Graceful Tree Conjecture for Infinite Trees

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Submitted: Sep 26, 2007; Accepted: May 19, 2009; Published: May 29, 2009

Abstract

One of the most famous open problems in graph theory is the Graceful Tree Conjecture, which states that every finite tree has a graceful labeling. In this paper, we define graceful labelings for countably infinite graphs, and state and verify a Graceful Tree Conjecture for countably infinite trees.

1 Introduction

The study of graph labeling was initiated by Rosa [9] in 1967. This involves labeling vertices or edges, or both, using integers subject to certain conditions. Ever since then, various kinds of graph labelings have been considered, and the most well-studied ones are graceful, magic and harmonious labelings. Not only interesting in its own right, graph labeling also finds a broad range of applications: the study of neofields, topological graph theory, coding theory, radio channel assignment, communication network addressing and database management. One should refer to the comprehensive survey by Gallian [6] for further details.

Rosa [9] considered the β-valuation which is commonly known as graceful labeling. A graceful labeling of a graph G with n edges is an injective function $f : V(G) \to \{0, 1, \ldots, n\}$ such that when each edge $xy \in E(G)$ is assigned the edge label, $|f(x) - f(y)|$, all the edge labels are distinct. A graph is graceful if it admits a graceful labeling. Graceful labeling was originally introduced to attack **Ringel's Conjecture** which says that a complete graph of order $2n + 1$ can be decomposed into $2n + 1$ isomorphic copies of any tree with n edges. Rosa showed that Ringel's Conjecture is true if every tree has a graceful labeling. This is known as the famous **Graceful Tree Conjecture** but such seemingly simple statement defies any effort to prove it [5]. Today, some known examples of graceful trees are: caterpillars [9] (a tree such that the removal of its end vertices leaves a path), trees with at most 4 end vertices [8], trees with diameter at most 5 [7], and trees with at most 27 vertices [1].

Most of the previous works on graph labeling focused on finite graphs only. Recently, Beardon [2], and later, Combe and Nelson [3] considered magic labelings of infinite graphs over integers and infinite abelian groups. Beardon showed that infinite graphs built by certain types of graph amalgamations possess bijective edge-magic $\mathbb{Z}$-labeling. An infinite graph makes constructing a magic labeling easier because both the graph and the labeling set are infinite. However, it is not known whether every countably infinite tree supports a bijective edge-magic $\mathbb{Z}$-labelings. Strongly motivated by their work, in this paper, we extend the definition of graceful labeling to countably infinite graphs and prove a version of the Graceful Tree Conjecture for countably infinite trees using graph amalgamation techniques.

This paper is organized as follows. In Section 2, we give a formal definition of graceful labeling. We also consider how to construct an infinite graph by means of amalgamation, and introduce the notions of bijective graceful $\mathbb{N}$-labeling and bijective graceful $\mathbb{N}/\mathbb{Z}^+$-labeling. Section 3 includes two examples on graceful labelings of the semi-infinite path which illustrate the main ideas in this paper. In Section 4, our main results are presented while further extensions are discussed in Section 5. In Section 6, we make use of the tools developed in Section 4 and characterize all countably infinite trees that have a bijective graceful $\mathbb{N}/\mathbb{Z}^+$-labeling (see Theorem 5). This, in turn, settles a Graceful Tree Conjecture for countably infinite trees.

2 Definitions and notations

All graphs considered in this paper are countable and simple (no loops or multiple edges). A graph is non-trivial if it has more than one vertex. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For $W \subset V(G)$, denote the neighbor of W (i.e. all vertices other than W that are adjacent to some vertex in W) by $N(W)$ and the subgraph of G induced by W (i.e. all vertices of W and all edges that are adjacent to only vertices in W) by $\mathbf{G}[W]$. Denote the set of natural numbers $\{0, 1, 2, 3, \ldots\}$ by $\mathbb{N}$ and the set of positive integers by $\mathbb{Z}^+$. A labeling of G is an injective function, say f, from $V(G)$ to $\mathbb{N}$. Such a vertex labeling induces an edge labeling from $E(G)$ to $\mathbb{Z}^+$ which is also denoted by f such that for every edge $e = xy \in E(G)$, $f(e) = |f(x) - f(y)|$. If this induced edge labeling is injective, then f is a graceful $\mathbb{N}$-labeling. Note that by this definition, every graph has a graceful $\mathbb{N}$-labeling by using $\{2^0 - 1, 2^1 - 1, 2^2 - 1, \ldots\}$ as labels. If f is graceful and is a bijection between $V(G)$ and $\mathbb{N}$, then f is a bijective graceful $\mathbb{N}$-labeling. If f is a bijective graceful $\mathbb{N}$-labeling and the induced edge labeling is a bijection, then f is a bijective graceful $\mathbb{N}/\mathbb{Z}^+$-labeling.

Consider any sequence G_n of graphs, and denote $V(G_n)$ by V_n and $E(G_n)$ by E_n. The sequence G_n is increasing if for each n, $V_n \subset V_{n+1}$ and $E_n \subset E_{n+1}$. An infinite graph, $\lim_n G_n$, is then defined to be the graph whose vertex set and edge set are $\bigcup_n V_n$ and $\bigcup_n E_n$ respectively. Note that if each G_n is countable, connected and simple, then so is $\lim_n G_n$.

We can build an infinite graph by joining an infinite sequence of graphs through the process of amalgamation described below. Let G and G' be any two graphs. We can assume that G and G' are disjoint (for otherwise, we replace G' by an isomorphic copy G'' that is disjoint from G and form the amalgamation of G and G''). Select a vertex v from G and a vertex v' from G'. The amalgamation of G and G', $G\#G'$, is obtained by taking the disjoint union of G and G' and identifying v with v'. The above amalgamation process can be generalized easily to identifying a set of vertices by removing multiple edges if necessary.

Now let $G'_0, G'_1, \ldots$ be an infinite sequence of graphs. Construct a new sequence G_n inductively by $G_0 = G'_0$ and $G_{n+1} = G_n \# G'_{n+1}$. Obviously, $\{G_n\}$ is increasing and their union $\lim_n G_n$ is an infinite graph. Using techniques similar to those introduced by Beardon [2], we are able to show that every infinite graph generated by certain types of graph amalgamations has a graceful labeling.

Further definitions and notations will be introduced as our discussions proceed. The graph theory terminology used in this paper can be found in the book by Diestel [4]. Throughout the paper, we use the term infinite to mean countably infinite.

3 Example: Semi-infinite Path

In this section, we will illustrate our graph labeling method and the key ideas behind by means of the semi-infinite path. Denote the semi-infinite path by P, with vertices: $v_0, v_1, v_2, \ldots$ and edges: $v_0v_1, v_1v_2, \ldots$. We will construct a certain graceful $\mathbb{N}$-labeling f of P inductively. Write $m_j = f(v_j)$ and $n_{j+1} = f(v_jv_{j+1}) = |m_j - m_{j+1}|$, $j = 0, 1, 2, \ldots$. We will always start with $f(v_0) = m_0 = 0$.

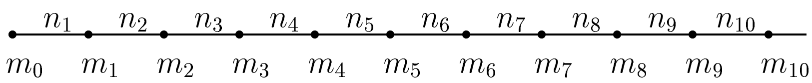


Figure 1

Bijective graceful $\mathbb{N}$-labeling of the semi-infinite path

Our goal is to label the vertices of P using $\mathbb{N}$ such that the vertex labels correspond one-to-one to the set of natural numbers and the edge labels are all distinct. We will proceed in a manner similar to that in [2].

Take m_2 to be the smallest integer in $\mathbb{N}$ not yet used for vertex labeling which is 1. Now, we can choose m_1 to be sufficiently large so that n_1 and n_2 are distinct and have not appeared in the edge labels. For example, $m_1 = 2$ will do, and we have $n_1 = 2$ and $n_2 = 1$.

Next, we should consider m_4 and define $m_4 = 3$. We may then choose $m_3 = 6$ and hence $n_3 = 5$ and $n_4 = 3$.

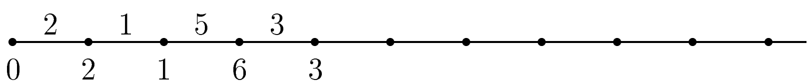


Figure 2

The above process can be repeated indefinitely. Since for each $k \in \mathbb{N}$, we can choose m_{2k} to be the smallest unused integer in $\mathbb{N}$, f is surjective. By construction, f is also injective and all edge labels are distinct. Hence, we have constructed a bijective graceful $\mathbb{N}$-labeling of the semi-infinite path.

Bijective graceful $\mathbb{N}/\mathbb{Z}^+$-labeling of the semi-infinite path

In the previous example, we require that all natural numbers appear in the vertex labels. A natural question arises: can we also require that all positive integers appear in the edge labels? As will be shown below, this is possible for the semi-infinite path.

We choose n_2 to be the smallest integer in $\mathbb{Z}^+$ not used in the edge labels. Hence, $n_2 = 1$. Now we would like to choose m_1 and m_2 that satisfy the following conditions:

(i) m_1 and m_2 are different from 0 (the vertex labels already used) and $n_2 = |m_1 - m_2| = 1$, and
(ii) $n_1 = |0 - m_1|$ is different from 1 (the edge labels already used).

This is always possible if we choose m_1 and m_2 to be sufficiently large so that n_1 has not appeared before. In this particular example, $m_1 = 3$ and $m_2 = 2$ will do, and we have $n_1 = 3$.

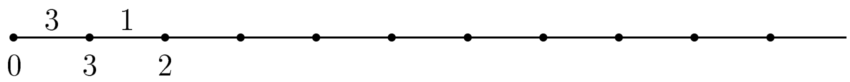


Figure 3

Next we choose m_4 to be the smallest integer in $\mathbb{N}$ not yet used in the vertex labels. So $m_4 = 1$. Now choose m_3 sufficiently large so that n_3 and n_4 have not appeared in the edge labels. Pick $m_3 = 6$, and we have $n_3 = 4$ and $n_4 = 5$.

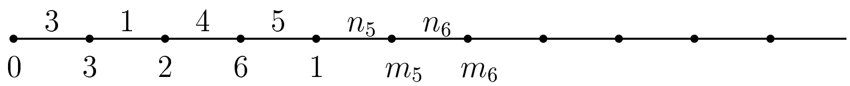


Figure 4

The above two labeling procedures can go on indefinitely (e.g. $n_6 = 2$,$m_5 = 7$, $m_6 = 9$ and $n_5 = 6$). Since for each $k \in \mathbb{N}$, we are able to choose n_{4k+2} and m_{4k+4} to be the smallest unused edge and vertex labels respectively, $f|_{E(P)} : E(P) \to \mathbb{Z}^+$ and $f|_{V(P)} : V(P) \to \mathbb{N}$ are surjective. By construction, f is also injective. Therefore, we have successfully constructed a bijective graceful $\mathbb{N}/\mathbb{Z}^+$-labeling of the semi-infinite path.

Summing up, the crucial element that makes bijective graceful $\mathbb{N}$-labeling of the semi-infinite path possible is that during the labeling process, one can find a vertex that is not adjacent to all the previously labelled vertices. Such a vertex can then be labelled using the smallest unused vertex label. Likewise, one can find an edge that is not incident to all the previously labelled vertices. Such edge can be labelled using the smallest unused edge label allowing one to construct a bijective graceful $\mathbb{N}/\mathbb{Z}^+$-labeling of the semi-infinite path.

4 Main Results

Here we put the ideas developed in the previous section into Lemma 2 and 3 which are key to our main results on graceful labelings of infinite graphs. First, we define type-1 and type-2 graph amalgamations. Let G and G' be any two disjoint graphs. Consider $v \in V(G)$ and $v' \in V(G')$. Suppose G' has a vertex u' that is not adjacent to v'. Then the amalgamation $G\#G'$ formed by identifying v and v' is called a **type-1 amalgamation**. Suppose G' has an edge e' that is not incident to v'. Then the amalgamation $G\#G'$ formed by identifying v and v' is called a **type-2 amalgamation**.

Before proving Lemma 2 and 3, we need the following lemma:

Lemma 1. *Let N_0 be a finite subset of $\mathbb{N}$. Consider the set of all non-constant linear polynomials $a_1x_1 + \ldots + a_kx_k$ in k variables x_i, where each $a_i \in \{-2, -1, 0, 1, 2\}$. Then there exists $m_1, \ldots, m_k \in \mathbb{N}$ such that no $a_1m_1 + \ldots + a_km_k$ is in N_0.*

Proof. Let A_k be the set of all non-constant linear polynomials $a_1x_1 + \ldots + a_kx_k$ where each $a_i \in \{-2, -1, 0, 1, 2\}$. Suppose $A_k(m_1, \ldots, m_k)$ is the set of integers obtained by evaluating all polynomials in A_k at $m_1, \ldots, m_k \in \mathbb{N}$. We will prove by induction. For $k = 1$, we can choose m_1 so that $-2m_1, -m_1, m_1, 2m_1$ are all outside N_0. Suppose the statement holds for every finite subset of $\mathbb{N}$ and for $k = 1, \ldots, n$. Now, consider linear polynomials of $n+1$ variables, $m_1, \ldots, m_n, m_{n+1}$, and any finite subset N_0 of $\mathbb{N}$. Choose m_{n+1} so that $-2m_{n+1}, -m_{n+1}, m_{n+1}, 2m_{n+1}$ are all outside N_0. By induction hypothesis, we can choose $m_1, \ldots, m_n$ so that $A_n(m_1, \ldots, m_n) \cap ((-2m_{n+1} + N_0) \cup (-m_{n+1} + N_0) \cup N_0 \cup (m_{n+1} + N_0) \cup (2m_{n+1} + N_0)) = \emptyset$. This implies that $A_{n+1}(m_1, \ldots, m_{n+1}) \cap N_0 = \emptyset$. Hence, the statement is true for $k = n+1$ and the proof is complete. □

Lemma 2. *Let G_0 be a finite graph and f_0 be a graceful $\mathbb{N}$-labeling of G_0. Let V_0 be the set of integers taken by f_0 on $V(G_0)$ and E_0 be the set of induced edge labels on $E(G_0)$. Suppose $m \in \mathbb{N} \setminus V_0$. Let G be any finite graph and form a type-1 amalgamated graph $G_0\#G$ by identifying a vertex v_0 of G_0 with a vertex v of G. Let u be a vertex in G not adjacent to v. Then f_0 can be extended to a graceful $\mathbb{N}$-labeling f of $G_0\#G$ so that $f(u) = m$.*

Proof. First define f to be f_0 on G_0 and $f(u) = m$. Write $m_0 = f_0(v_0)$. Since v is identified with v_0, we define $f(v) = m_0$. Let $v_1, \ldots, v_k$ be the vertices in G other than u and v. Define $f(v_i) = m_i$ for $i = 1, \ldots, k$ where m_i's are natural numbers to be determined. Now, each edge in G is of one of the forms: vv_i, uv_i or v_iv_j for $1 \leq i \neq j \leq k$ with edge labels $|m_0 - m_i|$, $|m - m_i|$, and $|m_i - m_j|$ respectively. Notice that the edge label of any edge $e \in E(G)$ is the absolute value of a non-constant linear polynomial $p_e(m_1, \ldots, m_k)$ with coefficients taken from the set $\{-1, 0, 1\}$. To make f injective, we want to choose m_i, for $i = 1, \ldots, k$, so that:

1. $m_i \neq m_j$ for $1 \leq i \neq j \leq k$,
2. $m_1, \ldots, m_k \notin V_0 \cup \{m\}$,
3. $p_e(m_1, \ldots, m_k) \notin E_0$, for all $e \in E(G)$,
4. $p_e(m_1, \ldots, m_k) \neq p_{e'}(m_1, \ldots, m_k)$ for all distinct $e, e' \in E(G)$, and
5. $p_e(m_1, \ldots, m_k) \neq -p_{e'}(m_1, \ldots, m_k)$ for all distinct $e, e' \in E(G)$.

This is possible by Lemma 1. □

Lemma 3. *Let G_0 be a finite graph and f_0 be a graceful $\mathbb{N}$-labeling of G_0. Let V_0 be the set of integers taken by f_0 on $V(G_0)$ and E_0 be the set of induced edge labels on $E(G_0)$. Suppose $n \in \mathbb{Z}^+ \setminus E_0$. Let G be any finite graph and form a type-2 amalgamated graph $G_0\#G$ by identifying a vertex v_0 of G_0 with a vertex v of G. Let xy be an edge in G not incident to v. Then f_0 can be extended to a graceful $\mathbb{N}$-labeling f of $G_0\#G$ so that $f(xy) = n$.*

Proof. The proof is almost identical to that of Lemma 2 except for some minor modifications. Let $m_v = f_0(v_0)$, and m_x and m_y be the labels of x and y respectively. By choosing

m_x and m_y sufficiently large, we can ensure that (i) $m_x, m_y \in \mathbb{N} \setminus V_0$, (ii) $|m_x - m_y| = n$, (iii) $|m_x - m_v| \notin E_0 \cup \{n\}$ if x is adjacent to v, and (iv) $|m_y - m_v| \notin E_0 \cup \{n\}$ if y is adjacent to v. Define f to be f_0 on G_0, $f(x) = m_x$ and $f(y) = m_y$. Let $v_1, \ldots, v_k$ be the vertices in G other than v, x and y. Define $f(v_i) = m_i$ for $i = 1, \ldots, k$ where m_i's are natural numbers to be determined. Now, each edge e in G except xy (and possibly vx and vy) is of one of the forms: vv_i, xv_i, yv_i or v_iv_j for $1 \leq i \neq j \leq k$ with edge labels $|m_v - m_i|$, $|m_x - m_i|$, $|m_y - m_i|$ and $|m_i - m_j|$ respectively. Notice that the edge label for every edge $e \in E(G)$ is the absolute value of a non-constant linear polynomial $p_e(m_1, \ldots, m_k)$ in the variables $m_1, \ldots, m_k$ with coefficients taken from the set $\{-1, 0, 1\}$. To make f injective, we want to choose m_i, for $i = 1, \ldots, k$, so that:

1. $m_i \neq m_j$ for $1 \leq i \neq j \leq k$,
2. $m_1, \ldots, m_k \notin V_0 \cup \{m_x\} \cup \{m_y\}$,

For all $e \in E(G)$,

3. $p_e(m_1, \ldots, m_k) \notin E_0 \cup \{n\}$,
4. $p_e(m_1, \ldots, m_k) \neq m_x - m_v$ if x is adjacent to v,
5. $p_e(m_1, \ldots, m_k) \neq m_v - m_x$ if x is adjacent to v,
6. $p_e(m_1, \ldots, m_k) \neq m_y - m_v$ if y is adjacent to v,
7. $p_e(m_1, \ldots, m_k) \neq m_v - m_y$ if y is adjacent to v,

For all distinct $e, e' \in E(G)$,

8. $p_e(m_1, \ldots, m_k) \neq p_{e'}(m_1, \ldots, m_k)$ for $i \neq j$, and
9. $p_e(m_1, \ldots, m_k) \neq -p_{e'}(m_1, \ldots, m_k)$ for $i \neq j$.

This is possible by Lemma 1. □

Now we present our main theorems that tell us what particular types of infinite graphs can have a bijective graceful $\mathbb{N}$-labeling or a bijective graceful $\mathbb{N}/\mathbb{Z}^+$-labeling.

Theorem 1. *Let $\{G'_n\}$ be an infinite sequence of finite graphs. Let $G_0 = G'_0$ and for each $n \in \mathbb{N}$, let $G_{n+1} = G_n \# G'_{n+1}$. If there are infinitely many type-1 amalgamations during the amalgamation process, then $\lim_n G_n$ has a bijective graceful $\mathbb{N}$-labeling.*

Proof. Let $n_0, n_1, n_2, \ldots$ be an increasing sequence such that $G_{n_k} \# G'_{n_k+1}$ is a type-1 amalgamation for each k.

Let f_0 be a graceful $\mathbb{N}$-labeling of G_0 such that 0 is a vertex label.

Suppose that we have constructed a graceful labeling of G_n. Let V_n and E_n be the set of vertex and edge labels of G_n respectively. It is obvious that we can extend f_n to a graceful $\mathbb{N}$-labeling f_{n+1} of $G_{n+1} = G_n \# G'_{n+1}$. Now consider the case when $n = n_k$ for some k. If $k+1 \in V_n$, then $k+1 \in V_{n+1}$. If $k+1 \notin V_n$, then by Lemma 2, we extend f_n in such a way that $k+1 \in f_{n+1}(V(G_{n+1})) = V_{n+1}$.

By repeating the above process indefinitely, we have $k+1 \in V_{n_k+1}$ for $k \in \mathbb{N}$. Hence, we obtain a bijective graceful $\mathbb{N}$-labeling of $\lim_n G_n$. $\square$

Theorem 2. *Let $\{G'_n\}$ be an infinite sequence of finite graphs. Let $G_0 = G'_0$ and for each $n \in \mathbb{N}$, let $G_{n+1} = G_n \# G'_{n+1}$. If there are infinitely many type-1 and type-2 amalgamations during the amalgamation process, then $\lim_n G_n$ has a bijective graceful $\mathbb{N}/\mathbb{Z}^+$-labeling.*

Proof. From the assumption, we have an increasing sequence $n_0, n_1, n_2, \ldots$ such that $G_{n_{2k}} \# G'_{n_{2k}+1}$ is a type-2 amalgamation and $G_{n_{2k+1}} \# G'_{n_{2k+1}+1}$ is a type-1 amalgamation for each k.

Let f_0 be a graceful $\mathbb{N}$-labeling of G_0 such that 0 is a vertex label.

Suppose that we have constructed a graceful labeling of G_n. Let V_n and E_n be the set of vertex and edge labels of G_n respectively. It is obvious that we can extend f_n to a graceful $\mathbb{N}$-labeling f_{n+1} of $G_{n+1} = G_n \# G'_{n+1}$. In the case that $n = n_{2k+1}$ for some k and $k+1 \notin V_n$, then by Lemma 2, we extend f_n in such a way that $k+1 \in f_{n+1}(V(G_{n+1})) = V_{n+1}$. On the other hand, if $k+1 \in V_n$, then $k+1 \in V_{n+1}$. If $n = n_{2k}$ for some k but $k+1 \notin E_n$, then by Lemma 3, we extend f_n in a way such that $k+1 \in f_{n+1}(E(G_{n+1})) = E_{n+1}$. When $k+1 \in E_n$, we clearly have $k+1 \in E_{n+1}$.

By repeating the above process indefinitely, we have $k+1 \in V_{n_{2k+1}+1}$ and $k+1 \in E_{n_{2k}+1}$ for $k \in \mathbb{N}$. Hence, we obtain a bijective graceful $\mathbb{N}/\mathbb{Z}^+$-labeling of $\lim_n G_n$. $\square$

5 Further extensions

The amalgamation process described above can be generalized to one that identifies a finite set of vertices in one graph with a finite set of vertices in another graph. Based on this more general amalgamation, we can derive the more general versions of Theorem 1 and 2. As a result, we are able to prove the following two propositions which are important for the characterizations of graphs that have a bijective graceful $\mathbb{N}$-labeling and graphs that have a bijective graceful $\mathbb{N}/\mathbb{Z}^+$-labeling.

Proposition 1. *Let G be an infinite graph. If every vertex of G has a finite degree, then G has a bijective graceful $\mathbb{N}$-labeling.*

Proof. We will show that G can be constructed inductively by type-1 amalgamations. Enumerate the vertices of G. Choose the first vertex v_0 in G and let $G_0 = G'_0 = \{v_0\}$. Since the degree of v_0 is finite, $|N(G_0)|$ is finite where $N(G_0)$ is the neighbor of G_0. Choose the first vertex $v_1 \in G$ such that $v_1 \notin G_0 \cup N(G_0)$. Let $G'_1 = \mathbf{G}[G_0 \cup N(G_0) \cup \{v_1\}]$.

Form a type-1 amalgamated graph $G_1 = G_0 \# G'_1$ by identifying G_0. Interestingly, we have $G_1 = G'_1$. Now choose the first vertex $v_2 \notin G_1 \cup N(G_1)$. Let $G'_2 = \mathbf{G}[G_1 \cup N(G_1) \cup \{v_2\}]$. Form a type-1 amalgamated graph $G_2 = G_1 \# G'_2$ by identifying G_1. By repeating the above process, we see that G_n is increasing and $G = \lim_n G_n$. Hence, by Theorem 1, G has a bijective graceful $\mathbb{N}$-labeling. $\square$

Proposition 2. *Let G be an infinite graph with infinitely many edges. If every vertex of G has a finite degree, then G has a bijective graceful $\mathbb{N}/\mathbb{Z}^+$-labeling.*

Proof. The proof is similar to that of Proposition 1. Here we form both type-1 and type-2 amalgamations instead and apply Theorem 2. $\square$

Although our discussions so far only make use of $\mathbb{N}$ for graph labeling, all the above results still hold for any infinite torsion-free abelian group $\mathbb{A}$ (written additively). An abelian group $\mathbb{A}$ is torsion-free if for all $n \in \mathbb{N}$ and for all $a \in \mathbb{A}$, $na \neq 0$. Here, $na = a + \ldots + a$ (n times). In such general settings, the absolute difference is no longer meaningful and we need to consider directed graphs without loops or multiple edges instead. Denote the directed edge from x to y by xy. Let $f(x)$ and $f(y)$ be the vertex labels of x and y respectively. We will define the edge label for xy to be $f(y) - f(x)$. Now we are ready for the more general versions of Theorem 1 and 2 but first we need the following three lemmas.

Lemma 4. *Let $\mathbb{A}$ be an infinite torsion-free abelian group and A_0 be a finite subset of $\mathbb{A}$. Then there exists $m \in \mathbb{A}$ such that for all $k \in \mathbb{Z} \backslash \{0\}$, $km \notin A_0$.*

Proof. Let $B = A_0 \cup -A_0$. Since B is finite, there exists $a \in \mathbb{A}$ such that $a \notin B$. Consider $C = \{a, 2a, 3a, \ldots\}$ in which all elements are distinct as $\mathbb{A}$ is torsion-free. Now, only finitely many elements of C can lie in B. Similarly, only finitely many elements of $-C$ lie in B. Therefore, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $na \notin B$ and $-na \notin B$. Take $m = Na$. We have for all $k \in \mathbb{Z} \backslash \{0\}$, $km \notin B$ and hence $km \notin A_0$. $\square$

Lemma 5. *Let $\mathbb{A}$ be an infinite torsion-free abelian group and A_0 be a finite subset of $\mathbb{A}$. Consider the set of all non-constant linear polynomials $a_1x_1 + \ldots + a_kx_k$ in k variables where each $a_i \in \{-2, -1, 0, 1, 2\}$. Then there exists $m_1, \ldots, m_k \in \mathbb{A}$ such that no $a_1m_1 + \ldots + a_km_k$ is in A_0.*

Proof. The proof is identical to that of Lemma 1. Here we use Lemma 4 to make sure that we can choose m so that $-2m, -m, m, 2m$ are all outside A_0. $\square$

Lemma 6. *Let $\mathbb{A}$ be an infinite abelian group. For any $m \in \mathbb{A}$, there exists infinitely many pairs $x, y \in \mathbb{A}$ such that $x - y = m$.*

Proof. Obvious. For each $y \in \mathbb{A}$, choose $x = y + m$. $\square$

Using Lemma 5 and 6, we can obtain results similar to Lemma 2 and 3 for any infinite torsion-free abelian group. The reason is that the polynomials we are dealing with are of the form described in Lemma 5. Lemma 6 ensures that we can choose m_x and m_y as desired for Lemma 3. As a result, we have the following generalizations of Theorems 1 and 2.

Theorem 3. *Suppose $\mathbb{A}$ is an infinite torsion-free abelian group. Let G'_n be an infinite sequence of finite graphs. Let $G_0 = G'_0$ and for each $n \in \mathbb{N}$, let $G_{n+1} = G_n \# G'_{n+1}$. If there are infinitely many type-1 amalgamations during the amalgamation process, then $\lim_n G_n$ has a bijective graceful $\mathbb{A}$-labeling.*

Theorem 4. *Suppose $\mathbb{A}$ is an infinite torsion-free abelian group. Let G'_n be an infinite sequence of finite graphs. Let $G_0 = G'_0$ and for each $n \in \mathbb{N}$, let $G_{n+1} = G_n \# G'_{n+1}$. If there are infinitely many type-1 and type-2 amalgamations during the amalgamation process, then $\lim_n G_n$ has a bijective graceful $\mathbb{A}/\mathbb{A} \setminus \{0\}$-labeling.*

We can generalize even further by examining the bijective graceful V or V/E-labeling where V and E are infinite subsets of an infinite abelian group. To illustrate this idea, let us consider an infinite graph with a bijective graceful $\mathbb{N}/\mathbb{Z}^+$-labeling. Now multiply each vertex label by q and then add r to it where $0 \leq r < q$. The result is a bijective graceful $(q\mathbb{N} + r)/q\mathbb{Z}^+$-labeling of the original graph. The reverse process can also be performed. This shows that bijective graceful $\mathbb{N}/\mathbb{Z}^+$-labeling and $(q\mathbb{N} + r)/q\mathbb{Z}^+$-labeling are equivalent. We will demonstrate the usefulness of such general notion of graceful labeling in the next section.

6 Graceful Tree Theorem for Infinite Trees

In this section, we make use of the tools developed earlier to characterize all infinite trees that have a bijective graceful $\mathbb{N}/\mathbb{Z}^+$-labeling. This in turn solves the Graceful Tree Conjecture for infinite trees. In order to characterize all infinite trees that have a bijective graceful $\mathbb{N}/\mathbb{Z}^+$-labeling, we shall divide the set of all infinite trees into four classes: (i) Infinite trees with no infinite degree vertices, (ii) Infinite trees with exactly one infinite degree vertex, (iii) Infinite trees with more than one but finitely many infinite degree vertices, and (iv) Infinite trees with infinitely many infinite degree vertices.

We shall show that bijective graceful $\mathbb{N}/\mathbb{Z}^+$-labeling exists for any infinite tree in class (i), (ii) and (iv). For any tree T in class (iii), we shall prove that such a labeling exists if and only if T contains a semi-infinite path or an once-subdivided infinite star. Here an once-subdivided infinite star is obtained from an infinite star by subdividing each edge once.

If we let $\mathcal{E}$ be the set of trees which have more than one but finitely many vertices of infinite degree and contain neither a semi-infinite path nor an once-subdivided infinite star, then we can state the **Graceful Tree Theorem for Infinite Trees** as follows.

Theorem 5. *An infinite tree has a bijective graceful $\mathbb{N}/\mathbb{Z}^+$-labeling if and only if it does not belong to $\mathcal{E}$.*

To prove Theorem 5, we first show that an infinite tree with a semi-infinite path or an once-subdivided infinite star has a bijective graceful $\mathbb{N}/\mathbb{Z}^+$-labeling. Note that an infinite tree contains an one-subdivided infinite star if and only if there is a vertex adjacent to infinite number of vertices of degree ≥ 2.

Proposition 3. *Let T be an infinite tree with a semi-infinite path. Then T has a bijective graceful $\mathbb{N}/\mathbb{Z}^+$-labeling.*

Proof. Denote the semi-infinite path by P. Enumerate $V(P)$ by $v_0, v_2, v_4, \ldots$ in a natural way, and enumerate $V(T-P)$ by $v_1, v_3, v_5, \ldots$. The infinite tree T can be constructed inductively by the following procedure.

0. Let $T_0 = \{v_0\}$. Set $i = 0$.

1. Consider the smallest odd k such that $v_k \notin V(T_i)$. Since T is a tree, there is a unique path G joining v_k to T_i. Let $T_{i+1} = T_i \# G$.
2. Consider the smallest even k such that $v_k \notin V(T_{i+1})$. Let $T_{i+2} = T_{i+1} \# v_{k-2} v_k v_{k+2}$, which is a type-1 and type-2 amalgamation.
3. $i = i + 2$. Goto step 1.

The above amalgamation process includes every vertex and edge of T in the limit, and we have $T = \lim_n T_n$. Now there are infinitely many type-1 and type-2 amalgamations. Therefore, by Theorem 2, T has a bijective graceful $\mathbb{N}/\mathbb{Z}^+$-labeling. $\square$

Proposition 4. *Let T be an infinite tree with an once-subdivided infinite star. Then T has a bijective graceful $\mathbb{N}/\mathbb{Z}^+$-labeling.*

Proof. Denote the once-subdivided infinite star by S. Let the center of S be v_0 which is adjacent to infinitely many vertices of degree ≥ 2, enumerate them by $v_2, v_4, v_6, \ldots$. Enumerate the rest of T by $v_1, v_3, v_5, \ldots$. The infinite tree T can be constructed inductively by the following procedure.

0. Let $T_0 = \{v_0\}$. Set $i = 0$.

1. Consider the smallest odd k such that $v_k \notin V(T_i)$. Since T is a tree, there is a unique path G joining v_k to T_i. Let $T_{i+1} = T_i \# G$.
2. Consider the smallest even k such that $v_k \notin V(T_{i+1})$. There exists $u \neq v_0$ which is adjacent to v_k. Let $T_{i+2} = T_{i+1} \# v_0 v_k u$, which is a type-1 and type-2 amalgamation.
3. $i = i + 2$. Goto step 1.

The above amalgamation process includes every vertex and edge of T in the limit, and we have $T = \lim_n T_n$. Now there are infinitely many type-1 and type-2 amalgamations. Therefore, by Theorem 2, T has a bijective graceful $\mathbb{N}/\mathbb{Z}^+$-labeling. $\square$

To prove Theorem 5, we will consider the four classes of infinite trees one by one and apply the following lemma.

Lemma 7. *Every infinite connected graph has a vertex of infinite degree or contains a semi-infinite path.*

Proof. Proposition 8.2.1 in [4]. $\square$

(i) Infinite trees with no infinite degree vertices

Proposition 5. *Every infinite tree with no infinite degree vertices has a bijective graceful $\mathbb{N}/\mathbb{Z}^+$-labeling.*

Proof. By Proposition 2. Another proof is by Proposition 3 and Lemma 7. $\square$

(ii) Infinite trees with exactly one infinite degree vertex

Lemma 8. *For any finite tree T of order $k+1$ and any vertex v_0 of T, T has a bijective graceful $\{0, n_1, \ldots, n_k\}/\{n_1, \ldots, n_k\}$-labeling with 0 being the label of v_0 and $n_1 < n_2 < \ldots < n_k$ $(n_i \in \mathbb{Z}^+)$.*

Proof. Pick any vertex $v_0 \in V(T)$ to be the root of T. Let $S(l)$ be the set of vertices in T that are at distance l from v_0. Let $T(l)$ be the subtree induced by all the vertices at distance $\leq l$ from v.

Label v_0 by 0 and the vertices of $S(1)$ by $\{1, 3, \ldots, 2p-1\}$ where $|S(1)| = p$ and obtain a labeling of $T(1)$.

Now suppose we have obtained a labeling for $T(l)$ such that the labels of $T(l-1)$ are all even and the labels of $S(l)$ are all odd. We would like to extend it to $T(l+1)$. The idea is to multiply the labels of $T(l)$ by $2q$ where q is a sufficiently large odd number and choose the labels for $S(l+1)$ using appropriate odd numbers.

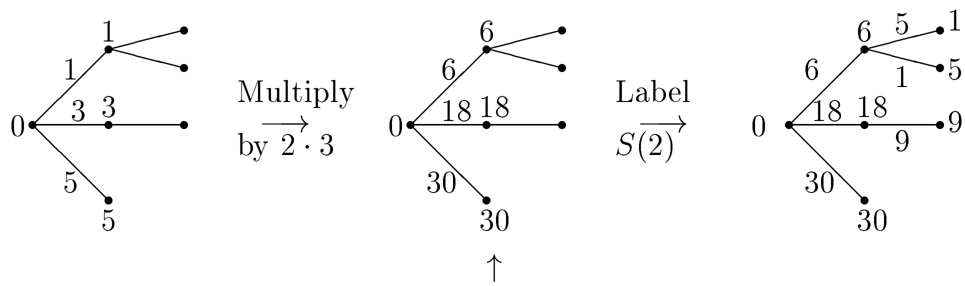
Let $v_1, v_2, \ldots, v_s$ be the vertices of $S(l)$ and their respective labels be $x_1, x_2, \ldots, x_s$ which are all odd. For each v_i, let $t_i = deg(v_i) - 1$ and $v_{i_1}, v_{i_2}, \ldots, v_{i_{t_i}}$ be its neighbors in $S(l+1)$. Multiply all the labels of $T(l)$ by $2q$ where q is an odd number to be determined. The labels of $T(l)$ now become all even and still satisfy the condition stated in the lemma. In particular, the labels for $v_1, v_2, \ldots, v_s$ now become $2qx_1, 2qx_2, \ldots, 2qx_s$.

Observe that the set of $2t_i + 1$ consecutive integers $\{qx_i - t_i, \ldots, qx_i - 1, qx_i, qx_i + 1, \ldots, qx_i + t_i\}$ contains at least t_i odd numbers. The labels of $v_{i_1}, v_{i_2}, \ldots, v_{i_{t_i}}$ can then be chosen from these odd numbers according to the rule: If x is used, so is $2qx_i - x$. Note that if t_i is odd, then qx_i is used.

Finally, to ensure the feasibility of the labeling, we require that: $0 < qx_1 - t_1, qx_1 + t_1 < qx_2 - t_2, \ldots, qx_{s-1} + t_{s-1} < qx_s - t_s$ or equivalently $q > \frac{t_1}{x_1}, q > \frac{t_2+t_1}{x_2-x_1}, \ldots, q > \frac{t_s+t_{s-1}}{x_s-x_{s-1}}$ which is always possible by choosing a sufficiently large odd number q. Hence we obtain a labeling of $T(l+1)$ satisfying the condition of the lemma.

By repeating the above procedure, we obtain a $\{0, n_1, \ldots, n_k\}/\{n_1, \ldots, n_k\}$-labeling of T with the desired properties. $\square$

We illustrate the above labeling procedure by the following example.



$$x_1 = 1,\ t_1 = 2,\ q > \tfrac{t_1}{x_1} = \tfrac{2}{1} = 2$$
$$x_2 = 3,\ t_2 = 1,\ q > \tfrac{t_2+t_1}{x_2-x_1} = \tfrac{2+1}{3-1} = \tfrac{3}{2}$$
$$x_3 = 5,\ t_3 = 0,\ q > \tfrac{t_3+t_2}{x_3-x_2} = \tfrac{0+1}{5-3} = \tfrac{1}{2}$$

Therefore, we can choose $q = 3$.

Figure 5

Proposition 6. *Every infinite tree with exactly one infinite degree vertex has a bijective graceful* $\mathbb{N}/\mathbb{Z}^+$*-labeling.*

Proof. Let T be the infinite tree. If T has a semi-infinite path or an once-subdivided infinite star, then by Proposition 3 and 4, T has a bijective graceful $\mathbb{N}/\mathbb{Z}^+$-labeling. Otherwise, by Lemma 7 and the fact that T has exactly one infinite degree vertex, T is the amalgamation of a finite tree T_0 and an infinite star S by identifying a root of T_0 with the center of S. By Lemma 8, T_0 has a bijective graceful $\{0, n_1, \ldots, n_k\}/\{n_1, \ldots, n_k\}$-labeling where the center of S is labelled as 0. Now label the leaves of S by $\mathbb{N} \setminus \{0, n_1, \ldots, n_k\}$. Therefore, T has a bijective graceful $\mathbb{N}/\mathbb{Z}^+$-labeling. $\square$

(iii) Infinite trees with more than one but finitely many infinite degree vertices

Lemma 9. *Let G be an amalgamation of a finite graph $G_0 = (V_0, E_0)$ and k infinite stars by identifying k distinct vertices of G_0 with the k centers of the infinite stars. Suppose that $|E_0| \geq |V_0| - 1$ and G has a bijective graceful* $\mathbb{N}$*-labeling. Then $k = 1$, the center of the infinite star is labelled 0, and $|E_0| = |V_0| - 1$.*

Proof. Denote the k centers of the infinite stars by $v_1, v_2, \ldots, v_k$. Consider v_1 and take a vertex v adjacent to v_1 such that the label of v is greater than that of any vertices of G_0. Let the label of v be n. Consider the subgraph H of G induced by the vertices labelled $\{0, 1, \ldots, n\}$,i.e. H is the subgraph contains G_0 and edges of the form v_iu_i where the label of u_i is less than or equal to n. Let n_i be the number of common edges between H and the infinite star centered at v_i. We have $|V(H)| = n + 1 = |V_0| + n_1 + \ldots + n_k$ and $|E(H)| = |E_0| + n_1 + \ldots + n_k$. We have $|E(H)| \geq |V(H)| - 1$ as $|E_0| \geq |V_0| - 1$. Since the edge labels of H are all distinct, $|E(H)|$ must be less than $|V(H)|$, implying that $|E_0| < |V_0|$. So $|E_0| = |V_0| - 1$ and $|E(H)| = |V(H)| - 1$. Now H has n edges which must be labelled by $\{1, 2, \ldots, n\}$. The edge labelled n must join the two vertices labelled 0 and n. Since the vertex labelled n is v, v_1 is labelled 0. Since the above argument applies to every v_i, we must have $k = 1$. □

Proposition 7. *Every infinite tree with more than one but finitely many vertices of infinite degree does* **not** *have a bijective graceful* $\mathbb{N}/\mathbb{Z}^+$*-labeling except when the tree contains a semi-infinite path or an once-subdivided infinite star.*

Proof. Let T be a infinite tree with more than one but finitely many vertices of infinite degree. Let U be the set of vertices of infinite degree. If T has a semi-infinite path or an once-subdivided infinite star, then by Proposition 3 and 4, T has a bijective graceful $\mathbb{N}/\mathbb{Z}^+$-labeling. Suppose not. Delete from T all degree 1 neighbors of v for all $v \in U$. The resulting graph T' is a finite tree. This means that T is an amalgamation of T' and $|U|$ infinite stars by identifying U in T' with the $|U|$ centers of the stars. By Lemma 9, T does not have a bijective graceful $\mathbb{N}$-labeling. □

(iv) Infinite trees with infinitely many vertices of infinite degree

Proposition 8. *Every infinite tree with infinitely many vertices of infinite degree has a bijective graceful* $\mathbb{N}/\mathbb{Z}^+$*-labeling.*

Proof. Let T be an infinite tree with infinitely many vertices of infinite degree. If T has a semi-infinite path, then by Proposition 3, T has a bijective graceful $\mathbb{N}/\mathbb{Z}^+$-labeling. Otherwise, the fact that T has infinitely many vertices of infinite degree implies that T contains an once-subdivided infinite star. By Proposition 4, T has a bijective graceful $\mathbb{N}/\mathbb{Z}^+$-labeling. □

The proof of the Graceful Tree Theorem for Infinite Trees is therefore complete.

Acknowledgment. The authors thank the referee for the extremely helpful and valuable suggestions.

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