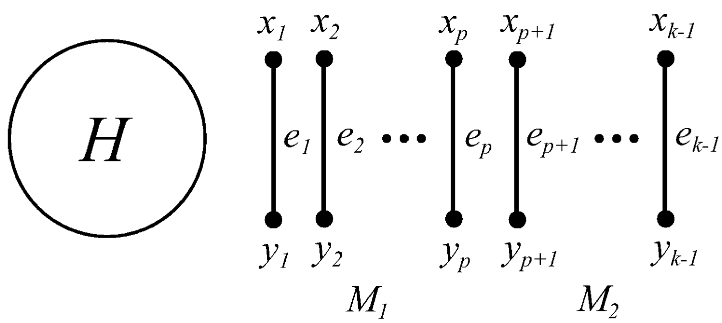

Figure 3: H and $M = M_1 \cup M_2$.

Let G' be a rainbow exactly r-edge-colored spanning subgraph of G that contains M. Since G' is rainbow and G' contains M, note that $E(G') \cap E(H) = \emptyset$ (i.e., H induces isolated vertices in G'). Here, we would like to count the number of colors in G. It is enough to count the number of edges of G' because $|color(E(G))| = |color(E(G'))| = |E(G')|$. Below, we consider only G' and the edges of H. Here, we give some notation. For two disjoint vertex sets A and B, we define $E'(A,B) = \{ab \in E(G') \mid a \in A, b \in B\}$. In the rest of the proof, for an edge $e = ab$, ab is often regarded as its vertex set $\{a, b\}$ when there is no fear of confusion.

Claim 3. *For any two distinct edges $e_i \in M_1$ and $e_j \in M$, $|E'(e_i, e_j)| \le 2$.*

Proof. By the definition of M_1, there exists an edge $f_1 \in E(H)$ such that $color(f_1) = color(e_i)$. If $|E'(e_i, e_j)| \ge 3$ then there are two independent edges f_2 and f_3 in $E'(e_i, e_j)$. Since G' contains M and G' is rainbow, $color(f_2), color(f_3) \notin color(M)$ and $color(f_2) \neq color(f_3)$. Since $color(f_1) = color(e_i)$, $color(f_1) \neq color(f_2)$ and $color(f_1) \neq color(f_3)$. Hence, we have a rainbow k-matching $M - e_i - e_j + f_1 + f_2 + f_3$, which is a contradiction. ■

Claim 4. *For any edge $e_i \in M_1$, let g_i be an edge in $E(H)$ such that $color(e_i) = color(g_i)$. Then $E'(e_i, V(H)) = E'(e_i, g_i)$ holds.*

Proof. Suppose that for some edge $f_1 \in E(H)$ with $color(e_i) = color(f_1)$ and for some edge $f_2 \in E'(e_i, V(H))$, these edges f_1, f_2 are independent. (See Figure 4.) By the definition of G', $color(f_2) \notin color(M)$. Thus, we have a rainbow k-matching $M - e_i + f_1 + f_2$, which is a contradiction. From this observation, the claim follows. ■

Claim 5. *If $E'(e_i, V(H)) \neq \emptyset$ for an edge $e_i \in M_1$, then the color of e_i induces a star in the graph H.*

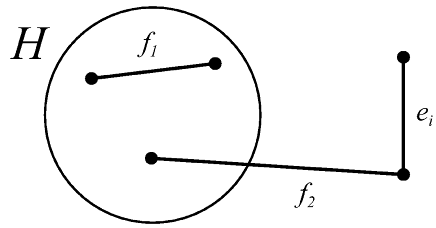

Figure 4: f_1, f_2 are independent.

Proof. Let $ab \in E'(e_i, V(H))$ such that $b \in V(H)$. By Claim 4, all the edges of H which have $color(e_i)$ in common are adjacent to the vertex b. Hence, the color of e_i induces a star with the center b in the graph H. ∎

Claim 6. *If H has a rainbow 2-matching f_1 and f_2 then $E'(e_i, e_j) = \emptyset$ for some edges $e_i, e_j \in M_1$ such that $color(f_1) = color(e_i)$ and $color(f_2) = color(e_j)$.*

Proof. By Claim 2, there are some edges $e_i, e_j \in M_1$ such that $color(f_1) = color(e_i)$ and $color(f_2) = color(e_j)$. Suppose that $E'(e_i, e_j) \neq \emptyset$. Let $f_3 \in E'(e_i, e_j)$. Then we have a rainbow k-matching $M - e_i - e_j + f_1 + f_2 + f_3$, which is a contradiction. ∎

Claim 7. *For any edge $e_i \in M_1$, $|E'(e_i, V(H))| \leq 2$.*

Proof. By the definition of M_1, there exists an edge $f_1 \in E(H)$ such that $color(f_1) = color(e_i)$. By Claim 4, $E'(e_i, V(H)) = E'(e_i, f_1)$. If $|E'(e_i, V(H))| \geq 3$, that is, $|E'(e_i, f_1)| \geq 3$, then there are two independent edges f_2 and f_3 in $E'(e_i, f_1)$ By the definition of G', $color(f_2)$, $color(f_3) \notin color(M)$ and $color(f_2) \neq color(f_3)$. Hence, we have a rainbow k-matching $M - e_i + f_2 + f_3$, which is a contradiction. ∎

Let $V_1 = \bigcup_{xy \in M_1} \{x, y\}$ and $V_2 = \bigcup_{xy \in M_2} \{x, y\}$. (See Figure 5.) We count the number of edges in $G' - V_2$.

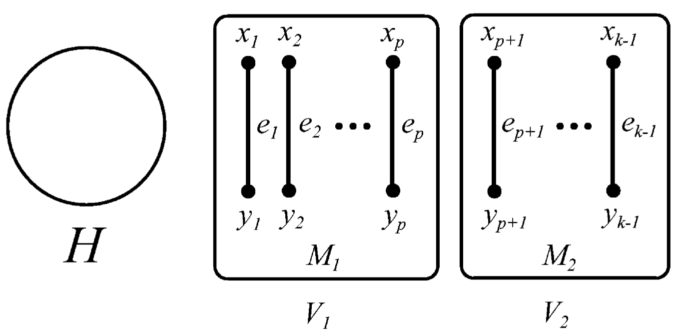

Figure 5: H and V_1, V_2.

Claim 8. $|E(G' - V_2)| \leq 2\binom{p}{2} + 3p - 2$.

Proof. By the definition of G', G' has no edges of H. Hence, by Claim 3 and Claim 7, we have $|E(G'-V_2)| = |E(G'[V_1])| + |E'(V_1, V(H))| \leq |M_1| + 2\binom{p}{2} + 2p = 2\binom{p}{2} + 3p$. Then, in view of the above inequality, it suffices to show that there exists some edge $e_i \in M_1$ such that $E'(e_i, e_j) = \emptyset$ for some edge $e_j \in M_1$ with $j \neq i$ or $E'(e_i, V(H)) = \emptyset$.

Suppose that for any edges $e_i, e_j \in M_1$, $E'(e_i, e_j) \neq \emptyset$ and $E'(e_i, V(H)) \neq \emptyset$. If $|V(H)| \geq 4$ then by Claim 5, H has a rainbow 2-matching. Thus, by Claim 6, there exist some edges $e_i, e_j \in M_1$ such that $E'(e_i, e_j) = \emptyset$, which is a contradiction. Therefore, we have $|V(H)| = 3$. Then, H is a triangle $\{a, b, c\}$. Hence it follows that $p = |M_1| = 1, 2,$ or 3 because $color(E(H)) = color(M_1)$.

If $p = 1$ then H is a monochromatic triangle. The color of the triangle H is $color(e_1)$. Since $E'(e_1, V(H)) \neq \emptyset$, the monochromatic triangle H contradicts Claim 5.

If $p = 2$ then we may assume that $color(ab) = color(e_1)$ and $color(ac) = color(bc) = color(e_2)$. By Claim 4, we may assume that $x_1a \in E'(e_1, V(H))$ and $x_2c \in E'(e_2, V(H))$. (See Figure 6.) If there exists an edge $f \in E'(y_1, e_2)$ then we have a rainbow k-matching

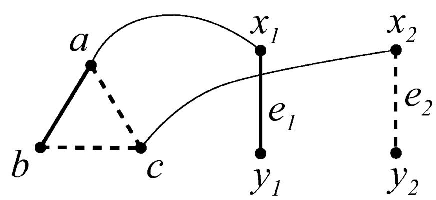

Figure 6: The case $p = 2$.

$M - e_1 - e_2 + ax_1 + bc + f$, which is a contradiction. Thus, $E'(y_1, e_2) = \emptyset$. If there exists an edge $f \in E'(y_2, e_1)$ then we have a rainbow k-matching $M - e_1 - e_2 + cx_2 + ab + f$, which is a contradiction. Thus, $E'(y_2, e_1) = \emptyset$. Hence, $E'(e_1, e_2) = \{x_1x_2\}$, which implies that we could decrease one edge in the above counting argument. Therefore, we may assume that $|E'(e_2, V(H))| = 2$. By Claim 4, $E'(e_2, V(H)) = \{cx_2, cy_2\}$. Then we have a rainbow k-matching $M - e_1 - e_2 + ab + cy_2 + x_1x_2$, which is a contradiction.

If $p = 3$ then we may assume that $color(ab) = color(e_1)$, $color(bc) = color(e_2)$, and $color(ac) = color(e_3)$. (See Figure 7.) Without loss of generality, we may as-

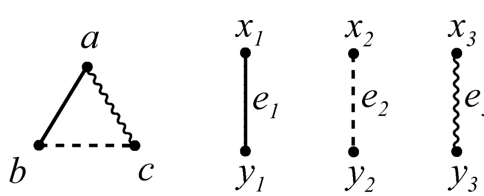

Figure 7: The case $p = 3$.

sume that $|E'(e_1, V(H))| = 2, |E'(e_2, V(H))| = 2, |E'(e_3, V(H))| \geq 1$, otherwise we can decrease two edges in the counting argument. By Claim 4, $E'(e_1, V(H)) = E'(e_1, ab)$,

$E'(e_2, V(H)) = E'(e_2, bc)$, and $E'(e_3, V(H)) = E'(e_3, ac)$. If the two edges in $E'(e_1, V(H))$ are independent, say, if $ax_1, by_1 \in E'(e_1, V(H))$, then we have a rainbow k-matching $M - e_1 + ax_1 + by_1$, which is a contradiction. Suppose that $ax_1, ay_1 \in E'(e_1, V(H))$. Without loss of generality, we may assume $x_1x_2 \in E'(e_1, e_2)$. Then we have a rainbow k-matching $M - e_1 - e_2 + x_1x_2 + ay_1 + bc$, which is a contradiction. Hence, we may assume that $ax_1, bx_1 \in E'(e_1, V(H))$. Similarly for e_2, we may assume that $bx_2, cx_2 \in E'(e_2, V(H))$. If there exists an edge $f \in E'(y_1, e_2)$ then we have a rainbow k-matching $M - e_1 - e_2 + ax_1 + bc + f$, which is a contradiction. Thus, $E'(y_1, e_2) = \emptyset$. If there exists an edge $f \in E'(y_2, e_1)$ then we have a rainbow k-matching $M - e_1 - e_2 + cx_2 + ab + f$, which is a contradiction. Thus, $E'(y_2, e_1) = \emptyset$. Hence, $E'(e_1, e_2) = \{x_1x_2\}$, which implies we can decrease one color in counting colors. Therefore, we may assume that $|E'(e_1, e_3)| = |E'(e_2, e_3)| = |E'(e_3, V(H)| = 2$. Similarly as for e_1, e_2, we may assume that $ax_3, cx_3 \in E'(e_3, V(H))$. If there exists an edge $f \in E'(y_2, e_3)$ then we have a rainbow k-matching $M - e_2 - e_3 + bx_2 + ac + f$, which is a contradiction. Thus, $E'(y_2, e_3) = \emptyset$, which implies $x_2x_3, x_2y_3 \in E'(e_2, e_3)$. Then we have a rainbow k-matching $M - e_2 - e_3 + ax_3 + bc + x_2y_3$, which is a contradiction. ■

Here, we classify the edges of M_2 as follows:

$$\begin{aligned} M_{2,1} &= \{e \in M_2 \mid |E'(e, V(H) \cup V_1)| \geq 2p + 1\}, \\ M_{2,2} &= M_2 - M_{2,1}. \end{aligned}$$

Note that by Claim 3, for any edge $e \in M_{2,1}$, $E'(e, V(H)) \neq \emptyset$. Let $V_{2,1} = \bigcup_{xy \in M_{2,1}} \{x, y\}$ and $V_{2,2} = \bigcup_{xy \in M_{2,2}} \{x, y\}$. (See Figure 8.)

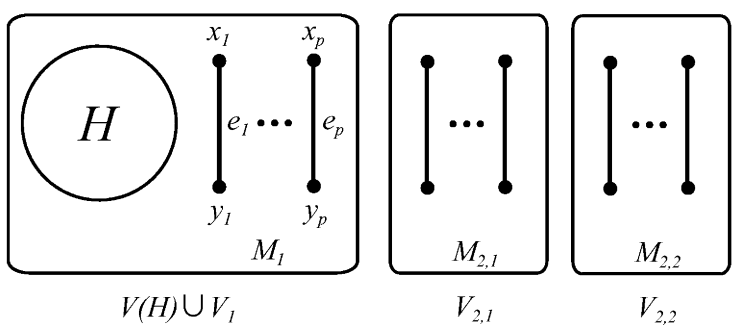

Figure 8: H, M_1, $M_{2,1}$, and $M_{2,2}$.

Claim 9. $|E'(V(H) \cup V_1, V_{2,1})| \leq (2p + |V(H)|)|M_{2,1}|$.

Proof. By Claim 3, for any edge $e \in M_{2,1}$, $|E'(e, V_1)| \leq 2p$. If there are two independent edges $f_1, f_2 \in E'(e, V(H))$ then we have a rainbow k-matching $M - e + f_1 + f_2$. Thus, $|E'(e, V(H))| \leq |V(H)|$ because $|V(H)| \neq 1$. Therefore $|E'(V_{2,1}, V_1 \cup V(H))| \leq (2p + |V(H)|)|M_{2,1}|$. ■

Claim 10. *Let e_i, e_j be two distinct edges in M_2. If both $E'(e_i, V(H))$ and $E'(e_j, V(H))$ are non-empty, and $|E'(e_i, e_j)| = 4$, then all edges in $E'(e_i, V(H))$ and $E'(e_j, V(H))$ are incident to exactly one vertex of $V(H)$.*

Proof. Suppose that for two distinct vertices $a, b \in V(H)$, $ax_i, bx_j \in E(G')$. Then we have a rainbow k-matching $M - e_i - e_j + ax_i + bx_j + y_iy_j$, which is a contradiction. ■

Claim 11. *Let e_i, e_j be two distinct edges in M_2. If $|E'(e_i, V_1)| \geq 2p-1$ and $E'(e_j, V(H)) \neq \emptyset$, then $|E'(e_i, e_j)| \leq 3$.*

Proof. Let $a \in V(H)$, and without loss of generality, we may assume $ax_j \in E'(e_j, V(H))$. Since $|V(H)| \geq 3$, $E(H - a) \neq \emptyset$. Let $bc \in E(H - a)$. Without loss of generality, we may assume that $color(bc) = color(e_1)$. (See Figure 9.) By Claim 3 and our assumption that

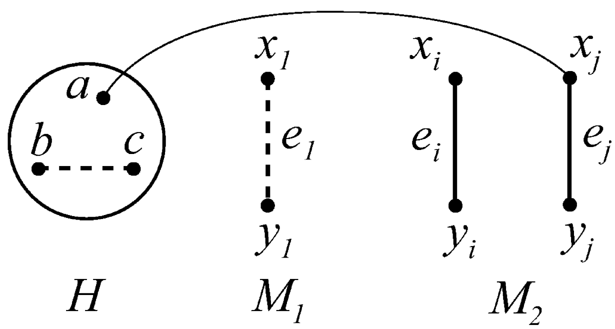

Figure 9: Proof of Claim 11.

$|E'(e_i, V_1)| \geq 2p - 1$, $E'(e_i, e) \neq \emptyset$ for any $e \in M_1$. Hence, without loss of generality, we may assume that $x_ix_1 \in E'(e_i, e_1)$. Suppose that $|E'(e_i, e_j)| = 4$. Then we have a rainbow k-matching $M - e_i - e_j - e_1 + bc + ax_j + x_ix_1 + y_iy_j$, which is a contradiction. ■

Claim 12. *For any two distinct edges $e_i, e_j \in M_{2,1}$, $|E'(e_i, e_j)| \leq 3$.*

Proof. By Claim 3 and the definition of $M_{2,1}$, $E'(e_i, V(H))$ and $E'(e_j, V(H))$ are not empty. Suppose that $|E'(e_i, e_j)| = 4$. By Claim 10, all edges in $E'(e_i, V(H))$ and $E'(e_j, V(H))$ are incident to exactly one vertex of $V(H)$. Thus, $|E'(e_i, V(H))| \leq 2$. Since $|E'(e_i, V_1 \cup V(H))| \geq 2p+1$ by the definition of $M_{2,1}$, we have $|E'(e_i, V_1)| \geq 2p - 1$. Hence by Claim 11, $|E'(e_i, e_j)| \leq 3$. ■

Claim 13. *For any edge $e_j \in M_{2,2}$, there is at most one edge $e \in M_{2,1}$ such that $|E'(e, e_j)| = 4$.*

Proof. Suppose that there are two distinct edges $e_s, e_t \in M_{2,1}$ such that $|E'(e_s, e_j)| = 4$ and $|E'(e_t, e_j)| = 4$. By Claim 3 and the definition of $M_{2,1}$, $E'(e_s, V(H))$ and $E'(e_t, V(H))$ are not empty. Let $x_sv \in E'(e_s, V(H))$ and $x_tv' \in E'(e_t, V(H))$. If $v \neq v'$ then we have a rainbow k-matching $M - e_s - e_t - e_j + vx_s + v'x_t + y_sx_j + y_ty_j$, which is a contradiction. Thus, $v = v'$ and $|E'(e_s, V(H))| \leq 2$. Then by the definition of $M_{2,1}$, we have $|E'(e_s, V_1)| \geq 2p-1$.

Hence, for any edge $e \in M_1$, $E'(e, e_s) \neq \emptyset$. Let $ab \in E(H - v)$. There is an edge $e \in M_1$, say e_1, such that $color(e_1) = color(ab)$. (See Figure 10.)

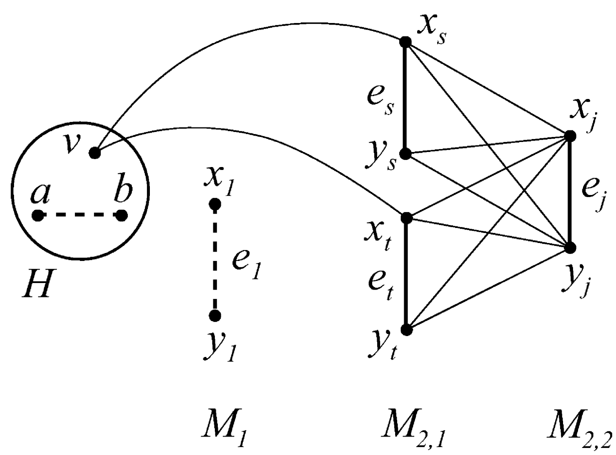

Figure 10: Proof of Claim 13.

Recall $E'(e_1, e_s) \neq \emptyset$. Utilizing this fact, we can easily find a rainbow k-matching. To see this, say, assume that $x_1x_s \in E'(e_1, e_s)$. Then we have a rainbow k-matching $M - e_s - e_t - e_j - e_1 + ab + vx_t + x_1x_s + y_sx_j + y_ty_j$, which is a contradiction. We can similarly get a contradiction in other cases. Thus the claim holds. ■

Claim 14. $|E'(V_{2,2}, V(H) \cup V_1 \cup V_{2,1})| \leq (2p + 3|M_{2,1}|)|M_{2,2}|$.

Proof. Let $e_j \in M_{2,2}$. By the definition of $M_{2,2}$, $|E'(e_j, V(H) \cup V_1))| \leq 2p$. If for any edge $e_i \in M_{2,1}$, $|E'(e_i, e_j)| \leq 3$ holds, then we have $|E'(e_j, V(H) \cup V_1 \cup V_{2,1})| \leq 2p + 3|M_{2,1}|$.

By Claim 13, there is at most one edge $e_i \in M_{2,1}$ such that $|E'(e_i, e_j)| = 4$. Suppose that there exists exactly one edge $e_i \in M_{2,1}$ such that $|E'(e_i, e_j)| = 4$. By Claim 3 and the definition of $M_{2,1}$, $E'(e_i, V(H)) \neq \emptyset$. Let $x_iv \in E'(e_i, V(H))$. Suppose $E'(e_j, V(H)) \neq \emptyset$. Then by Claim 10, all edges in $E'(e_i, V(H))$ and $E'(e_j, V(H))$ are incident to v. Thus, $|E'(e_i, V(H))| \leq 2$. By the definition of $M_{2,1}$, $|E'(e_i, V(H) \cup V_1)| \geq 2p + 1$. Hence $|E'(e_i, V_1)| \geq 2p - 1$. Therefore, by Claim 11, $|E'(e_i, e_j)| \leq 3$, which is a contradiction. Hence we may assume that $E'(e_j, V(H)) = \emptyset$. Then, by Claim 11, $|E'(e_j, V_1)| \leq 2p - 2$. Thus, $|E'(e_j, V(H) \cup V_1 \cup V_{2,1})| \leq 2p - 2 + 3(|M_{2,1}| - 1) + 4 = 2p + 3|M_{2,1}| - 1$.

Consequently, for any $e_j \in M_{2,2}$, we have $|E'(e_j, V(H) \cup V_1 \cup V_{2,1})| \leq 2p + 3|M_{2,1}|$. Hence, the Claim holds. ■

Recall that $r = |color(G)| = |color(G')| = |E(G')|$. We prove that

$$|E(G')| < max\{\binom{2k-3}{2} + 2, \binom{k-2}{2} + (k-2)(n-k+2) + 2\}$$

by the above Claims.

Now, we have

$$\begin{aligned}|E(G')| &= |E(G'[V(H)\cup V_1])| + |E'(V(H)\cup V_1, V_{2,1})| + |E(G'[V_{2,1}]))| \\ &\quad +|E'(V(H)\cup V_1\cup V_{2,1}, V_{2,2})| + |E(G'[V_{2,2}]))|.\end{aligned}$$

By Claim 8, $|E(G'[V(H)\cup V_1])| \leq 2\binom{p}{2} + 3p - 2$. By Claim 9, $|E'(V(H)\cup V_1, V_{2,1})| \leq (2p + |V(H)|)|M_{2,1}|$. By Claim 12, $|E(G'[V_{2,1}]))| \leq 3\binom{|M_{2,1}|}{2} + |M_{2,1}|$. By Claim 14, $|E'(V(H)\cup V_1\cup V_{2,1}, V_{2,2})| \leq (2p + 3|M_{2,1}|)|M_{2,2}|$. Also, the number of edges of $G'[V_{2,2}]$ is upper bounded by the number of edges of the complete graph on $V_{2,2}$. Since $|V_{2,2}| = 2|M_{2,2}|$, it follows that $|E(G'[V_{2,2}])| \leq \binom{2|M_{2,2}|}{2}$. Therefore, we have

$$\begin{aligned}|E(G')| &\leq 2\binom{p}{2} + 3p - 2 + (2p + |V(H)|)|M_{2,1}| + 3\binom{|M_{2,1}|}{2} + |M_{2,1}| \\ &\quad +(2p + 3|M_{2,1}|)|M_{2,2}| + \binom{2|M_{2,2}|}{2}.\end{aligned}$$

Let $q = |M_{2,1}|$ and $h = |V(H)|$, then $|M_{2,2}| = |M| - |M_1| - |M_{2,1}| = k - 1 - p - q$. Hence,

$$\begin{aligned}|E(G')| &\leq 2\binom{p}{2} + 3p - 2 + (2p + h)q + 3\binom{q}{2} + q \\ &\quad +(2p + 3q)(k - 1 - p - q) + \binom{2(k - 1 - p - q)}{2} \\ &= \frac{1}{2}q^2 + (h - k + p + \frac{3}{2})q + p^2 + (5 - 2k)p + 2k^2 - 5k + 1 \qquad (1) \\ &= \frac{1}{2}(q + h - k + p + \frac{3}{2})^2 - \frac{1}{2}(h - k + p + \frac{3}{2})^2 + p^2 + (5 - 2k)p + 2k^2 - 5k + 1 \\ &= \frac{1}{2}(q + h - k + p + \frac{3}{2})^2 + \frac{1}{2}p^2 + (\frac{7}{2} - h - k)p \\ &\quad -\frac{1}{2}h^2 + (k - \frac{3}{2})h + \frac{3}{2}k^2 - \frac{7}{2}k - \frac{1}{8}.\end{aligned}$$

Let $F(q, p)$ be a function with two parameter q and p as follows:

$$\begin{aligned}F(q, p) &= \frac{1}{2}(q + h - k + p + \frac{3}{2})^2 + \frac{1}{2}p^2 + (\frac{7}{2} - h - k)p \\ &\quad -\frac{1}{2}h^2 + (k - \frac{3}{2})h + \frac{3}{2}k^2 - \frac{7}{2}k - \frac{1}{8}. \qquad (2)\end{aligned}$$

For this function $F(q, p)$, we do quadratic optimization by fixing the parameter p, that is, we assume $F(q) = F(q, p)$ is a quadratic function with the parameter q. Note that $0 \leq q = |M_{2,1}| \leq |M_2| = |M| - |M_1| = k - 1 - p$. Then, this function is maximum when $q = 0$ or $q = k - 1 - p$. From (2), the corresponding value to the axis of symmetry of $F(q)$ is $q = -h + k - p - \frac{3}{2}$.

If the middle value of the range $0 \leq q \leq k - 1 - p$ is less than the corresponding value to the axis of symmetry, that is,

$$\frac{0 + k - 1 - p}{2} \leq -h + k - p - \frac{3}{2}, \qquad (3)$$

then $F(q)$ is maximum when $q = 0$. Since $h = |V(H)| \geq 3$, this together with (3) shows $p \leq k - 2 - 2h \leq k - 8$. Hence, if $1 \leq p \leq k - 8$ then $F(q)$ is maximum when $q = 0$. From (1),

$$F(0,p) = p^2 + (5-2k)p + 2k^2 - 5k + 1 = (p + \frac{5-2k}{2})^2 + k^2 - \frac{21}{4}.$$

This is a function with the parameter p whose axis of symmetry is $p = (2k-5)/2$. Since, the middle value of the range $1 \leq p \leq k-8$ is less than the corresponding value to the axis of symmetry, $F(0,p)$ is maximum when $p = 1$.

$$F(0,1) = 2k^2 - 7k + 7 < 2k^2 - 7k + 8 = \binom{2k-3}{2} + 2.$$

Hence $|E(G')| < max\{\binom{2k-3}{2} + 2, \binom{k-2}{2} + (k-2)(n-k+2) + 2\}$. Therefore, we may assume that $F(q,p)$ is maximum when $q = k - 1 - p$.

$$\begin{aligned} F(k-1-p,p) &= \frac{1}{2}p^2 + (\frac{7}{2} - h - k)p + (k-1)h + \frac{3}{2}k^2 - \frac{7}{2}k \\ &= \frac{1}{2}(p + \frac{7}{2} - h - k)^2 + \frac{1}{2}(\frac{7}{2} - h - k)^2 + (k-1)h + \frac{3}{2}k^2 - \frac{7}{2}k. \end{aligned}$$

This is a function with the parameter p whose axis of symmetry is $p = k + h - 7/2$. Since the middle value of the range $1 \leq p \leq k-1$ is less than the corresponding value to the axis of symmetry, $F(k-1-p,p)$ is maximum when $p = 1$.

$$F(k-1-p,1) = \frac{3}{2}k^2 - \frac{9}{2}k + 4 + (k-2)h.$$

Since $n = |V(H)| + 2|M| = h + 2(k-1)$, that is $h = n - 2k + 2$, we have

$$\begin{aligned} F(k-1-p,1) &= \frac{3}{2}k^2 - \frac{9}{2}k + 4 + (k-2)(n-2k+2) \\ &= \frac{1}{2}k^2 - \frac{5}{2}k + 4 + (k-2)(n-k+2) \\ &< \frac{1}{2}k^2 - \frac{5}{2}k + 5 + (k-2)(n-k+2) \\ &= \binom{k-2}{2} + (k-2)(n-k+2) + 2. \end{aligned}$$

Hence $|E(G')| < max\{\binom{2k-3}{2} + 2, \binom{k-2}{2} + (k-2)(n-k+2) + 2\}$. ■

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