Anti-Ramsey numbers for graphs with independent cycles

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Abstract

An edge-colored graph is called *rainbow* if all the colors on its edges are distinct. Let $\mathcal{G}$ be a family of graphs. The *anti-Ramsey number* $AR(n, \mathcal{G})$ for $\mathcal{G}$, introduced by Erdős et al., is the maximum number of colors in an edge coloring of K_n that has no rainbow copy of any graph in $\mathcal{G}$. In this paper, we determine the anti-Ramsey number $AR(n, \Omega_2)$, where Ω_2 denotes the family of graphs that contain two independent cycles.

1 Introduction

An edge-colored graph is called *rainbow* if any of its two edges have distinct colors. Let $\mathcal{G}$ be a family of graphs. The *anti-Ramsey number* $AR(n, \mathcal{G})$ for $\mathcal{G}$ is the maximum number of colors in an edge coloring of K_n that has no rainbow copy of any graph in $\mathcal{G}$. The *Turán number* $ex(n, \mathcal{G})$ is the maximum number of edges of a simple graph without a copy of any graph in $\mathcal{G}$. Clearly, by taking one edge of each color in an edge coloring of K_n, one can show that $AR(n, \mathcal{G}) \leq ex(n, \mathcal{G})$. When $\mathcal{G}$ consists of a single graph H, we write $AR(m, H)$ and $ex(n, H)$ for $AR(m, \{H\})$ and $ex(n, \{H\})$, respectively.

Anti-Ramsey numbers were introduced by Erdős et al. in [5], and showed to be connected not so much to Ramsey theory than to Turán numbers. In particular, it was proved that $AR(n, H) - ex(n, \mathscr{H}) = o(n^2)$, where $\mathscr{H} = \{H - e : e \in E(H)\}$. By the asymptotic of Turán numbers, we have $AR(n,H)/\binom{n}{2} \to 1-(1/d)$ as $n \to \infty$, where $d+1 = \min\{\chi(H-e) : e \in E(H)\}$. So the anti-Ramsey number $AR(n,H)$ is determined asymptotically for graphs H with $\min\{\chi(H-e) : e \in E(H)\} \geq 3$. The case $\min\{\chi(H-e) : e \in E(H)\} = 2$ remains harder.

The anti-Ramsey numbers for some special graph classes have been determined. As conjectured by Erdős et al. [5], the anti-Ramsey number for cycles, $AR(n, C_k)$, was determined for $k \leq 6$ in [1, 5, 8], and later completely solved in [11]. The anti-Ramsey number for paths, $AR(n, P_{k+1})$, was determined in [13]. Independently, the authors of [10] and [12] considered the anti-Ramsey number for complete graphs. The anti-Ramsey numbers for other graph classes have been studied, including small bipartite graphs [2], stars [6], subdivided graphs [7], trees of order k [9], and matchings [12]. The bipartite analogue of the anti-Ramsey number was studied for even cycles [3] and for stars [6].

Denote by Ω_k the family of (multi)graphs that contain k vertex disjoint cycles. Vertex disjoint cycles are said to be *independent cycles*. The family of (multi)graphs not belonging to Ω_k is denoted by $\overline{\Omega}_k$. Clearly, $\overline{\Omega}_1$ is just the family of forests. In this paper, we consider the anti-Ramsey numbers for the family Ω_k. It was proved in [5] that $AR(n, C_3) = n-1$. In fact, from the appendix of [5], we have $AR(n, \Omega_1) = n-1$. Using the extremal structures theorem for graphs in $\overline{\Omega}_2$ [4], we determine the anti-Ramsey number $AR(n, \Omega_2)$ for $n \geq 6$. The bounds of $AR(n, \Omega_k)$, $k \geq 3$, are discussed.

Let G be a graph and c be an edge coloring of G. A *representing subgraph* of c is a spanning subgraph of G, such that any two edges of which have distinct colors and every color of G is in the subgraph. For an edge $e \in E(G)$, denote by $c(e)$ the color assigned to the edge e.

2 Extremal structures theorem for graphs in $\overline{\Omega}_2$

First, we present extremal structures for the graphs which do not contain two independent cycles.

Theorem 2.1 [4] *Let G be a multigraph without two independent cycles. Suppose that $\delta(G) \geq 3$ and there is no vertex contained in all the cycles of G. Then one of the following six assertions holds (see Figure 1).*

(1) *G has three vertices and multiple edges joining every pair of the vertices.*

(2) *G is a K_4 in which one of the triangles may have multiple edges.*

(3) *$G \cong K_5$.*

(4) *G is K_5^- such that some of the edges not adjacent to the missing edge may be multiple edges.*

(5) *G is a wheel whose spokes may be multiple edges.*

(6) *G is obtained from $K_{3,p}$ by adding edges or multiple edges joining vertices in the first class.*

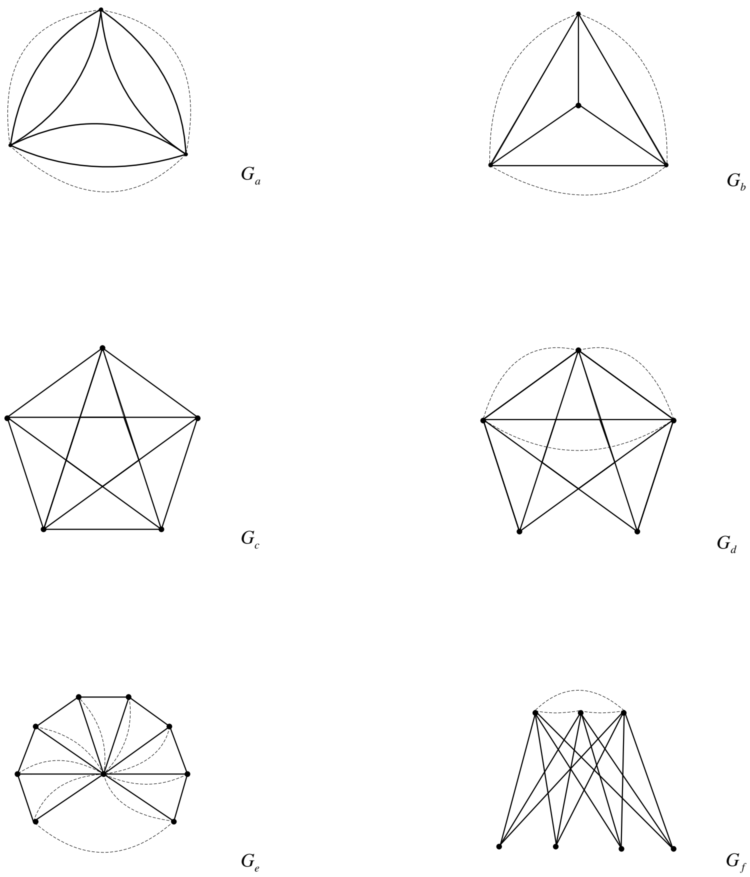

Figure 1: The graphs G_a, G_b, G_c, G_d, G_e and G_f

In general, we have the following result.

Theorem 2.2 [4] *A multigraph G does not contain two independent cycles if and only if either it contains a vertex x_0 such that $G - x_0$ is a forest, or it can be obtained from a subdivision G_0 of a graph listed in Figure 1 by adding a forest and at most one edge joining each tree of the forest to G_0.*

More precisely, from the theorem above, we have the following lemmas.

Lemma 2.3 *Let G be a simple graph of order n and size m. If G contains a vertex x_0 such that $G - x_0$ is a forest, then $m \leq 2n - 3$.*

Lemma 2.4 *Let G be a simple graph of order n and size m. Suppose that G can be obtained from a subdivision G_0 of a graph listed in Figure 1 by adding a forest and at most one edge joining each tree of the forest to G_0. Then*

(1). *if* G_0 *is a subdivision of* G_a, $m \leq 2n-3$.

(2). *if* G_0 *is a subdivision of* G_b, $m \leq 2n-2$.

(3). *if* G_0 *is a subdivision of* G_c, $m \leq n+5$.

(4). *if* G_0 *is a subdivision of* G_d, $m \leq 2n-1$. *Furthermore, the equality holds if and only if* G *contains five distinct vertices* u, v, w, x, y *such that* $G[\{u, v, w, x, y\}] = K_5^-$, $uv \notin E(G)$, *and each vertex* $z \in V(G) - \{u, v, w, x, y\}$ *is adjacent to just two vertices of* $\{w, x, y\}$.

(5). *if* G_0 *is a subdivision of* G_e, $m \leq 2n-2$.

(6). *if* G_0 *is a subdivision of* G_f, $m \leq 2n+p-3$. *Furthermore, when* $p=3$, *the equality holds if and only if* G *can be obtained from* $K_{3,3}$ *by adding two edges joining vertices in the first class, and each vertex not in* $K_{3,3}$ *is adjacent to just two vertices of the first class.*

3 Anti-Ramsey numbers for Ω_2

Let G be a graph of order n. An edge coloring c of K_n is *induced* by G if c assigns distinct colors to the edges of G and assigns one additional color to all the edges of $\overline{G}$. Clearly, an edge coloring of K_n induced by G has $|E(G)|+1$ colors (unless $G = K_n$). Given two vertex disjoint graphs G and H, denote by $G+H$ the graph obtained from $G \cup H$ by joining every vertex of G to all the vertices of H. We have the following result.

Theorem 3.1 *For any* $n \geq 7$, $AR(n, \Omega_2) = 2n-2$.

Proof. Lower bound

Let $G \cong K_2 + \overline{K}_{n-2}$. Suppose c is an edge coloring of K_n induced by G. For any graph $H \in \Omega_2$ of order at most n, any copy of H in K_n must contain at least two edges not in G. Then the edge coloring c of K_n has no rainbow graph in Ω_2. This immediately yields the lower bound $AR(n, \Omega_2) \geq 2n-2$.

Upper bound

In order to prove the upper bound, here we only need to show that any $(2n-1)$-edge-coloring of K_n always contains a rainbow subgraph belonging to the family Ω_2. Suppose that there is a $(2n-1)$-edge-coloring c of K_n which does not contain any rainbow subgraph belonging to the family Ω_2. Let G be a representing graph of c. Then G does not contain two independent cycles. From Theorem 2.2 and Lemma 2.3, we have that G can be obtained from a subdivision G_0 of a graph listed in Figure 1 by adding a forest and at most one edge joining each tree of the forest to G_0. Since $|E(G)| = 2n-1$, from Lemma 2.4 we have that G_0 is a subdivision of G_d or G_f. To complete the proof, we distinguish the following cases.

Case 1. G_0 is a subdivision of G_d.

Since $|E(G)| = 2n-1$, from Lemma 2.4, we may assume that G contains five distinct vertices u, v, w, x, y such that $G[\{u, v, w, x, y\}] = K_5^-$ and $uv \notin E(G)$, and take a vertex $z \in V(G) - \{u, v, w, x, y\}$ with $N(z) = \{x, y\}$. Furthermore, since $n \geq 7$, from Lemma 2.4, there is a vertex $s \in V(G) - \{u, v, w, x, y, z\}$ adjacent to just two vertices of $\{w, x, y\}$.

Now, considering the possible neighborhood of the vertex s, we distinguish the following subcases.

Subcase 1.1 The vertex s is not adjacent to both x and y.

By the symmetry of x and y, without loss of generality, we assume that s is adjacent to just the vertices x and w.

Since the cycle $xyzx$ is rainbow, we have

$$c(uv) \in \{c(uw), c(wv), c(xy), c(yz), c(xz)\},$$

otherwise the union of the cycles $uvwu$ and $xyzx$ is a rainbow graph belonging to the family Ω_2. So the cycle $uvyu$ is rainbow, and the union of the cycles $uvyu$ and $xswx$ is a rainbow graph belonging to the family Ω_2. A contradiction.

Subcase 1.2 The vertex s is adjacent to both x and y.

Since the cycle $ywvy$ is rainbow, we have

$$c(sz) \in \{c(sx), c(xz), c(wv), c(yw), c(yv)\},$$

otherwise the union of the cycles $ywvy$ and $xszx$ is a rainbow graph belonging to the family Ω_2.

Since the cycle $xwux$ is rainbow, we have

$$c(sz) \in \{c(sy), c(yz), c(wu), c(ux), c(wx)\},$$

otherwise the union of the cycles $xwux$ and $yszy$ is a rainbow graph belonging to the family Ω_2, a contradiction, since the two sets $\{c(sx), c(xz), c(wv), c(yw), c(yv)\}$ and $\{c(sy), c(yz), c(wu), c(ux), c(wx)\}$ have no common elements.

Case 2. G_0 is a subdivision of G_f.

From Lemma 2.4, $p \geq 2$. If $p = 2$, since $|E(G)| = 2n-1$, G_0 must be a subdivision of G_d, and we only need to go back to the previous case. So we may assume that $p \geq 3$. Denote by u, v, w all the vertices in the first class of G_f. Note that for each edge x_1x_2 of G_f, it may be subdivided to a path connecting the vertices x_1 and x_2 in G. For convenience, we still use the notation x_1x_2 to denote the corresponding path in G.

Suppose $p \geq 4$. Let x, y, z, s be four distinct vertices in the second class of G_f. If $c(zs) \notin \{c(wz), c(ws), c(ux), c(uy), c(vx), c(vy)\}$, then the union of the cycles $wzsw$ and $uxvyu$ is a rainbow graph belonging to the family Ω_2. So $c(zs) \in \{c(wz), c(ws), c(ux),$

$c(uy), c(vx), c(vy)\}$. Then either the union of the cycles $uzsu$ and $vxwyv$ or the union of the cycles $vzsv$ and $uxwyu$ is a rainbow graph belonging to the family Ω_2.

So, let $p = 3$ and denote by x, y, z all the vertices in the second class of G_f. Since $|E(G)| = 2n-1$, from Lemma 2.4, there are at least two edges joining vertices of u, v and w. Without loss of generality, assume that $uv, vw \in E(G)$. Since $n \geq 7$, from Lemma 2.4, there is a vertex $s \in V(G) - \{x, y, z, u, v, w\}$ that is adjacent to just two vertices of $\{u, v, w\}$.

If $c(yz) \notin \{c(wz), c(wy), c(ux), c(uv), c(vx)\}$, then the union of the cycles $wyzw$ and $uxvu$ is a rainbow graph belonging to the family Ω_2. So we have $c(yz) \in \{c(wz), c(wy), c(ux), c(uv), c(vx)\}$. Then the cycle $yzuy$ is rainbow. Since the cycle $xwvx$ is rainbow, we have $c(yz) = c(xv)$, otherwise the union of the cycles $yzuy$ and $xwvx$ is a rainbow graph belonging to the family Ω_2. By the analog analysis, we have $c(xy) = c(vz)$.

Now, considering the possible neighborhood of the vertex s, we only need to distinguish the following subcases.

Subcase 2.1 The vertex s is adjacent to just the vertices v and w.

Since $c(yz) = c(xv)$, we have that the union of the cycles $yzuy$ and $swvs$ is a rainbow graph belonging to the family Ω_2, a contradiction.

Subcase 2.2 The vertex s is adjacent to just the vertices u and w.

Since $c(yz) = c(xv)$, we have

$$c(sv) \in \{c(ws), c(wv), c(uy), c(uz), c(yz)\},$$

otherwise the union of the cycles $swvs$ and $yzuy$ is a rainbow graph belonging to the family Ω_2. By the analog analysis, from $c(xy) = c(vz)$, we have

$$c(sv) \in \{c(us), c(uv), c(xy), c(xw), c(yw)\},$$

a contradiction, since the two sets $\{c(ws), c(wv), c(uy), c(uz), c(yz)\}$ and $\{c(us), c(uv), c(xy), c(xw), c(yw)\}$ have no common elements.

This completes the proof. ∎

4 The value of $AR(6, \Omega_2)$

In this section, we present an 11-edge-coloring of K_6 which does not contain any graphs in Ω_2. Let $V(K_6) = \{u, v, w, x, y, z\}$. Define an 11-edge-coloring ϕ of K_6 as follows. Let $G = K_6 - uv - uz - vz - wz$. Clearly, the size of G is just 11. Color the edges of G with distinct colors. Then color the edges uv and wz with the same color in $\{\phi(xy), \phi(uw), \phi(wv)$, color the edge uz with the color $\phi(wv)$, and color the edge vz with the color $\phi(uw)$. It is easy to verify that the edge coloring ϕ of K_6 does not contain any graph in the family Ω_2. This implies the lower bound $AR(6, \Omega_2) \geq 11$. In fact, using the same analysis as in the previous section, we can show that any 12-edge-coloring of K_6 contains a rainbow graph belonging to the family Ω_2. To complete the section, we have the following result.

Theorem 4.1 *$AR(6, \Omega_2) = 11$.*

5 Bounds of anti-Ramsey numbers for Ω_k

Unlike graphs in the family $\overline{\Omega}_2$, we have no more information about graphs in the family $\overline{\Omega}_k$ for $k \geq 3$. So we cannot treat the family Ω_k ($k \geq 3$) as we did for the case Ω_2. Fortunately, the bound of $ex(n, \overline{\Omega}_k)$ presents an upper bound of $AR(n, \overline{\Omega}_k)$ for $k \geq 3$. Let $f(n, k) = (2k-1)(n-k)$ and

$$g(n,k) = \begin{cases} f(n,k) + (24k-n)(k-1), & \text{if } n \leq 24k; \\ f(n,k), & \text{if } n \geq 24k. \end{cases}$$

Lemma 5.1 [4] *Every graph G of order $n \geq 3k$, $k \geq 2$, and size at least $g(n,k)$ contains k independent cycles except when $n \geq 24k$ and $G \cong K_{2k-1} + \overline{K}_{n-2k+1}$.*

This easily yields $AR(n, \Omega_k) < g(n,k)$. Let $G \cong K_{2k-2} + \overline{K}_{n-2k+2}$. Clearly, the edge coloring of K_n induced by G has no rainbow graph in Ω_k. Then we have the following result.

Theorem 5.2 *For any integer n and k, $n \geq 3k$, $k \geq 2$,*

$$\binom{2k-2}{2} + (2k-2)(n-2k+2) + 1 \leq AR(n, \Omega_k) \leq g(n,k) - 1.$$

When n is large enough, i.e., $n \geq 24k$, the gap between the upper bound and the lower bound is just $n - 2k - 1$. From Theorem 3.1, we know the left equality holds for $n \geq 7$ and $k = 2$. In fact, though we cannot prove it, we feel that the value of $AR(n, \Omega_k)$ would be very near to the lower bound rather than the upper bound.

Conjecture 5.3 *For any integer n and k, $n \geq 3k$, $k \geq 2$,*

$$AR(n, \Omega_k) = \binom{2k-2}{2} + (2k-2)(n-2k+2) + 1.$$

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