Certificates of factorisation for a class of triangle-free graphs

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Abstract

The chromatic polynomial $P(G,\lambda)$ gives the number of λ-colourings of a graph. If $P(G,\lambda) = P(H_1,\lambda)P(H_2,\lambda)/P(K_r,\lambda)$, then the graph G is said to have a chromatic factorisation with chromatic factors H_1 and H_2. It is known that the chromatic polynomial of any clique-separable graph has a chromatic factorisation. In this paper we construct an infinite family of graphs that have chromatic factorisations, but have chromatic polynomials that are not the chromatic polynomial of any clique-separable graph. A certificate of factorisation, that is, a sequence of rewritings based on identities for the chromatic polynomial, is given that explains the chromatic factorisations of graphs from this family. We show that the graphs in this infinite family are the only graphs that have a chromatic factorisation satisfying this certificate and having the odd cycle C_{2n+1}, $n \geq 2$, as a chromatic factor.

1 Introduction

The chromatic polynomial, $P(G,\lambda) \in \mathbb{Z}[\lambda]$, gives the number of proper λ-colourings of a graph G. This polynomial was first studied by Birkhoff [1, 2] in an effort to algebraically prove the four colour theorem. Since then the chromatic polynomial has been extensively studied in both graph theory and statistical mechanics. There has been considerable interest in *chromatic roots* (roots of the chromatic polynomial); see the surveys by Woodall [6] and Jackson [3].

This paper continues the study of algebraic properties of the chromatic polynomial that we began in [5].

As a first step in the study of the algebraic structure of the chromatic polynomial, we considered the factorisation of the chromatic polynomial of a graph G into chromatic polynomials of lower degree. We say $P(G,\lambda)$ has a *chromatic factorisation* if

$$P(G,\lambda) = \frac{P(H_1,\lambda)P(H_2,\lambda)}{P(K_r,\lambda)} \tag{1}$$

where H_1 and H_2 are graphs of lower order than G, neither H_1 nor H_2 is isomorphic to K_r and $0 \leq r \leq \min\{\chi(H_1), \chi(H_2)\}$. By convention $P(K_0,\lambda) := 1$. We say G has a *chromatic factorisation*, if $P(G,\lambda)$ has a chromatic factorisation, and that the *chromatic factors* of G are H_1 and H_2.

A graph is *clique-separable* if it is either disconnected or if it can be obtained by identifying two graphs at some clique. Two graphs are said to be *chromatically equivalent* if they have the same chromatic polynomial. A graph is *quasi-clique-separable* if it chromatically equivalent to a clique-separable graph. If G is quasi-clique-separable, then G has a chromatic factorisation. In [5] we demonstrated that there exist *strongly non-clique-separable* graphs — graphs that are not quasi-clique-separable — that have chromatic factorisations. We found 512 chromatic polynomials of strongly non-clique-separable graphs of order at most 10 that have chromatic factorisations. We introduced the concept of a *certificate of factorisation*, which is a sequence of steps that explains the chromatic factorisation of a given chromatic polynomial. A schema for certificates was introduced and certificates were given for all strongly non-clique-separable graphs of order at most 9 that have a chromatic factorisation [5].

The graphs that have chromatic factorisations that satisfy this schema all have a common structural property; they are almost clique-separable, that is graphs that can obtained by adding a single edge to, or removing a single edge from, a clique-separable graph. In this paper we construct an infinite family of strongly non-clique-separable graphs. Graphs in this family not only have the property of being almost clique-separable; these graphs are also triangle-free. We give a certificate of factorisation for graphs belonging to this family. We then show that any graph that has a chromatic factorisation that satisfies this certificate and has an odd cycle of length at least five as a chromatic factor must belong to this family.

We assume the reader is familiar with [5]. The basic definitions and properties of the chromatic polynomial given in [5] will be used in this article. Section 2 establishes some properties on the number of triangles in graphs that have chromatic factorisations. These properties are used in Section 3 where we give a certificate of factorisation and prove that any non-clique-separable graph that factorises in the form of this certificate contains no triangles if one of the chromatic factors contains no triangles. In Section 4 we give an infinite family of strongly non-clique-separable graphs that have a chromatic factorisation and give a certificate of factorisation for these factorisations.

2 Graphs having a Chromatic Factorisation

In this section we consider the number of triangles in strongly non-clique-separable graphs that have chromatic factorisations.

Lemma 1 *If G is a strongly non-clique-separable graph and $P(G, \lambda)$ satisfies (1) with chromatic factors H_1 and H_2, then either H_1 or H_2 does not contain a clique of size at least r.*

Proof Suppose, in order to obtain a contradiction, both H_1 and H_2 contain an r-clique. As H_1 and H_2 are chromatic factors, neither of these graphs is isomorphic to K_r. So the graph obtained by identifying an r-clique in H_1 and an r-clique in H_2 is chromatically equivalent to G. But then G, a strongly non-clique-separable graph, is chromatically equivalent to a clique-separable graph, a contradiction. $\square$

Corollary 2 *If G is a strongly non-clique-separable graph and $P(G, \lambda)$ satisfies (1), then $r \geq 3$.*

Proof Let H_1 and H_2 be the chromatic factors of G. The proof considers the cases $r = 1$ and $r = 2$.

Suppose $r = 1$. Then both H_1 and H_2 have at least one vertex, and thus a clique of size one, which contradicts Lemma 1.

Suppose $r = 2$. Now as $\chi(H_i) \geq r = 2$ for $i = 1, 2$, both H_1 and H_2 have at least one edge. Thus each of these graphs contain a clique of size at least two, which contradicts Lemma 1. $\square$

The Stirling number of the first kind is denoted by $s(n, k)$ where $s(n, k)$ is the coefficient of λ^k in the expansion of the falling factorial $\lambda(\lambda - 1) \ldots (\lambda - n + 1)$. The Stirling number $s(r, r-2)$ is the coefficient of λ^{r-2} in the expansion of $P(K_r, \lambda)$, and we use this in the proof of Theorem 4.

Fact 3 *The Stirling number $s(r, r-2)$ is*

$$\begin{aligned} s(r, r-2) &= \sum_{i=2}^{r-1} \left(i \times \sum_{j=1}^{i-1} j \right) = \sum_{i=2}^{r-1} \left(i \times \left(\frac{i(i-1)}{2} \right) \right) = \sum_{i=2}^{r-1} \left(\frac{i^2(i-1)}{2} \right) \\ &= \frac{1}{2} \left(\left(\frac{(r-1)^4}{4} + \frac{(r-1)^3}{2} + \frac{(r-1)^2}{4} \right) - \left(\frac{(r-1)^3}{3} + \frac{(r-1)^2}{2} + \frac{r-1}{6} \right) \right) \\ &= \frac{r^4}{8} - \frac{5r^3}{12} + \frac{3r^2}{8} - \frac{r}{12}. \end{aligned}$$

We now show that, if G has a chromatic factorisation, its number of triangles behaves as if G is clique-separable, even if it is not. This will be used later, in Section 4.

Theorem 4 *If $P(G, \lambda) = P(H_1, \lambda)P(H_2, \lambda)/P(K_r, \lambda)$, $r \geq 3$, then G has $t_1 + t_2 - \binom{r}{3}$ triangles, where t_1 and t_2 are the number of triangles in H_1 and H_2 respectively.*

Proof The first three terms of the chromatic polynomial are

$$P(G, \lambda) = \lambda^n - m\lambda^{n-1} + \left(\binom{m}{2} - t \right) \lambda^{n-2} + \ldots$$

where the graph G has n vertices, m edges and t triangles. Let n_i and m_i be the number of vertices and edges in graph H_i, $i = 1, 2$. Then

$$
\begin{aligned}
P(G,\lambda) &= \frac{P(H_1,\lambda)P(H_2,\lambda)}{P(K_r,\lambda)} \\
&= \frac{(\lambda^{n_1} - m_1\lambda^{n_1-1} + (\binom{m_1}{2} - t_1)\lambda^{n_1-2} + \ldots)(\lambda^{n_2} - m_2\lambda^{n_2-1} + (\binom{m_2}{2} - t_2)\lambda^{n_2-2} + \ldots)}{P(K_r,\lambda)}
\end{aligned}
$$

which by Fact 3 becomes

$$
\begin{aligned}
P(G,\lambda) &= \frac{\lambda^{n_1+n_2} - (m_1+m_2)\lambda^{n_1+n_2} + (\binom{m_1}{2} + \binom{m_2}{2} - m_1m_2 - (t_1+t_2))\lambda^{n_1+n_2-2} + \ldots}{\lambda(\lambda^{r-1} - \frac{r(r-1)}{2}\lambda^{r-2} + (\frac{r^4}{8} - \frac{5r^3}{12} + \frac{3r^2}{8} - \frac{r}{12})\lambda^{r-3} + \ldots)} \\
&= \lambda^{n_1+n_2-r} - (m_1 + m_2 - \frac{r(r-1)}{2})\lambda^{n_1+n_2-r-1} + \\
&\quad \left(\binom{m_1}{2} + \binom{m_2}{2} + m_1m_2 - (t_1+t_2) - \frac{r^4}{8} + \frac{5r^3}{12} - \frac{3r^2}{8} + \frac{r}{12}\right. \\
&\quad \left. - (m_1+m_2)\frac{r(r-1)}{2} + \frac{r^2(r-1)^2}{4}\right)\lambda^{n_1+n_2-r-2} + \ldots \qquad (2)
\end{aligned}
$$

Now from (2) G has $m_1 + m_2 - r(r-1)/2$ edges. Let t_G be the number of triangles in G. Then

$$
\begin{aligned}
\binom{m_1 + m_2 - \frac{r(r-1)}{2}}{2} - t_G &= \binom{m_1}{2} + \binom{m_2}{2} + m_1m_2 - (t_1+t_2) \\
&\quad - \frac{r^4}{8} + \frac{5r^3}{12} - \frac{3r^2}{8} + \frac{r}{12} - (m_1+m_2)\frac{r(r-1)}{2} + \frac{r^2(r-1)^2}{4}
\end{aligned}
$$

So

$$
\begin{aligned}
t_G &= \binom{m_1 + m_2 - \frac{r(r-1)}{2}}{2} - \binom{m_1}{2} - \binom{m_2}{2} - m_1 m_2 + (t_1 + t_2) \\
&\quad + \frac{r^4}{8} - \frac{5r^3}{12} + \frac{3r^2}{8} - \frac{r}{12} + (m_1 + m_2)\frac{r(r-1)}{2} - \frac{r^2(r-1)^2}{4} \\
&= \binom{m_1}{2} + \binom{m_2}{2} + m_1 m_2 - (m_1 + m_2)\frac{r(r-1)}{2} + \frac{r^2(r-1)^2}{8} \\
&\quad + \frac{r(r-1)}{4} - \binom{m_1}{2} - \binom{m_2}{2} - m_1 m_2 + (t_1 + t_2) + \frac{r^4}{8} - \frac{5r^3}{12} \\
&\quad + \frac{3r^2}{8} - \frac{r}{12} + (m_1 + m_2)\frac{r(r-1)}{2} - \frac{r^2(r-1)^2}{4} \\
&= \frac{r(r-1)}{4} + (t_1 + t_2) + \frac{r^4}{8} - \frac{5r^3}{12} + \frac{3r^2}{8} - \frac{r}{12} - \frac{r^2(r-1)^2}{8} \\
&= (t_1 + t_2) - \frac{r(r-1)(r-2)}{6} \\
&= t_1 + t_2 - \binom{r}{3} \qquad \Box
\end{aligned} \tag{3}
$$

Now by Lemma 1 one of the chromatic factors of a chromatic factorisation of a strongly non-clique-separable graph graph has no r-clique. We now consider the case where one of these chromatic factors, say H_1, has no triangle.

Corollary 5 *If $P(G, \lambda)$ satisfies (1) with $r = 3$ and G is a strongly non-clique-separable graph, then exactly one of H_1 or H_2 has at least one triangle. If H_2 is the chromatic factor that has at least one triangle, then H_2 has exactly one more triangle than G.*

Proof By Lemma 1 as G is not chromatically equivalent to any clique-separable graph, one of the chromatic factors, say H_1, contains no triangles. Thus (3) becomes

$$t(G) = t_2 - \binom{3}{3} = t_2 - 1. \tag{4}$$

So H_2 contains exactly one more triangle that G, and certainly has at least one triangle. $\Box$

3 A Certificate of Factorisation for $r = 3$

We now give some more specific results on the number of triangles in graphs that satisfy a particular certificate of factorisation. In Section 4 these results are used to demonstrate that an infinite family of graphs have chromatic factorisations that satisfy this certificate.

A certificate of factorisation is a sequence of steps that explains the chromatic factorisation of a given chromatic polynomial. A schema for some certificates and a number of classes of certificates were given in [5]. In this section we present a certificate of factorisation belonging to this schema for the case where:

- $r = 3$ in (1), that is $P(G,\lambda) = P(H_1,\lambda)P(H_2,\lambda)/P(K_3,\lambda)$,
- G is a non-clique-separable graph with connectivity 2, and
- there exists $uv \notin E(G)$ such that such that $G + uv$ and G/uv are both clique-separable graphs each having H_1 as a chromatic factor.

Without loss of generality, it is assumed that:

- H_1 contains no triangles and
- H_2 contains at least one triangle by Corollary 5.

This case is illustrated in Figure 1. (In this figure we use the standard approach of representing the chromatic polynomial of a graph by the graph itself.) In this case $G+uv$

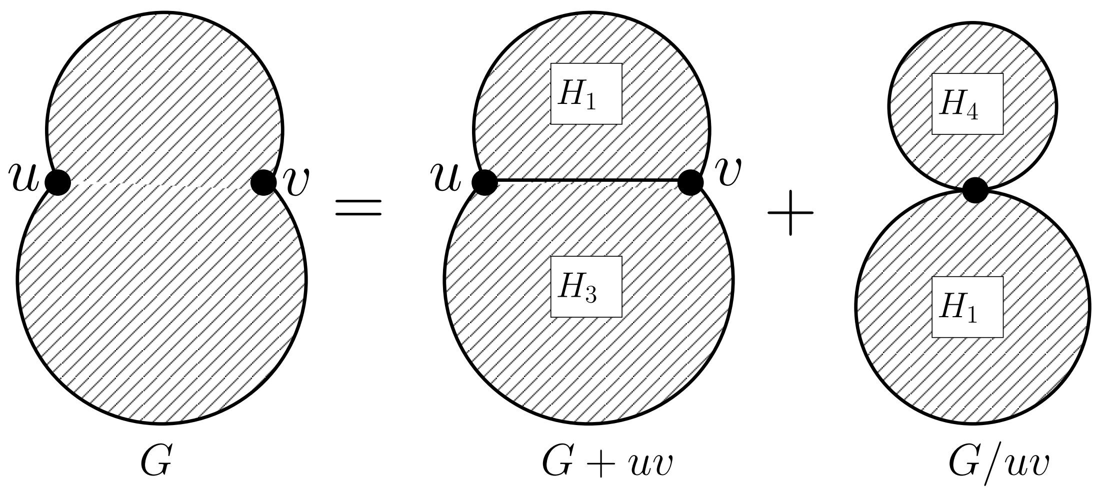

Figure 1: $P(G,\lambda) = P(G + uv,\lambda) + P(G/uv,\lambda)$

is isomorphic to a 2-gluing of H_1 and some graph H_3, and G/uv is isomorphic to a 1-gluing of H_1 and some graph H_4. Thus,

$$\begin{aligned} P(G,\lambda) &= P(G + uv,\lambda) + P(G/uv,\lambda) \\ &= \frac{P(H_1,\lambda)P(H_3,\lambda)}{P(K_2,\lambda)} + \frac{P(H_1,\lambda)P(H_4,\lambda)}{P(K_1,\lambda)}. \end{aligned} \tag{5}$$

Now, H_1 and H_3 in $G + uv$ contract to H_4 and H_1 respectively in G/uv (see Figure 1). Thus, it is clear that

$$H_1 \cong H_3/uv \tag{6}$$

and

$$H_4 \cong H_1/uv. \tag{7}$$

Thus (5) becomes

$$\begin{aligned}
P(G,\lambda) &= \frac{P(H_1,\lambda)P(H_3,\lambda)}{P(K_2,\lambda)} + \frac{P(H_1,\lambda)P(H_1/uv,\lambda)}{P(K_1,\lambda)} \\
&= P(H_1,\lambda)\left(\frac{P(H_3,\lambda)}{P(K_2,\lambda)} + \frac{P(H_1/uv,\lambda)}{P(K_1,\lambda)}\right) \\
&= \frac{P(H_1,\lambda)}{P(K_3,\lambda)}\left(\frac{P(K_3,\lambda)P(H_3,\lambda)}{P(K_2,\lambda)} + \frac{P(H_1/uv,\lambda)P(K_3,\lambda)}{P(K_1,\lambda)}\right) \\
&= \frac{P(H_1,\lambda)}{P(K_3,\lambda)}\left(\frac{P(K_3,\lambda)P(H_3,\lambda)}{P(K_2,\lambda)} + \frac{P(H_1/uv,\lambda)P(K_3,\lambda)P(K_2,\lambda)}{P(K_2,\lambda)P(K_1,\lambda)}\right). \qquad (8)
\end{aligned}$$

Now if there exists $wx \notin E(H_2)$ such that $H_2 + wx$ is isomorphic to a 2-gluing of H_3 and K_3, and H_2/wx is isomorphic to a $(2,1)$-gluing of the graphs H_1/uv, K_3 and K_2, then (8) becomes

$$\begin{aligned}
P(G,\lambda) &= \frac{P(H_1,\lambda)}{P(K_3,\lambda)}\left(P(H_2+wx,\lambda) + P(H_2/wx,\lambda)\right) \\
&= \frac{P(H_1,\lambda)P(H_2,\lambda)}{P(K_3,\lambda)}.
\end{aligned}$$

Thus the certificate for such a factorisation is as follows:

$$\begin{aligned}
P(G,\lambda) &= P(G+uv,\lambda) + P(G/uv,\lambda) \\
&= \frac{P(H_1,\lambda)P(H_3,\lambda)}{P(K_2,\lambda)} + \frac{P(H_1,\lambda)P(H_1/uv,\lambda)}{P(K_1,\lambda)} \\
&= P(H_1,\lambda)\left(\frac{P(H_3,\lambda)}{P(K_2,\lambda)} + \frac{P(H_1/uv,\lambda)}{P(K_1,\lambda)}\right) \\
&= \frac{P(H_1,\lambda)}{P(K_3,\lambda)}\left(\frac{P(K_3,\lambda)P(H_3,\lambda)}{P(K_2,\lambda)} + \frac{P(H_1/uv,\lambda)P(K_3,\lambda)}{P(K_1,\lambda)}\right) \\
&= \frac{P(H_1,\lambda)}{P(K_3,\lambda)}\left(\frac{P(K_3,\lambda)P(H_3,\lambda)}{P(K_2,\lambda)} + \frac{P(H_1/uv,\lambda)P(K_3,\lambda)P(K_2,\lambda)}{P(K_2,\lambda)P(K_1,\lambda)}\right) \\
&= \frac{P(H_1,\lambda)}{P(K_3,\lambda)}\left(P(H_2+wx,\lambda) + P(H_2/wx,\lambda)\right) \qquad (9) \\
&= \frac{P(H_1,\lambda)P(H_2,\lambda)}{P(K_3,\lambda)}.
\end{aligned}$$

Certificate 1

In the remainder of this section, some properties of graphs with chromatic factorisations that satisfy Certificate 1 will be examined.

Theorem 6 *If G is a non-clique-separable graph that has a chromatic factorisation that satisfies Certificate 1 and the chromatic factor H_1 contains no triangles, then G contains no triangles.*

Proof Now H_1 contains no triangles by assumption. But $H_1 \cong H_3/uv$, $uv \in E(H_3)$, $uv \notin E(G)$, so H_3/uv contains no triangles. Thus any triangle in H_3 must contain the edge uv, and $H_3 \setminus uv$ contains no triangles.

Recall $G + uv$ is the graph obtained by a 2-gluing of H_1 and H_3 on edge uv. Now H_1 and $H_3 \setminus uv$ contain no triangles. It follows that G contains no triangles. □

An immediate consequence of Theorem 6 is

Theorem 7 *If the chromatic factor H_1 in Certificate 1 contains no triangles, then the chromatic factor H_2 in Certificate 1 contains exactly one triangle.*

Proof By Corollary 5, as both G and H_1 are triangle-free, $t_2 = 1$. □

In summary, some necessary properties for graphs, G, H_1, H_2, H_3, satisfying Certificate 1 are:

- G contains no triangles,
- H_1 contains no triangles,
- H_2 contains exactly one triangle,
- $\min\{\chi(H_1), \chi(H_2)\} \geq 3$,
- $H_1 \cong H_3/uv$,
- $H_2 + wx$ is isomorphic to a 2-gluing of K_3 and H_3.

4 A Factorisable Family

In this section we show that there exists an infinite family of strongly non-clique-separable graphs that have chromatic factorisations that satisfy Certificate 1. These have $H_1 = C_{2n+1}$, $n \geq 2$, which may be considered the simplest graphs containing no triangles and with chromatic number at least three. We then show that graphs belonging to this infinite family are the only graphs that have a chromatic factorisation that satisfies Certificate 1 where C_{2n+1}, $n \geq 2$, is a chromatic factor.

Theorem 8 *There exists an infinite family of graphs $\mathcal{G}$ such that every $G \in \mathcal{G}$ satisfies Certificate 1 with $H_1 = C_{2n+1}$, $n \geq 2$.*

Proof Let $n \geq 2$ and let $G \in \mathcal{G}$ be the graph (a K_4-subdivision) with $V = \{0, 1, \ldots, 4n\}$ and $E = \{(i, i+1) : 0 \leq i \leq 4n-1 \cup \{(0, 4n), (0, 2n+1), (2n, 4n)\}$ (see Figure 2). Let

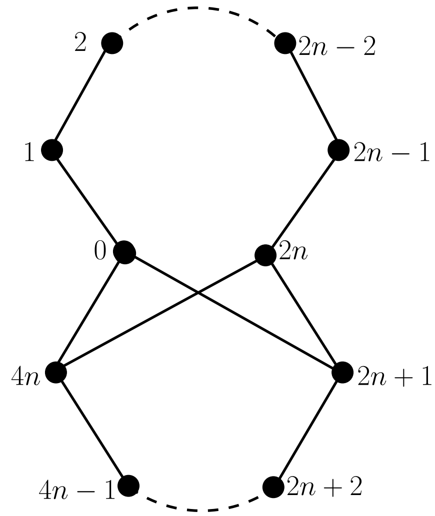

Figure 2: Graph G isomorphic to $C_{4n+1} + (0, 2n+1) + (2n, 4n)$, $n \geq 2$.

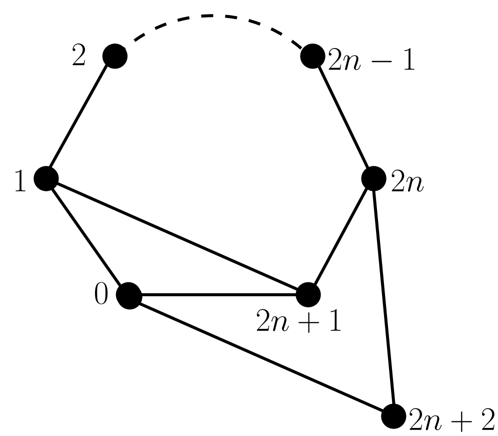

Figure 3: Graph H_2.

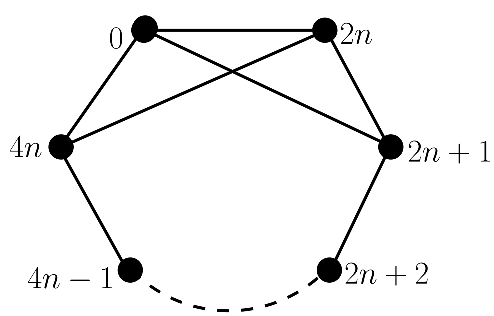

Figure 4: Graph H_3

$H_1 = C_{2n+1}$, H_2 be the graph in Figure 3 and $H_3 = C_{2n+2} + (0, 2n+1) + (2n, 4n)$ as displayed in Figure 4. By addition-identification,

$$P(G, \lambda) = P(G + (0, 2n), \lambda) + P(G/(0, 2n), \lambda). \tag{10}$$

Now $G + (0, 2n)$ is isomorphic to a 2-gluing of $H_1 = C_{2n+1}$ and $H_3 = C_{2n+2} + (2n, 4n) + (0, 2n+1)$, and $G/(0, 2n)$ is isomorphic to a 1-gluing of C_{2n} and C_{2n+1}, so (10) becomes

$$\begin{aligned}
P(G, \lambda) &= \frac{P(C_{2n+1}, \lambda)P(H_3, \lambda)}{P(K_2, \lambda)} + \frac{P(C_{2n}, \lambda)P(C_{2n+1}, \lambda)}{P(K_1, \lambda)} \\
&= P(C_{2n+1}, \lambda)\left(\frac{P(H_3, \lambda)}{P(K_2, \lambda)} + \frac{P(C_{2n}, \lambda)}{P(K_1, \lambda)}\right) \\
&= \frac{P(C_{2n+1}, \lambda)}{P(K_3, \lambda)}\left(\frac{P(H_3, \lambda)P(K_3, \lambda)}{P(K_2, \lambda)} + \frac{P(C_{2n}, \lambda)P(K_3, \lambda)}{P(K_1, \lambda)}\right) \\
&= \frac{P(C_{2n+1}, \lambda)}{P(K_3, \lambda)}\left(\frac{P(H_3, \lambda)P(K_3, \lambda)}{P(K_2, \lambda)} + \frac{P(C_{2n}, \lambda)P(K_3, \lambda)P(K_2, \lambda)}{P(K_2, \lambda)P(K_1, \lambda)}\right). \qquad (11)
\end{aligned}$$

Now $H_2 + (0, 2n)$ is isomorphic to the 2-gluing of $H_3 = C_{2n+2} + (2n, 4n) + (0, 2n+1)$ and K_3 on the edge $(2n, 4n)$. Furthermore $H_2/(0, 2n)$ is isomorphic to the graph obtained by a $(2, 1)$-gluing of C_{2n}, K_3 and K_2. So (11) becomes

$$\begin{aligned}
P(G, \lambda) &= \frac{P(C_{2n+1}, \lambda)}{P(K_3, \lambda)}\left(P(H_2 + (0, 2n), \lambda) + P(H_2/(0, 2n), \lambda)\right) \\
&= \frac{P(C_{2n+1}, \lambda)P(H_2, \lambda)}{P(K_3, \lambda)}. \qquad (12)
\end{aligned}$$

Thus, Certificate 1 is a certificate of factorisation for $G \in \mathcal{G}$ with $H_1 = C_{2n+1}$, $n \geq 2$, and H_2 being the graph in Figure 3. $\square$

Lemma 9 *All graphs in the family $\mathcal{G}$ are strongly non-clique-separable graphs.*

Proof It is clear that any $G \in \mathcal{G}$ is a non-clique-separable graph (see Figure 2). In fact each G is isomorphic to $K_4(1, 1, 1, 1, 2n-1, 2n)$, the graph obtained by replacing two disjoint edges in K_4 by paths of length $2n-1$ and $2n$ respectively. As the graph $K_4(s, s, s, s, t, u)$, for $t, u > s$, is chromatically unique [4], each $G \in \mathcal{G}$ is chromatically unique. Thus all graphs in this family are strongly non-clique-separable. $\square$

A specialisation of Certificate 1 for $G \in \mathcal{G}$ is given in Certificate 2. In this certificate $H_1 \cong C_{2n+1}$, $n \geq 2$, and H_2 is the graph in Figure 3. We now show that any Certificate 1 factorisation with $H_1 \ncong K_3$ an odd cycle must have this form.

$$
\begin{aligned}
P(G,\lambda) &= P(G+(0,2n),\lambda) + P(G/(0,2n),\lambda) \\
&= \frac{P(C_{2n+1},\lambda)P(H_3,\lambda)}{P(K_2,\lambda)} + \frac{P(C_{2n},\lambda)P(C_{2n+1},\lambda)}{P(K_1,\lambda)} \\
&= P(C_{2n+1},\lambda)\left(\frac{P(H_3,\lambda)}{P(K_2,\lambda)} + \frac{P(C_{2n},\lambda)}{P(K_1,\lambda)}\right) \\
&= \frac{P(C_{2n+1},\lambda)}{P(K_3,\lambda)}\left(\frac{P(H_3,\lambda)P(K_3,\lambda)}{P(K_2,\lambda)} + \frac{P(C_{2n},\lambda)P(K_3,\lambda)}{P(K_1,\lambda)}\right) \\
&= \frac{P(C_{2n+1},\lambda)}{P(K_3,\lambda)}\left(\frac{P(H_3,\lambda)P(K_3,\lambda)}{P(K_2,\lambda)} + \frac{P(C_{2n},\lambda)P(K_3,\lambda)P(K_2,\lambda)}{P(K_2,\lambda)P(K_1,\lambda)}\right) \\
&= \frac{P(C_{2n+1},\lambda)}{P(K_3,\lambda)}\left(P(H_2+(0,2n),\lambda) + P(H_2/(0,2n,\lambda))\right) \\
&= \frac{P(C_{2n+1},\lambda)P(H_2,\lambda)}{P(K_3,\lambda)}
\end{aligned}
$$

Certificate 2.
A specialisation of Certificate 1 for $G \in \mathcal{G}$ where G is the graph in Figure 2, H_2 is the graph in Figure 3 and H_3 is the graph in Figure 5.

Theorem 10 *If G is a strongly non-clique-separable graph and $P(G,\lambda)$ has a chromatic factorisation that satisfy Certificate 1 with $P(H_1,\lambda) = P(C_{2n+1},\lambda)$, then H_2 is isomorphic to the graph in Figure 3 and $P(G,\lambda)$ has the chromatic factorisation given in Certificate 2.*

Proof Let $H_1 = C_{2n+1}$. Suppose there exists a non-clique-separable graph G and graph H_2 such that

$$
P(G,\lambda) = \frac{P(C_{2n+1},\lambda)P(H_2,\lambda)}{P(K_3,\lambda)} \tag{13}
$$

and $P(G,\lambda)$ has a chromatic factorisation in the form stated in Certificate 1 for some $uv \notin E(G)$, $wx \notin E(H_2)$ and graph H_3. Then by Theorem 6 the graph G contains no triangles, and by Theorem 7 the graph H_2 contains exactly one triangle.

Now $G+uv$ is a 2-gluing of graphs C_{2n+1} and H_3 on the edge uv. Furthermore G and C_{2n+1} contain no triangles, so the only triangles in $G+uv$ must be in H_3. From (6), $H_3/uv \cong C_{2n+1}$, which is triangle-free. Thus, all triangles in H_3 must contain the edge uv. Hence, there are three possibilities for H_3. It is the cycle graph C_{2n+2} with one the following sets of additional edges where a and b are the vertices adjacent to u and v, respectively, in C_{2n+2}:

- $\{av\}$,
- $\emptyset$, that is, no additional edges,
- $\{av, bu\}$.

These graphs are displayed in Figure 5. Note that $C_{2n+2} + av \cong C_{2n+2} + bu$.

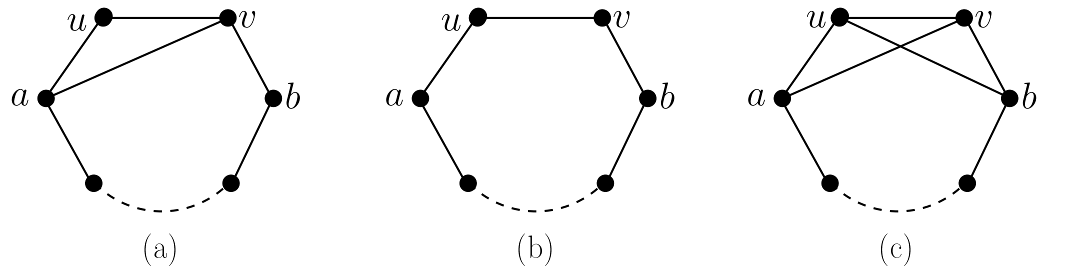

Figure 5: Three candidate graphs for H_3: (a) $C_{2n+2}+av$, (b) C_{2n+2} and (c) $C_{2n+2}+av+bu$.

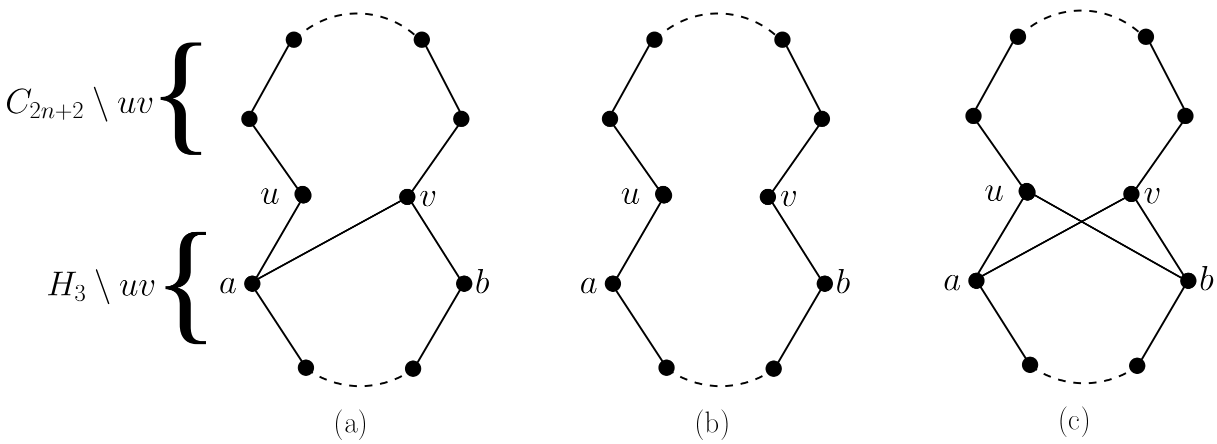

Figure 6: G where H_3 is: (a) $C_{2n+2} + av$, (b) C_{2n+2} and (c) $C_{2n+2} + av + bu$.

Case 1 Suppose $H_3 \cong C_{2n+2} + av$. Then $G + uv$ is isomorphic to a 2-gluing of C_{2n+1} and $C_{2n+2} + av$. But this means that G (see Figure 6(a)) is isomorphic to a 2-gluing of C_{2n+2} and C_{2n+1}, which makes G clique-separable, a contradiction.

Case 2 Suppose $H_3 \cong C_{2n+2}$. Now in order to satisfy (9) in Certificate 1, there must exist a 2-gluing of C_{2n+2} and K_3 that is isomorphic to $H_2 + e$ for some $e \notin E(H_2)$. Let H' be the 2-gluing of C_{2n+2} and K_3 on some edge $cd \in E(C_{2n+2})$. Then there exists $e \in E(H')$ such that $H' \setminus e \cong H_2$. As both H_2 and H' each have exactly one triangle, e must be some edge on C_{2n+2} excluding cd. But then H'/e is isomorphic to the 2-gluing of C_{2n+1} and K_3, which is not chromatically equivalent to any 1-gluing of K_3 and C_{2n} as required by Certificate 1, a contradiction.

Case 3 Finally suppose $H_3 \cong C_{2n+2} + av + bu$. Now in order to satisfy (9) in Certificate 1, there must exist a 2-gluing of $C_{2n+2} + av + bu$ and K_3 that is isomorphic to $H_2 + e$ for some $e \notin E(H_2)$.

Firstly, suppose H' is the graph obtained by a 2-gluing of H_3 and K_3 on the edge uv. Then H' contains three triangles each sharing the common edge uv. In order to satisfy (9) there exists $e \in E(H')$ such that $H' \setminus e \cong H_2$. But H_2 contains exactly one triangle. Now if e is an edge not in any of the triangles in H' then $H' \setminus e$ contains 3 triangles, a contradiction. In addition, if e is an edge in any of the triangles in H', then either $e = uv$ and $H' \setminus e$ contains no triangles, or $e \neq uv$ and $H' \setminus e$ contains 2 triangles, also a contradiction. Thus $H' \setminus e \ncong H_2$ for all $e \in E(H')$. Thus H' cannot be obtained by a 2-gluing of $C_{2n+2} + av + bu$ and K_3 on edge uv.

Suppose H' is obtained by a 2-gluing of $C_{2n+2} + av + bu$ and K_3 on an edge $e \in E(H_3) \setminus \{uv, bv, av, au, bu\}$. Then H' contains 3 triangles, two of which share a common edge uv, with the third being edge-disjoint from the others. Thus, the only edge whose removal from H' will reduce the number of triangles to one is uv. However, H'/uv is isomorphic to a 2-gluing of C_{2n+1} and K_3, which is not chromatically equivalent to a 1-gluing of K_3 and C_{2n} as required by Certificate 1. Thus, H' cannot be obtained by a 2-gluing of $C_{2n+2} + av + bu$ and K_3 on any edge $e \in E(H_3) \setminus \{uv, bv, av, au, bu\}$.

Suppose H' is obtained by a 2-gluing of $C_{2n+2} + av + bu$ and K_3 on an edge $e \in \{bv, av, au, bu\}$, say bv without loss of generality. Then H' contains 3 triangles. The only edge whose removal reduces the number of triangles in H' to one is either uv or bv. Now H'/uv is isomorphic to a 2-gluing of C_{2n+1} and K_3 which is not chromatically equivalent to a 1-gluing of K_3 and C_{2n} as required by Certificate 1 and thus is not suitable. However, H'/bv is isomorphic to a $(2, 1)$-gluing of graphs C_{2n}, K_3 and K_2, which is chromatically equivalent to a 1-gluing of K_3 and C_{2n} as required by Certificate 1.

Thus $C_{2n+2} + av + bu$ is the only H_3 and $H' \setminus uv$ is the only H_2 satisfying Certificate 1 (up to isomorphism).

Now $G + uv$ is isomorphic to a 2-gluing of C_{2n+1} and $C_{2n+2} + av + bu$ on the edge uv. Thus G is isomorphic to the graph with $V = \{0, 1, \ldots, 4n\}$ and $E = \{(i, i+1) : 0 \leq i \leq 4n - 1\} \cup \{(0, 2n+1), (2n, 4n)\}$ (see Figure 2).

Thus the chromatic factorisation given in Certificate 2 is the only chromatic factorisation satisfying Certificate 1 where $H_1 = C_{2n+1}$, $n \geq 2$. □

5 Conclusion

In [5] we introduced the idea of certificates of factorisation in order to explain the chromatic factorisations of strongly non-clique-separable graphs. We noted that graphs that have chromatic factorisations that satisfy some certificate, also have common structural properties. In this article we gave an example of a family of such graphs.

We constructed an infinite family of strongly non-clique-separable graphs that have chromatic factorisations. Graphs in this family, $K_4(1, 1, 1, 1, 2n-1, 2n)$ where $n \geq 2$, are not only almost clique-separable, but are also triangle-free. A certificate of factorisation was given for graphs belonging to this family. Some properties of the number of triangles in graphs that have chromatic factorisations and in their chromatic factors were proved. These properties were used to show that members of this infinite family of graphs are the only graphs that have chromatic factorisations that satisfy Certificate 1 when an odd cycle (excluding K_3) is a chromatic factor.

Not all strongly non-clique-separable graphs that have chromatic factorisations belong to the infinite family we have constructed. It would be interesting to determine other properties of strongly non-clique-separable graphs having chromatic factorisations.

An interesting problem is to determine the length of a minimal certificate of factorisation for a given graph. It is clear that any clique-separable graph has a minimal certificate of factorisation of length 1, and any quasi-clique-separable graph that is not clique-separable has a minimal certificate of factorisation of length 2. In this paper we gave a certificate of factorisation of length 8 for graphs belonging to the family $K_4(1,1,1,1,2n-1,2n)$. However, it is not known if this is the shortest certificate of factorisation for graphs belonging to this family.

Another related question concerns the length of the shortest certificate of factorisation for strongly non-clique-separable graphs. In [5] we gave several certificates for a number of classes of strongly non-clique-separable graphs. The shortest certificate given had seven steps. It is an open question whether shorter certificates of factorisation for strongly non-clique-separable graphs exist.

Acknowledgement

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