Maximum Multiplicity of a Root of the Matching Polynomial of a Tree and Minimum Path Cover

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Abstract

We give a necessary and sufficient condition for the maximum multiplicity of a root of the matching polynomial of a tree to be equal to the minimum number of vertex disjoint paths needed to cover it.

1 Introduction

All the graphs in this paper are simple. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$ respectively. A *matching* of a graph G is a set of pairwise disjoint edges of G. Recall that for a graph G on n vertices, the *matching polynomial* $\mu(G,x)$ of G is given by

$$\mu(G,x) = \sum_{k\geq 0} (-1)^k p(G,k) x^{n-2k},$$

where $p(G,k)$ is the number of matchings with k edges in G. Let $\mathrm{mult}(\theta, G)$ denote the multiplicity of θ as a root of $\mu(G,x)$.

The following results are well known. The proofs can be found in [2, Theorem 4.5 on p. 102].

Theorem 1.1. *The maximum multiplicity of a root of the matching polynomial $\mu(G,x)$ is at most the minimum number of vertex disjoint paths needed to cover the vertex set of G.*

Consequently,

Theorem 1.2. *If G has a Hamiltonian path, then all roots of its matching polynomial are simple.*

The above is the source of motivation for our work. It is natural to ask when does equality holds in Theorem 1.1. In this note, we give a necessary and sufficient condition for the maximum multiplicity of a root of the matching polynomial of a tree to be equal to the minimum number of vertex disjoint paths needed to cover it. Before stating the main result, we require some terminology and basic properties of matching polynomials.

It is well known that the roots of the matching polynomial are real. If $u \in V(G)$, then $G \setminus u$ is the graph obtained from G by deleting the vertex u and the edges of G incident to u. It is known that the roots of $G \setminus u$ *interlace* those of G, that is, the multiplicity of a root changes by at most one upon deleting a vertex from G. We refer the reader to [2] for an introduction to matching polynomials.

Lemma 1.3. *Suppose θ is a root of $\mu(G,x)$ and u is a vertex of G. Then*

$$\operatorname{mult}(\theta, G) - 1 \leq \operatorname{mult}(\theta, G \setminus u) \leq \operatorname{mult}(\theta, G) + 1.$$

As a consequence of Lemma 1.3, we can classify the vertices in a graph by assigning a 'sign' to each vertex (see [3]).

Definition 1.4. Let θ be a root of $\mu(G,x)$. For any vertex $u \in V(G)$,

- u is θ-*essential* if $\operatorname{mult}(\theta, G \setminus u) = \operatorname{mult}(\theta, G) - 1$,
- u is θ-*neutral* if $\operatorname{mult}(\theta, G \setminus u) = \operatorname{mult}(\theta, G)$,
- u is θ-*positive* if $\operatorname{mult}(\theta, G \setminus u) = \operatorname{mult}(\theta, G) + 1$.

Clearly, if $\operatorname{mult}(\theta, G) = 0$ then there are no θ-essential vertices since the multiplicity of a root cannot be negative. Nevertheless, it still makes sense to talk about θ-neutral and θ-positive vertices when $\operatorname{mult}(\theta, G) = 0$. The converse is also true, i.e. any graph G with $\operatorname{mult}(\theta, G) > 0$ must have at least one θ-essential vertex. This was proved in [3, Lemma 3.1].

A further classification of vertices plays an important role in establishing some structural properties of a graph:

Definition 1.5. Let θ be a root of $\mu(G,x)$. For any vertex $u \in V(G)$, u is θ-*special* if it is not θ-essential but has a neighbor that is θ-essential.

If G is connected and not all of its vertices are θ-essential, then G must contain a θ-special vertex. It turns out that a θ-special vertex must be θ-positive (see [3, Corollary 4.3]).

We now introduce the following definition which is crucial in describing our main result.

Definition 1.6. Let G be a graph and $\mathcal{Q} = \{Q_1, \ldots, Q_m\}$ be a set of vertex disjoint paths that cover G. Then $\mathcal{Q}$ is said to be (θ, G)-*extremal* if it satisfies the following:

(a) θ is a root of $\mu(Q_i, x)$ for all $i = 1, \ldots, m$;

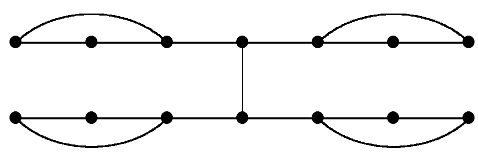
(b) for every edge $e = \{u, v\} \in E(G)$ with $u \in Q_r$ and $v \in Q_s$, $r \neq s$, either u is θ-special in Q_r or v is θ-special in Q_s.

Our main result is the following:

Theorem 1.7. *Let T be a tree and $\mathcal{Q} = \{Q_1, \ldots, Q_m\}$ be a set of vertex disjoint paths covering T. Then m is the maximum multiplicity of a root of the matching polynomial $\mu(T, x)$, say* $\mathrm{mult}(\theta, T) = m$ *for some root θ, if and only if $\mathcal{Q}$ is (θ, T)-extremal.*

The following example shows that Theorem 1.7 cannot be extended to general graphs.

Example 1.8. *Consider the following graph G:*



Let P_7 denote the path on 7 vertices. Note that $\mathrm{mult}(\sqrt{3}, G) = 2$ and $\mu(P_7, x) = x^7 - 6x^5 + 10x^3 - 4x$. By Theorem 1.1, the maximum multiplicity of a root of $\mu(G, x)$ is 2. Also, G can be covered by two paths on 7 vertices. However, $\sqrt{3}$ is not a root of $\mu(P_7, x)$.

2 Basic Properties

In this section, we collect some useful results proved in [2] and [3]. Recall that if $u \in V(G)$, then $G \setminus u$ is the graph obtained from G by deleting vertex u and the edges of G incident to u. We also denote the graph $(G \setminus u) \setminus v$ by $G \setminus uv$. Note that the resulting graph does not depend on the order of which the vertices are deleted.

If $e \in E(G)$, the graph $G - e$ is the graph obtained from G by deleting the edge e. The matching polynomial satisfies the following basic identities, see [2, Theorem 1.1 on p. 2].

Proposition 2.1. *Let G and H be graphs with matching polynomials $\mu(G, x)$ and $\mu(H, x)$, respectively. Then*

(a) $\mu(G \cup H, x) = \mu(G, x)\mu(H, x)$,

(b) $\mu(G, x) = \mu(G - e, x) - \mu(G \setminus uv, x)$ *where $e = \{u, v\}$ is an edge of G,*

(c) $\mu(G, x) = x\mu(G \setminus u, x) - \sum_{v \sim u} \mu(G \setminus uv, x)$ *for any vertex u of G.*

Suppose P is a path in G. Let $G \setminus P$ denote the graph obtained from G by deleting the vertices of P and all the edges incident to these vertices. It is known that the multiplicity of a root decreases by at most one upon deleting a path, see [3, Corollary 2.5].

Lemma 2.2. *For any root θ of $\mu(G, x)$ and a path P in G,*

$$\mathrm{mult}(\theta, G \setminus P) \geq \mathrm{mult}(\theta, G) - 1.$$

If equality holds, we say that the path P is *θ-essential* in G. Godsil [3] proved that if a vertex v is not θ-essential in G, then no path with v as an end point is θ-essential. In other words,

Lemma 2.3. *If P is a θ-essential path in G, then its endpoints are θ-essential in G.*

The next result of Godsil [3, Corollary 4.3] implies that a θ-special vertex must be θ-positive.

Lemma 2.4. *A θ-neutral vertex cannot be joined to any θ-essential vertex.*

3 Gallai-Edmonds Decomposition

It turns out that θ-special vertices play an important role in the Gallai-Edmonds decomposition of a graph. We now define such a decomposition. For any root θ of $\mu(G, x)$, partition the vertex set $V(G)$ as follows:

$$\begin{aligned} D_\theta(G) &= \{u : u \text{ is } \theta\text{-essential in G}\} \\ A_\theta(G) &= \{u : u \text{ is } \theta\text{-special in G}\} \\ C_\theta(G) &= V(G) - D_\theta(G) - A_\theta(G). \end{aligned}$$

We call these sets of vertices the *θ-partition classes* of G. The Gallai-Edmonds Structure Theorem is usually stated in terms of the structure of maximum matchings of a graph with respect to its θ-partition classes when $\theta = 0$. Its proof essentially follows from the following assertions (for more information, see [5, Section 3.2]):

Theorem 3.1 (Gallai-Edmonds Structure Theorem)**.**
Let G be any graph and let $D_0(G)$, $A_0(G)$ and $C_0(G)$ be the 0-partition classes of G.

(i) (*The Stability Lemma*) *Let $u \in A_0(G)$ be a 0-special vertex in G. Then*

- *$v \in D_0(G)$ if and only if $v \in D_0(G \setminus u)$;*
- *$v \in A_0(G)$ if and only if $v \in A_0(G \setminus u)$;*
- *$v \in C_0(G)$ if and only if $v \in C_0(G \setminus u)$.*

(ii) (*Gallai's Lemma*) *If every vertex of G is 0-essential then* $\mathrm{mult}(0, G) = 1$.

For any root θ of $\mu(G,x)$, it was shown by Neumaier [6, Corollary 3.3] that the analogue of Gallai's Lemma holds when G is a tree. A different proof was given by Godsil (see [3, Corollary 3.6]).

Theorem 3.2 ([3], [6]). *Let T be a tree and let θ be a root of $\mu(T,x)$. If every vertex of T is θ-essential then* $\mathrm{mult}(\theta, G) = 1$.

On the other hand, it was proved in [3, Theorem 5.3] that if θ is any root of $\mu(T,x)$ where T is tree and $u \notin D_\theta(T)$, then $v \in D_\theta(T)$ if and only if $v \in D_\theta(T \setminus u)$. It turns out that this assertion is incorrect (see Example 4.2 below). However, using the idea of the proof of Theorem 5.3 in [3], we shall prove the Stability Lemma for trees with any given root of its matching poynomial. Note that the Stability Lemma is a weaker statement than Theorem 5.3 in [3]. Together with Theorem 3.2, this yields the Gallai-Edmonds Structure Theorem for trees with general root θ. Recently, Chen and Ku [1] had proved the Gallai-Edmonds Structure Theorem for general graph with any root θ. However, our proof of the special case for trees, which uses an eigenvector argument, is different from the the one given in [1]. We believe that different proofs can be illuminating. For the sake of completeness, we include the proof in the next section.

Theorem 3.3 (The Stability Lemma for Trees). *Let T be a tree and let θ be a root of $\mu(T,x)$. Let $u \in A_\theta(T)$ be a θ-special vertex in T. Then*

- *$v \in D_\theta(T)$ if and only if $v \in D_\theta(T \setminus u)$;*
- *$v \in A_\theta(T)$ if and only if $v \in A_\theta(T \setminus u)$;*
- *$v \in C_\theta(T)$ if and only if $v \in C_\theta(T \setminus u)$.*

It is well known that the matching polynomial of a graph G is equal to the characteristic polynomial of G if and only if G is a forest. To prove Theorem 3.3, the following characterization of θ-essential vertices in a tree via eigenvectors is very useful. Recall that a vector $f \in \mathbb{R}^{|V(G)|}$ is an eigenvector of a graph G with eigenvalue θ if and only if for every vertex $u \in V(G)$,

$$\theta f(u) = \sum_{v \sim u} f(v). \tag{1}$$

Proposition 3.4 ([6, Theorem 3.4]). *Let T be a tree and let θ be a root of its matching polynomial. Then a vertex u is θ-essential if and only if there is an eigenvector f of T such that $f(u) \neq 0$.*

In fact Proposition 3.4 can be deduced from Lemma 5.1 of [3]. An immediate consequence of Proposition 3.4 is that if f is an eigenvector of T such that $f(u) \neq 0$, then there exists another eigenvector g such that $g(u) = \alpha$ for any given non-zero real number α. Moreover, both g and f have the same support, i.e. $\{i : f(i) \neq 0\} = \{i : g(i) \neq 0\}$.

Corollary 3.5 ([6, Theorem 3.4]). *Let T be a tree. If u is θ-special then it is joined to at least two θ-essential vertices.*

Proof. By definition, u has a θ-essential neighbor, say w. By Proposition 3.4, there exists an eigenvector f of T corresponding to θ such that $f(w) \neq 0$. By Proposition 3.4 again, $f(u) = 0$ and so $\sum_{v \sim u} f(v) = 0$. This implies that $f(v) \neq 0$ on at least two neighbors of u. By Proposition 3.4, both of them are θ-essential. $\square$

The following assertion follows from Theorem 3.2 and Proposition 3.4.

Corollary 3.6 ([6, Corollary 3.3]). *Let T be a tree and let θ be a root of $\mu(T, x)$. Suppose every vertex of T is θ-essential. Then every non-zero θ-eigenvector of T has no zero entries.*

We also require the following partial analogue of the Stability Lemma for general root obtained by Godsil in [3].

Proposition 3.7 ([3, Theorem 4.2]). *Let θ be a root of $\mu(G, x)$ with non-zero multiplicity and let u be a θ-positive vertex in G. Then*

(a) *if v is θ-essential in G then it is θ-essential in $G \setminus u$;*

(b) *if v is θ-positive in G then it is θ-essential or θ-positive in $G \setminus u$;*

(c) *if v is θ-neutral in G then it is θ-essential or θ-neutral in $G \setminus u$.*

4 Proof of the Stability Lemma for Trees

This section is devoted to the proof of Theorem 3.3, which will follow from the following theorem.

Theorem 4.1. *Let T be a tree and let θ be a root of $\mu(T, x)$. Then there exists a θ-eigenvector f of T such that $f(x) \neq 0$ for every θ-essential vertex x in T. Moreover, if v is θ-essential in $T \setminus u$ where u is θ-special in T, then v is θ-essential in T.*

Proof. If every vertex of T is θ-essential, then the result follows from Corollary 3.6. Therefore, we may assume that T has a θ-special vertex, say u. We proceed by induction on the number of vertices.

Suppose $b_1, \dots, b_s$ are all the neighbors of u in T. Then each b_i belongs to different components of $T \setminus u$, say $b_i \in V(C_i)$ where $C_1, \dots C_s$ are components of $T \setminus u$.

First, we partition the set $\{1, \dots, s\}$ as follows:

$$
\begin{aligned}
A &= \{i : b_i \text{ is } \theta\text{-essential in } C_i\}, \\
B &= \{i : i \notin A, \theta \text{ is a root of } \mu(C_i, x)\}, \\
C &= \{1, \dots, s\} \setminus (A \cup B).
\end{aligned}
$$

By the inductive hypothesis, for each $i \in A$, there exists a θ-eigenvector f_i of C_i such that $f_i(x) \neq 0$ for every θ-essential vertex x in C_i. In particular, $f_i(b_i) \neq 0$ for all $i \in A$. By Proposition 3.7, any θ-essential vertex in T is also θ-essential in $T \setminus u$, so for each $i \in A$,

$$f_i(x) \neq 0 \quad \text{if } x \text{ is } \theta\text{-essential in } T \text{ and } x \in V(C_i). \tag{2}$$

For every $i \in A$, choose $\alpha_i \in \mathbb{R}$ such that

$$\alpha_i \neq 0 \quad \text{and} \quad \sum_{i \in A} \alpha_i = 0.$$

Such a choice is always possible since $|A| \geq 2$ by Corollary 3.5. Now, for each $i \in A$, there is an eigenvector g_i of C_i such that $g_i(b_i) = \alpha_i \neq 0$ with both g_i and f_i having the same support. In particular, it follows from (2) that for each $i \in A$,

$$g_i(x) \neq 0 \quad \text{if } x \text{ is } \theta\text{-essential in } T \text{ and } x \in V(C_i). \tag{3}$$

Also, for each $i \in B$, by the inductive hypothesis, we can choose an eigenvector g_i so that $g_i(x) \neq 0$ for every θ-essential vertex x in C_i. By Proposition 3.7 again, (3) also holds for every g_i with $i \in B$. However, note in passing that $g_i(b_i) = 0$ for all $i \in B$ since b_i is not θ-essential in C_i (by Proposition 3.4).

Next, for each $i \in C$, set g_i to be the zero vector on $V(C_i)$. Note that (3) is satisfied vacuously for every g_i with $i \in C$ since there are no θ-essential vertices in the corresponding C_i.

Finally, we extend these g_i's to an eigenvector of T as follows: define $g \in \mathbb{R}^{|V(T)|}$ by

$$g(x) = \begin{cases} g_i(x) & \text{if } x \in V(C_i) \quad \text{for some } i \in A \cup B \cup C, \\ 0 & \text{if } x = u. \end{cases}$$

Since (3) holds for every g_i, we must have $g(x) \neq 0$ for every θ-essential vertex x of T. It is also readily verified that conditions in (1) are satisfied so that g is indeed a θ-eigenvector of T, as desired.

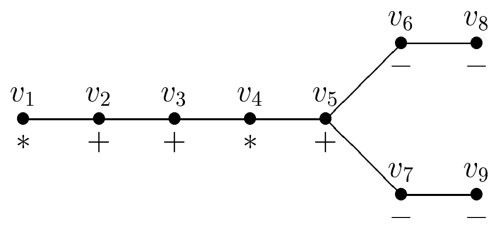
Moreover, by our construction, if x is θ-essential in $T \setminus u$, then $g(x) \neq 0$. By Proposition 3.4, x must be θ-essential in T, proving the second assertion of the theorem

$\square$

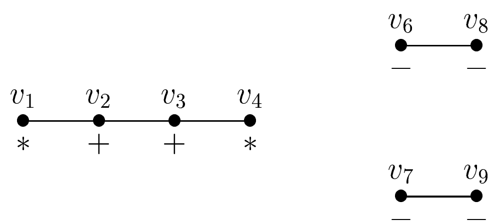
Proof of Theorem 3.3.

Recall that u is a given θ-special vertex of T. By Proposition 3.7, it remains to show that if v is θ-essential in $T \setminus u$ then v is θ-essential in T. But this is just the second assertion of the preceding theorem. This proves the Stability Lemma for trees.

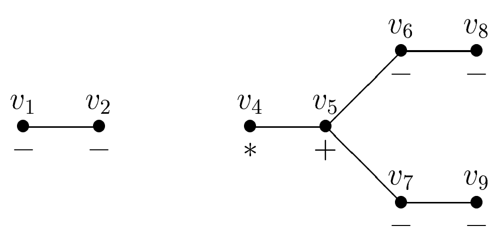
Example 4.2. *Let T be the following tree :*



The vertices are labeled $v_1, \ldots, v_9$ and the symbols $*$, $+$, $-$ below each vertex indicates whether it is θ-neutral or θ-positive or θ-essential respectively where $\theta = 1$. Note that $\mu(T, x) = x^9 - 8x^7 + 20x^5 - 18x^3 + 5x$ and $\mathrm{mult}(1, T) = 1$. As the vertex v_5 is adjacent to a θ-essential vertex, v_5 is θ-special in T. By Theorem 3.3, upon deleting v_5 from T, all other vertices are 'stable' with respect to their θ-partition classes:



However, this is generally not true if we delete a non-special vertex, for example, deleting v_3 from T gives the following:



5 Roots of Paths

In this section, we prove some basic properties about roots of paths.

Lemma 5.1. *Let P_n denote the path on n vertices, $n \geq 2$. Then $\mu(P_n, x)$ and $\mu(P_{n-1}, x)$ have no common root.*

Proof. Note that $\mu(P_1, x) = x$ and $\mu(P_2, x) = x^2 - 1$, and so they have no common root. Suppose $\mu(P_n, x)$ and $\mu(P_{n-1}, x)$ have a common root for some $n \geq 3$. Let n be the least positive integer for which $\mu(P_n, x)$ and $\mu(P_{n-1}, x)$ have a common root, say θ. Then $\mu(P_{n-1}, x)$ and $\mu(P_{n-2}, x)$ have no common root.

First we show that $\theta \neq 0$. Note that for any graph G, the multiplicity of 0 as a root of its matching polynomial is the number of vertices missed by some maximum matching.

Therefore, if n is even then P_n has a perfect matching, so 0 cannot be a root of $\mu(P_n, x)$. It follows that if n is odd then 0 cannot be a root of $\mu(P_{n-1}, x)$. So $\theta \neq 0$.

Let $\{v_1, v_2\}$ be an edge in P_n where v_1 is an endpoint of the path P_n. Note that $P_n \setminus v_1 = P_{n-1}$ and $P_n \setminus v_1 v_2 = P_{n-2}$. By part (c) of Proposition 2.1, $\mu(P_n, x) = x\mu(P_{n-1}, x) - \mu(P_{n-2}, x)$, so θ is a root of $\mu(P_{n-1}, x)$ and $\mu(P_{n-2}, x)$, which is a contradiction. Hence $\mu(P_n, x)$ and $\mu(P_{n-1}, x)$ have no common root. $\square$

Corollary 5.2. *Let θ be a root of $\mu(P_n, x)$. Then the endpoints of P_n are θ-essential.*

Proof. Suppose v is an endpoint of P_n. If v is θ-neutral or θ-positive in P_n then θ is a root of $\mu(P_{n-1}, x)$, a contrary to Lemma 5.1. $\square$

Corollary 5.3. *Let θ be a root of $\mu(P_n, x)$. Then P_n has no θ-neutral vertices. Moreover, every θ-positive vertex in P_n is θ-special.*

Proof. Let v be a vertex of P_n such that it is not θ-essential. In view of Lemma 2.4, it is enough to show that v has a θ-essential neighbor. By Corollary 5.2, v cannot be an endpoint of P_n. Then $P_n \setminus v$ consists of two disjoint paths, say Q_1 and Q_2. Let u_1 be the endpoint of Q_1 such that it is a neighbor of v in P_n.

Consider the paths Q_1 and $Q_1 v$ in P_n. Since v is not θ-essential, by Lemma 2.3, $Q_1 v$ is not θ-essential in P_n. So the path $Q_2 = P_n \setminus Q_1 v$ has θ as a root of its matching polynomial.

If Q_1 is not θ-essential in P_n then the path $P_n \setminus Q_1$ would also have θ as a root of its matching polynomial. Since $P_n \setminus Q_1$ and Q_2 differ by exactly one vertex, this contradicts Lemma 5.1. Therefore, Q_1 is a θ-essential path in P_n. By Lemma 2.3, u_1 must be θ-essential in P_n. Since u_1 is joined to v, we deduce from Lemma 2.4 that v must be θ-special. $\square$

6 Proof of Main Result

We begin by proving the following special case.

Proposition 6.1. *Let T be a tree and* $\mathrm{mult}(\theta, T) = 2$. *Let $\mathcal{Q} = \{Q_1, Q_2\}$ be a set of vertex disjoint paths that cover T. Then $\mathcal{Q}$ is (θ, T)-extremal.*

Proof. Since T is a tree, there is an edge $\{u, v\} \in E(T)$ with $u \in V(Q_1)$ and $v \in V(Q_2)$. By Lemma 2.2, $\mathrm{mult}(\theta, Q_1) = \mathrm{mult}(\theta, T \setminus Q_2) \geq \mathrm{mult}(\theta, T) - 1 = 1$. Similarly, $\mathrm{mult}(\theta, Q_2) \geq 1$. Therefore θ is a root of $\mu(Q_1, x)$ and $\mu(Q_2, x)$.

It remains to show that either u is θ-special in Q_1 or v is θ-special in Q_2. If all vertices in T are θ-essential, then $\mathrm{mult}(\theta, T) = 1$ by Theorem 3.2, which is impossible. So, there must be a θ-special vertex in T, say w.

Suppose $w = u$. We shall prove that w is also θ-special in Q_1. Note that $\mathrm{mult}(\theta, T \setminus w) = 3$. If w is an endpoint of Q_1 then $T \setminus w$ is a disjoint union of two paths $Q_1 \setminus w$ and Q_2. Since $Q_1 \setminus w$ and Q_2 cover $T \setminus w$, we deduce from Theorem 1.1 that $\mathrm{mult}(\theta, T \setminus w) \leq 2$, a contradiction. So w is not an endpoint of Q_1. Removing w from Q_1 would result in two disjoint paths, say R_1 and R_2. Note that $T \setminus w$ is the disjoint union of R_1, R_2 and Q_2. By part (a) of Proposition 2.1, $\mathrm{mult}(\theta, T \setminus w) = \mathrm{mult}(\theta, R_1) + \mathrm{mult}(\theta, R_2) + \mathrm{mult}(\theta, Q_2)$. By Theorem 1.2 and the fact that $\mathrm{mult}(\theta, T \setminus w) = 3$, we conclude that $\mathrm{mult}(\theta, R_1) = \mathrm{mult}(\theta, R_2) = \mathrm{mult}(\theta, Q_2) = 1$. Therefore, $\mathrm{mult}(\theta, Q_1 \setminus w) = \mathrm{mult}(\theta, R_1) + \mathrm{mult}(\theta, R_2) = 2$. This means that w must be θ-positive in Q_1. By Corollary 5.3, w is θ-special in Q_1, as desired.

The case $w = v$ can be proved similarly.

Therefore, we may assume that $w \neq u, v$. We now proceed by induction on the number of vertices. Without loss of generality, we may assume that $w \in V(Q_1)$. As before, it can be shown that w is not an endpoint of Q_1. So removing w from Q_1 results in two disjoint paths, say S_1 and S_2. We may assume that $u \in V(S_2)$. Then $T \setminus w$ is a disjoint union of S_1 and T' where T' is the tree induced by S_2 and Q_2. By Theorem 1.2, $\mathrm{mult}(\theta, S_1) \leq 1$. Since S_2 and Q_2 cover T', by Theorem 1.1, $\mathrm{mult}(\theta, T') \leq 2$. As w is θ-special, $\mathrm{mult}(\theta, T \setminus w) = 3$. By part (a) of Proposition 2.1, $\mathrm{mult}(\theta, T \setminus w) = \mathrm{mult}(\theta, S_1) + \mathrm{mult}(\theta, T')$. We deduce that $\mathrm{mult}(\theta, S_1) = 1$ and $\mathrm{mult}(\theta, T') = 2$. By induction, either u is θ-special in S_2 or v is θ-special in Q_2. In the latter, we are done. Therefore, we may assume that u is θ-special in S_2. So u is not θ-essential in $Q_1 \setminus w$. Since $\mathrm{mult}(\theta, Q_1 \setminus w) = \mathrm{mult}(\theta, S_1) + \mathrm{mult}(\theta, S_2) = 2$, w is θ-positive in Q_1, so w is θ-special in Q_1 by Corollary 5.3. By the Stability Lemma for trees (Theorem 3.3), u is not θ-essential in Q_1. By Corollary 5.3, u is θ-special in Q_1.

Note that the base cases of our induction occur when $w = u$ or $w = v$. ☐

Theorem 6.2. *Let T be a tree and $\mathrm{mult}(\theta, T) = m$. Suppose $\mathcal{Q} = \{Q_1, \dots, Q_m\}$ be a set of vertex disjoint paths that cover T. Then $\mathcal{Q}$ is (θ, T)-extremal.*

Proof. We shall prove this by induction on $m \geq 1$. The theorem is trivial if $m = 1$. If $m = 2$, then the result follows from Proposition 6.1. So let $m \geq 3$. Since T is a tree, there exist two paths, say Q_1 and Q_m, such that exactly one vertex in Q_1 is joined to other paths in $\mathcal{Q}$ and exactly one vertex in Q_m is joined to other paths in $\mathcal{Q}$. To be precise, let T' denote the tree induced by $Q_2, \dots, Q_{m-1}$. Then there is only one edge joining Q_1 to T' and only one edge joining Q_m to T'.

By Theorem 1.2, $\mathrm{mult}(\theta, T \setminus Q_1) \geq \mathrm{mult}(\theta, T) - 1 = m - 1$. Let T'' be the tree induced by T' and Q_m, that is $T'' = T \setminus Q_1$. Now T'' can be covered by $Q_2, \dots, Q_m$. By Theorem 1.1, $\mathrm{mult}(\theta, T'') \leq m - 1$. Therefore, $\mathrm{mult}(\theta, T'') = m - 1$ by Lemma 2.2. Moreover, $m - 1$ is the maximum multiplicity of a root of $\mu(T'', x)$. By induction, $\{Q_2, \dots, Q_m\}$ is $(\theta, T \setminus Q_1)$-extremal.

By a similar argument, $\{Q_1, \dots, Q_{m-1}\}$ is $(\theta, T \setminus Q_m)$-extremal. Hence $\mathcal{Q}$ is (θ, T)-extremal. ☐

Theorem 6.3. *Let F be a forest and $\mathcal{Q} = \{Q_1, \ldots, Q_m\}$ be a set of vertex disjoint paths that cover F. Suppose $\mathcal{Q}$ is (θ, F)-extremal. Then* $\mathrm{mult}(\theta, F) = m$ *and θ is a root of $\mu(F, x)$ with the maximum multiplicity.*

Proof. Since F can be covered by m vertex disjoint paths, by Theorem 1.1, we must have $\mathrm{mult}(\alpha, F) \leq m$ for any root α of $\mu(F, x)$. It remains to show that $\mathrm{mult}(\theta, F) \geq m$.

An edge $\{u, v\}$ of F is said to be $\mathcal{Q}$-*crossing* if u and v belong to different paths in $\mathcal{Q}$. If F contains no $\mathcal{Q}$-crossing edges then F consists of m disjoint paths $Q_1, \ldots, Q_m$. Clearly, $\mathrm{mult}(\theta, F) = \sum_{i=1}^{m} \mathrm{mult}(\theta, Q_i) = m$, as required. So we may assume that there exists an edge $\{u, v\} \in E(F)$ such that $u \in V(Q_1)$ and $v \in V(Q_2)$. Since $\mathcal{Q}$ is (θ, F)-extremal, either u is θ-special in Q_1 or v is θ-special in Q_2.

We now proceed by induction on the number of vertices. Suppose u is θ-special in Q_1. Since u is not an endpoint of Q_1, $Q_1 \setminus u$ consists of two disjoint paths, say R_1 and R_2. Since $\mathrm{mult}(\theta, Q_1 \setminus u) = 2$ and $\mathrm{mult}(\theta, R_i) \leq 1$ for each $i = 1, 2$ (by Theorem 1.2), we deduce that $\mathrm{mult}(\theta, R_i) = 1$ for each $i = 1, 2$. Note that $\{R_1, R_2, Q_3, \ldots, Q_m\}$ is a set of disjoint paths that cover $F \setminus u$. Recall that there are no θ-neutral vertices in Q_1. Moreover, by the Stability Lemma for trees (Theorem 3.3), every θ-positive vertex in Q_1 remains θ-positive in $Q_1 \setminus u$ and every θ-essential vertex in Q_1 remains θ-essential in $Q_1 \setminus u$. So every θ-special vertex in Q_1 remains θ-special in $Q_1 \setminus u$. Consequently, $\{R_1, R_2, Q_3, \ldots, Q_m\}$ is $(\theta, F \setminus u)$-extremal. By induction, $\mathrm{mult}(\theta, F \setminus u) = m + 1$ and θ is a root of $\mu(F \setminus u, x)$ with maximum multiplicity. It follows from Lemma 1.3 that $\mathrm{mult}(\theta, F) \geq \mathrm{mult}(\theta, F \setminus u) - 1 = m$, as desired.

The case when v is θ-special in Q_2 can be settled by a similar argument. Note that the base cases of our induction occur when F has no crossing edges. $\square$

Our main result Theorem 1.7 now follows immediately from Theorem 6.2 and Theorem 6.3.

7 Conclusion

Theorem 1.7 gives a characterisation of trees for which the matching polynomial bound on the size of a minimum path cover is tight. Denote the class of such trees by $\mathcal{T}$. Then $T \in \mathcal{T}$ if and only if there exists a (θ, T)-extremal path cover for some root θ of its matching polynomial. In general, the conditions of (θ, T)-extremality are not easy to verify. However, these conditions can be useful when we wish to show that a given tree T does not belong to $\mathcal{T}$. Indeed, suppose $T \in \mathcal{T}$ and let $\{Q_1, \ldots, Q_m\}$ be a (θ, T)-extremal path cover of T. Then θ must be a common root of $\mu(Q_i, x)$ and $\mu(T, x)$ for all $i = 1, \ldots, m$. This places some restrictions on θ as well as the lengths of the paths Q_i. Together with additional structural information about T, it may be possible to derive a contradiction. As a future avenue of enquiry, we believe it might be interesting to determine which types of trees do not belong to $\mathcal{T}$.

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References

[1] W. Chen and C.Y. Ku, An analogue of the Gallai-Edmonds Structure Theorem for nonzero roots of the matching polynomial, *J. Combin. Theory Ser. B* (2009), doi:10.1016/j.jctb.2009.05.001.

[2] C. D. Godsil, *Algebraic Combinatorics*, Chapman and Hall, New York (1993).

[3] C. D. Godsil, Algebraic matching theory, *The Electronic Journal of Combinatorics* **2** (1995), # R8.

[4] C. D. Godsil, Problems in Algebraic Combinatorics, *The Electronic Journal of Combinatorics* **2** (1995), # F1.

[5] L. Lovász and M. D. Plummer, *Matching Theory*, Annals Discrete Math. **29**, North-Holland, Amsterdam (1986).

[6] A. Neumaier, The second largest eigenvalue of a tree, *Linear Algebra Appl.* **48** (1982), 9–25.