f-Vectors of 3-Manifolds

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Dedicated to Anders Björner on the occasion of his 60th birthday

Abstract

In 1970, Walkup [46] completely described the set of f-vectors for the four 3-manifolds S^3, $S^2 \veebar S^1$, $S^2 \times S^1$, and $\mathbb{R}\mathbf{P}^3$. We improve one of Walkup's main restricting inequalities on the set of f-vectors of 3-manifolds. As a consequence of a bound by Novik and Swartz [35], we also derive a new lower bound on the number of vertices that are needed for a combinatorial d-manifold in terms of its β_1-coefficient, which partially settles a conjecture of Kühnel. Enumerative results and a search for small triangulations with bistellar flips allow us, in combination with the new bounds, to completely determine the set of f-vectors for twenty further 3-manifolds, that is, for the connected sums of sphere bundles $(S^2 \times S^1)^{\# k}$ and twisted sphere bundles $(S^2 \veebar S^1)^{\# k}$, where $k = 2, 3, 4, 5, 6, 7, 8, 10, 11, 14$. For many more 3-manifolds of different geometric types we provide small triangulations and a partial description of their set of f-vectors. Moreover, we show that the 3-manifold $\mathbb{R}\mathbf{P}^3 \# \mathbb{R}\mathbf{P}^3$ has (at least) two different minimal g-vectors.

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1 Introduction

Let M be a (compact) 3-manifold (without boundary). According to Moise [34], M can be triangulated as a (finite) simplicial complex. If a triangulation of M has *face vector* $f = (f_0, f_1, f_2, f_3)$, then by Euler's equation, $f_0 - f_1 + f_2 - f_3 = 0$. By double counting the edges of the triangle-facet incidence graph, $2f_2 = 4f_3$. So it follows that

$$f = (f_0, f_1, 2f_1 - 2f_0, f_1 - f_0). \tag{1}$$

In particular, the number of vertices f_0 and the number of edges f_1 determine the complete f-vector of the triangulation.

Theorem 1 (Walkup [46]) *For every* 3*-manifold* M *there is a largest integer* $\Gamma(M)$ *such that*

$$f_1 \geq 4f_0 - 10 + \Gamma(M) \tag{2}$$

for every triangulation of M *with* f_0 *vertices and* f_1 *edges (with the inequality being tight for at least one triangulation of* M*). Moreover there is a smallest integer* $\Gamma^*(M) \geq \Gamma(M)$ *such that for every pair* (f_0, f_1) *with* $f_0 \geq 0$ *and*

$$\binom{f_0}{2} \geq f_1 \geq 4f_0 - 10 + \Gamma^*(M) \tag{3}$$

there is a triangulation of M *with* f_0 *vertices and* f_1 *edges. Specifically,*

(a) $\Gamma^* = \Gamma = 0$ *for* S^3,

(b) $\Gamma^* = \Gamma = 10$ *for* $S^2 \underline{\times} S^1$,

(c) $\Gamma^* = 11$ *and* $\Gamma = 10$ *for* $S^2 \times S^1$*, where, with the exception* $(9, 36)$*, all pairs* (f_0, f_1) *with* $f_0 \geq 0$ *and* $4f_0 \leq f_1 \leq \binom{f_0}{2}$ *occur,*

(d) $\Gamma^* = \Gamma = 17$ *for* $\mathbb{R}\mathbf{P}^3$*, and*

(e) $\Gamma^*(M) \geq \Gamma(M) \geq 18$ *for all other* 3*-manifolds* M*.*

By definition, $\Gamma(M)$ and $\Gamma^*(M)$ are topological invariants of M, with $\Gamma(M)$ determining the range of pairs (f_0, f_1) for which triangulations of M can occur, whereas $\Gamma^*(M)$ ensures that for all pairs (f_0, f_1) in the respective range there indeed are triangulations with the corresponding f-vectors.

Remark 2 *Walkup originally stated Theorem 1 in terms of the constants* $\gamma = -10 + \Gamma$ *and* $\gamma^* = -10 + \Gamma^*$*. As we will see in Section 3, our choice of the constant* $\Gamma(M)$ *(as well as of* $\Gamma^*(M)$*) is more naturally related to the* g_2*-entries of the* g*-vectors of triangulations of a* 3*-manifold* M*:* $\Gamma(M)$ *is the smallest* g_2 *that is possible for all triangulations of* M*.*

In the following section, we review some of the basic facts on f- and g-vectors of triangulated d-manifolds and how they change under (local) modifications of the triangulation. Moreover, we derive a new bound on the minimal number of vertices for a triangulable d-manifold depending on its β_1-coefficient. In Section 3, we discuss the f- and g-vectors of 3-manifolds in more detail and introduce tight-neighborly triangulations. Section 4 is devoted to the proof of an improvement of a bound by Walkup and to the notion of g_2-irreducible triangulations. In Section 5 we completely enumerate all g_2-irreducible triangulations of 3-manifolds with $g_2 \leq 20$ and all potential g_2-irreducible triangulations of 3-manifolds with $f_0 \leq 15$. Section 6 presents small triangulations of different geometric types, in particular, examples of Seifert manifolds from the six Seifert geometries as well as triangulations of hyperbolic 3-manifolds. With the help of these small triangulations we establish upper bounds on the invariants Γ and Γ^* of the respective manifolds. For the 3-manifold $\mathbb{RP}^3 \# \mathbb{RP}^3$ we show that it has (at least) two different minimal g-vectors. Finally, we extend Walkup's Theorem 1 by completely characterizing the set of f-vectors of the twenty 3-manifolds $(S^2 \times S^1)^{\# k}$ and $(S^2 \underline{\times} S^1)^{\# k}$ with $k = 2, 3, 4, 5, 6, 7, 8, 10, 11, 14$. (In dimensions $d \geq 4$, a complete description of the set of f-vectors is only known for the six 4-manifolds S^4 [32], $S^3 \times S^1$, $\mathbb{CP}^2$, $K3$-surface, $(S^2 \times S^1)^{\# 2}$ [44], and $S^3 \underline{\times} S^1$ [10].) In Section 7 we compare the invariant $\Gamma(M)$ to Matveev's complexity measure $c(M)$.

2 Face Numbers and (Local) Modifications

Let K be a triangulation of a d-manifold M with f-vector $f(K) = (f_0(K), \dots, f_d(K))$ (and with $f_{-1}(K) = 1$), that is, $f_i(K)$ denotes the number of i-dimensional faces of K. For simplicity, we write $f = (f_0, \dots, f_d)$, and we define numbers h_i by

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d+1-i}{d+1-k} f_{i-1}. \tag{4}$$

The vector $h = (h_0, \dots, h_{d+1})$ is called the *h-vector* of K. Moreover, the *g-vector* $g = (g_0, \dots, g_{\lfloor (d+1)/2 \rfloor})$ of K is defined by $g_0 = 1$ and $g_k = h_k - h_{k-1}$, for $k \geq 1$, which gives

$$g_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d+2-i}{d+2-k} f_{i-1}. \tag{5}$$

In particular,

$$g_1 = f_0 - (d+2), \tag{6}$$

$$g_2 = f_1 - (d+1) f_0 + \binom{d+2}{2}. \tag{7}$$

Let $\mathcal{H}^d$ be the class of triangulated d-manifolds that can be obtained from the boundary of the $(d+1)$-simplex by a sequence of the following three operations:

- S Subdivide a facet with one new vertex in the interior of the facet.
- H Form a handle (oriented or non-oriented) by identifying a pair of facets in $K \in \mathcal{H}^d$ and removing the interior of the identified facet in such a way that the resulting complex is still a simplicial complex (i.e., the distance in the 1-skeleton of K between every pair of identified vertices must be at least three).
- $\#$ Form the connected sum of $K_1, K_2 \in \mathcal{H}^d$ by identifying a pair of facets, one from each complex, and then removing the interior of the identified facet.

For the operations S, H, and $\#$ the resulting triangulations depend on the particular choices of the facets and, in the case of H and $\#$, on the respective identifications. However, all triangulated d-manifolds in the class $\mathcal{H}^d$ are of the following topological types: the d-sphere S^d, connected sums $(S^{d-1}\times S^1)^{\#k}$ of the orientable sphere product $S^{d-1}\times S^1$ for $k\geq 1$, or connected sums $(S^{d-1}\underline{\times} S^1)^{\#l}$ of the twisted sphere product $S^{d-1}\underline{\times} S^1$ for $l\geq 1$.

Let K, K_1, and K_2 be arbitrary triangulated d-manifolds with f-vectors

$$\begin{aligned} f(K) &= (f_0(K),\dots,f_d(K)),\\ f(K_1) &= (f_0(K_1),\dots,f_d(K_1)),\\ f(K_2) &= (f_0(K_2),\dots,f_d(K_2)), \end{aligned}$$

and $f_{-1}(K)=f_{-1}(K_1)=f_{-1}(K_2)=1$. Again, let SK be the triangulated d-manifold obtained from K by performing the subdivision operation S on some facet of K, HK be the triangulated d-manifold obtained from K by performing the handle addition operation H on some (admissible) pair of facets of K, and $K_1\#\pm K_2$ be the triangulated d-manifold obtained from K_1 and K_2 by the connected sum operation $\#$ on some pair of facets of K_1 and K_2. Then the f-vectors of SK, HK, and $K_1\#\pm K_2$ have entries

$$\begin{aligned} f_k(SK) &= f_k(K)+\binom{d+1}{k}, \quad \text{for} \quad 0\leq k\leq d-1, && (8)\\ f_d(SK) &= f_d(K)+d, && (9)\\ f_k(HK) &= f_k(K)-\binom{d+1}{k+1}, \quad \text{for} \quad 0\leq k\leq d-1, && (10)\\ f_d(HK) &= f_d(K)-2, && (11)\\ f_k(K_1\#\pm K_2) &= f_k(K_1)+f_k(K_2)-\binom{d+1}{k+1}, \quad \text{for} \quad 0\leq k\leq d-1, && (12)\\ f_d(K_1\#\pm K_2) &= f_d(K_1)+f_d(K_2)-2. && (13) \end{aligned}$$

In particular, it follows that

$$\begin{aligned} g_1(SK) &= g_1(K)+1, && (14)\\ g_k(SK) &= g_k(K), \quad \text{for} \quad 2\leq k\leq \lfloor (d+1)/2\rfloor, && (15)\\ g_1(HK) &= g_1(K)-(d+1), && (16)\\ g_k(HK) &= g_k(K)+(-1)^k\binom{d+2}{k}, \quad \text{for} \quad 2\leq k\leq \lfloor (d+1)/2\rfloor, && (17)\\ g_1(K_1\#\pm K_2) &= g_1(K_1)+g_1(K_2)+1, && (18)\\ g_k(K_1\#\pm K_2) &= g_k(K_1)+g_k(K_2), \quad \text{for} \quad 2\leq k\leq \lfloor (d+1)/2\rfloor. && (19) \end{aligned}$$

Conjecture 3 (Kalai [18]) *Let K be a connected triangulated d-manifold with $d\geq 3$. Then*

$$g_2(K)\geq \binom{d+2}{2}\beta_1(K;\mathbb{Q}). \tag{20}$$

In [44], Swartz verified Kalai's conjecture for all $d \geq 3$ when $\beta_1(K;\mathbb{Q}) = 1$, and for orientable K when $d \geq 4$ and $\beta_2(K,\mathbb{Q}) = 0$.

Theorem 4 (Novik and Swartz [35]) *Let $\mathbb{K}$ be any field and let K be a (connected) triangulation of a $\mathbb{K}$-orientable $\mathbb{K}$-homology d-dimensional manifold with $d \geq 3$. Then*

$$g_2(K) \geq \binom{d+2}{2}\beta_1(K;\mathbb{K}). \tag{21}$$

Furthermore, if $g_2 = \binom{d+2}{2}\beta_1(K;\mathbb{K})$ and $d \geq 4$, then $K \in \mathcal{H}^d$.

Since any d-manifold (without boundary) is orientable over $\mathbb{K}$ if $\mathbb{K}$ has characteristic two, and in this case $\beta_1(\mathbb{K}) \geq \beta_1(\mathbb{Q})$ (universal coefficient theorem), this theorem proves Conjecture 3.

Combining (21) and (7) with the obvious inequality $f_1 \leq \binom{f_0}{2}$ yields

$$\begin{aligned} \binom{d+2}{2}\beta_1 \quad \leq \quad g_2 \quad &= \quad f_1 - (d+1)f_0 + \binom{d+2}{2} \\ &\leq \quad \binom{f_0}{2} - (d+1)f_0 + \binom{d+2}{2} \end{aligned}$$

or, equivalently,

$$f_0^2 - (2d+3)f_0 + (d+1)(d+2)(1-\beta_1) \quad \geq \quad 0. \tag{22}$$

Theorem 5 *Let $\mathbb{K}$ be any field and let K be a $\mathbb{K}$-orientable triangulated d-manifold with $d \geq 3$. Then*

$$f_0(K) \geq \left\lceil \tfrac{1}{2}\Big((2d+3) + \sqrt{1 + 4(d+1)(d+2)\beta_1(K;\mathbb{K})}\Big)\right\rceil. \tag{23}$$

Inequality (22) can also be written in the form $\binom{f_0-d-1}{2} \geq \binom{d+2}{2}\beta_1$. Its proof settles Kühnel's conjectured bounds $\binom{f_0-d+j-2}{j+1} \geq \binom{d+2}{j+1}\beta_j$ (cf. [25], with $1 \leq j \leq \lfloor \frac{d-1}{2} \rfloor$) in the cases with $j = 1$.

According to Brehm and Kühnel [4], we further have for all $(j-1)$-connected but not j-connected combinatorial d-manifolds K, with $1 \leq j < d/2$, that

$$f_0(K) \geq 2d + 4 - j. \tag{24}$$

While the bound (23) becomes trivial for manifolds with $\beta_1 = 0$, with the d-sphere S^d admitting triangulations in the full range $f_0(S^d) \geq d+2$, the inequality (24) yields stronger restrictions for higher-connected manifolds. In contrast, for all non-simply connected combinatorial d-manifolds K the bound (24) uniformly gives

$$f_0(K) \geq 2d + 3, \tag{25}$$

whereas the bound (23) explicitly depends on the β_1-coefficient.

In the case $\beta_1 = 1$, the bounds (23) and (24) coincide with (25) and are sharp for

- $S^{d-1} \times S^1$ if d is even [19, 22],
- $S^{d-1} \mathbin{\underline{\times}} S^1$ if d is odd [19, 22],

while $f_0(S^{d-1} \times S^1) \geq 2d+4$ for d odd and $f_0(S^{d-1} \mathbin{\underline{\times}} S^1) \geq 2d+4$ for d even; see [1, 10].

If K is a triangulated 2-manifold with Euler characteristic $\chi(K)$, then by Heawood's inequality [14],

$$f_0 \geq \left\lceil \tfrac{1}{2}\left(7 + \sqrt{49 - 24\chi(K)}\right) \right\rceil. \tag{26}$$

For an orientable surface K of genus g the Euler characteristic of K is $2 - 2g$. Therefore $f_0 \geq \lceil \frac{1}{2}(7 + \sqrt{1 + 48g}) \rceil$, whereas $\chi(K) = 2 - u$ for a non-orientable surface K of genus u and hence $f_0 \geq \lceil \frac{1}{2}(7 + \sqrt{1 + 24u}) \rceil$. These bounds all coincide with

$$f_0 \geq \left\lceil \tfrac{1}{2}\left(7 + \sqrt{1 + 48\tfrac{\beta_1(K;\mathbb{Z}_2)}{2}}\right) \right\rceil \tag{27}$$

or, equivalently, $\binom{f_0-3}{2} \geq \binom{4}{2}\frac{\beta_1(K;\mathbb{Z}_2)}{2}$, where the factor $\frac{1}{2}$ on the right hand side compensates the doubling of homology in the middle homology of even dimensional manifolds by Poincaré duality; see [25] for Kühnel's conjectured higherdimensional analogues of this bound.

Heawood's bound (26) is sharp, except in the cases of the orientable surface of genus 2, the Klein bottle, and the non-orientable surface of genus 3. Each of these requires an extra vertex to be added. The construction of series of examples of vertex-minimal triangulations was completed in 1955 for all non-orientable surfaces by Ringel [39] and in 1980 for all orientable surfaces by Jungerman and Ringel [17].

Question 6 *Is inequality (23) sharp for all but finitely many connected sums $(S^{d-1} \times S^1)^{\#k}$ of sphere products as well as for all but finitely many connected sums $(S^{d-1} \mathbin{\underline{\times}} S^1)^{\#k}$ of twisted sphere products in every fixed dimension $d \geq 3$?*

Problem 7 *Construct series of vertex-minimal triangulations of $(S^{d-1} \times S^1)^{\#k}$ and of $(S^{d-1} \mathbin{\underline{\times}} S^1)^{\#k}$ for $d \geq 3$. Can the examples be chosen to lie in the class $\mathcal{H}^d$?*

The only known series of such vertex-minimal triangulations are the ones mentioned above of the d-sphere $S^d \cong (S^{d-1} \times S^1)^{\#0} \cong (S^{d-1} \mathbin{\underline{\times}} S^1)^{\#0}$, triangulated as the boundary of the $(d+1)$-simplex with $d+2$ vertices, and for $k = 1$ the vertex-minimal triangulations of $(S^{d-1} \times S^1)^{\#1}$ and $(S^{d-1} \mathbin{\underline{\times}} S^1)^{\#1}$.

A first sporadic vertex-minimal 4-dimensional example with $k = 3$ was recently discovered by Bagchi and Datta [2]. They construct a triangulation of $(S^3 \mathbin{\underline{\times}} S^1)^{\#3}$ in $\mathcal{H}^d$ with 15 vertices and $g_2 = 45$, both the minimums required by (21) and (23). For 3-dimensional examples with $k = 2, 3, 4, 5, 6, 7, 8, 10, 11, 14$, see Theorem 31 below.

Besides subdivisions, handle additions, and connected sums, *bistellar flips* (also called Pachner moves [37]) are a very useful class of local modifications of triangulations.

Definition 8 [37] *Let K be a triangulated d-manifold. If A is a $(d-i)$-face of K, $0 \leq i \leq d$, such that the link of A in K, $\mathrm{Lk}\,A$, is the boundary $\partial(B)$ of an i-simplex B that is not a face of K, then the operation Φ_A on K defined by*

$$\Phi_A(K) := (K \backslash (A * \partial(B))) \cup (\partial(A) * B)$$

is a bistellar i-move *(with $*$ the join operation for simplicial complexes).*

In particular, the subdivision operation S from above on any facet A of K coincides with the bistellar 0-move on this facet.

If K' is obtained from K by a bistellar i-move, $0 \leq i \leq \lfloor (d-1)/2 \rfloor$, then

$$g_{i+1}(K') = g_{i+1}(K) + 1 \tag{28}$$

$$g_k(K') = g_k(K) \quad \text{for all} \quad k \neq i+1. \tag{29}$$

If d is even and $i = \frac{d}{2}$, then

$$g_k(K') = g_k(K) \quad \text{for all} \quad k. \tag{30}$$

Bistellar flips can be used to navigate through the set of triangulations of a d-manifold, with the objective of obtaining a small, or perhaps even vertex-minimal, triangulation of this manifold. A simulated annealing type strategy for this aim is described in [3]. The reference [3] also contains further background on combinatorial topology aspects of bistellar flips. A basic implementation of the bistellar flip heuristics is [26]. The `bistellar` client of the `polymake`-system [13] allows for fast computations for rather large triangulations, as we will need in Section 6.

3 Face Numbers and (Local) Modifications for 3-Manifolds

Let K be a triangulated 3-manifold with f-vector $f = (f_0, f_1, 2f_1 - 2f_0, f_1 - f_0)$. The relations (6), (7), (14)–(19) then read,

$$g_1 = f_0 - 5, \tag{31}$$

$$g_2 = f_1 - 4f_0 + 10 \tag{32}$$

and

$$g_1(SK) = g_1(K) + 1, \tag{33}$$

$$g_2(SK) = g_2(K), \tag{34}$$

$$g_1(HK) = g_1(K) - 4, \tag{35}$$

$$g_2(HK) = g_2(K) + 10, \tag{36}$$

$$g_1(K_1 \# \pm K_2) = g_1(K_1) + g_1(K_2) + 1, \tag{37}$$

$$g_2(K_1 \# \pm K_2) = g_2(K_1) + g_2(K_2). \tag{38}$$

For a 3-manifold M, $\Gamma(M)$ is the smallest g_2 that is possible for all triangulations of M. Hence, the following lemma follows immediately from (38) and Theorem 1.

Lemma 9 *Let M, M_1, and M_2 be 3-manifolds. Then*

$$\begin{aligned}\Gamma(M_1\#\pm M_2) &\leq \Gamma(M_1)+\Gamma(M_2), &(39)\\ \Gamma(M\#(S^2\!\times\! S^1)^{\#k}) &\leq \Gamma(M)+10k, &(40)\\ \Gamma(M\#(S^2\,\underline{\times}\, S^1)^{\#k}) &\leq \Gamma(M)+10k. &(41)\end{aligned}$$

As a consequence of Theorem 4 and Lemma 9:

Corollary 10 *For every $k\in\mathbb{N}$,*

$$\Gamma((S^2\!\times\! S^1)^{\#k}) = \Gamma(S^2\,\underline{\times}\, S^1)^{\#k}) = 10k. \qquad (42)$$

Conjecture 11 *Let M, M_1, and M_2 be 3-manifolds. Then*

$$\begin{aligned}\Gamma(M_1\#\pm M_2) &= \Gamma(M_1)+\Gamma(M_2), &(43)\\ \Gamma(M\#(S^2\!\times\! S^1)^{\#k}) &= \Gamma(M)+10k, &(44)\\ \Gamma(M\#(S^2\,\underline{\times}\, S^1)^{\#k}) &= \Gamma(M)+10k. &(45)\end{aligned}$$

While the latter two equalities above would follow from the first, it may be the case that only these two special cases hold.

By Theorem 5, inequality (23) holds for all $\mathbb{K}$-orientable triangulated 3-manifolds K, that is,

$$f_0(K) \geq \left\lceil \tfrac{1}{2}\Big(9+\sqrt{1+80\beta_1(K;\mathbb{K})}\Big)\right\rceil. \qquad (46)$$

We next consider the class of connected sums $(S^2\!\times\! S^1)^{\#k}$ and $(S^2\,\underline{\times}\, S^1)^{\#k}$ with $\beta_1=k$ for $k\in\mathbb{N}$. For this class, inequality (22) can be interpreted as an upper bound on the number k for which the corresponding connected sums can have triangulations with f_0 vertices. If inequality (22) is sharp, then $f_1=\binom{f_0}{2}$, i.e., such a triangulation must be *neighborly* with complete 1-skeleton. We therefore call triangulations of connected sums of the sphere bundles $S^2\!\times\! S^1$ and $S^2\,\underline{\times}\, S^1$ for which inequality (22) is tight *tight-neighborly*. In the case of equality,

$$\frac{(f_0-9)f_0}{20} = k-1, \qquad (47)$$

the right hand side of (47) is integer, and therefore, the left hand side is integer as well. This is possible if and only if

$$f_0 \equiv 0,4,5,9 \bmod 20, \qquad (48)$$

with the additional requirement that $f_0\geq 5$. Table 1 gives the possible parameters (f_0,k) for tight-neighborly triangulations. The first two pairs are $(f_0,k)=(5,0)$ and $(f_0,k)=(9,1)$, for which we have the triangulation of S^3 as the boundary $\partial\Delta^4$ of the 4-simplex Δ^4 and Walkup's unique 9-vertex triangulation [46] of $S^2\,\underline{\times}\, S^1$, respectively. There is no triangulation of $S^2\!\times\! S^1$ with 9 vertices.

Table 1: Parameters for tight-neighborly triangulations

f_0	k
$20m$	$20m^2 - 9m + 1$
$4 + 20m$	$20m^2 - m$
$5 + 20m$	$20m^2 + m$
$9 + 20m$	$20m^2 + 9m + 1$

Question 12 *Are there* 3*-dimensional tight-neighborly triangulations for* $k > 1$*?*

The first two cases would be $(f_0, k) = (20, 12)$ and $(f_0, k) = (24, 19)$.

Tight-neighborly triangulations are possible candidates for "tight triangulations" in the following sense (cf., [20, 23]): A simplicial complex K with vertex-set V is *tight* if for any subset $W \subseteq V$ of vertices the induced homomorphism

$$H_*(\langle W \rangle \cap K; \mathbb{K}) \to H_*(K; \mathbb{K})$$

is injective, where $\langle W \rangle$ denotes the face of the $(|V| - 1)$-simplex $\Delta^{|V|-1}$ spanned by W.

Obviously, we can extend the concept of tight-neighborly triangulations to any dimension $d \geq 2$: Triangulations of connected sums of sphere bundles $S^{(d-1)} \times S^1$ and $S^{(d-1)} \underline{\times} S^1$ are *tight-neighborly* if inequality (22) is tight.

By Theorem 4, every triangulation K of a $\mathbb{K}$-orientable $\mathbb{K}$-homology d-dimensional manifold with $d \geq 4$ for which (22) is tight lies in $\mathcal{H}^d$ and therefore is a tight-neighborly connected sum of sphere bundles $S^{(d-1)} \times S^1$ or $S^{(d-1)} \underline{\times} S^1$.

Conjecture 13 *Tight-neighborly triangulations are tight.*

The conjecture holds for surfaces (i.e., for $d = 2$) [20, Sec. 2D], for $k = 0$ (that is, for the triangulation of S^d as the boundary of the $(d + 1)$-simplex) [20, Sec. 3A], and for $k = 1$, in which case there is a unique and tight triangulation with $2d + 3$ vertices in every dimension $d \geq 2$ (see [19, 33, 46] for existence, [1, 10] for uniqueness, and [20, Sec. 5B] for tightness).

For the sporadic Bagchi-Datta example [2] we used the computational methods from [23] to determine the tightness.

Proposition 14 *The tight-neighborly* 4*-dimensional* 15*-vertex example of Bagchi and Datta with* $k = 3$ *is tight.*

Most recently, Conjecture 13 was settled in even dimensions $d \geq 4$ by Effenberger [11]. In particular, this also yields the tightness of the Bagchi-Datta example.

4 g_2-Irreducible Triangulations

The main idea behind the proof of Theorem 1 is that triangulations which minimize g_2 have several special combinatorial properties. A *missing facet* of a triangulated d-manifold K is a subset σ of the vertex set of cardinality $d+1$ such that $\sigma \notin K$, but every proper subset of σ is a face of K.

Definition 15 *Let K be a triangulation of a 3-manifold M. Then K is g_2-*minimal *if $g_2(K') \geq g_2(K)$ for all other triangulations K' of M, i.e., $g_2(K) = \Gamma(M)$. The triangulation K is g_2-*irreducible *if the following hold:*

1. *K is g_2-minimal.*
2. *K is not the boundary of the 4-simplex.*
3. *K does not have any missing facets.*

The reason for introducing the third condition is the following folk theorem. For a complete proof, see [1, Lemma 1.3].

Theorem 16 *Let K be a triangulated 3-manifold. Then K has a missing facet if and only if K equals $K_1 \# K_2$ or HK'.*

So, a triangulation K which realizes the minimum g_2 for a particular 3-manifold M is either g_2-irreducible, or is of the form $K_1 \# K_2$ or HK', where the component triangulations realize their minimum g_2. The remainder of this section is devoted to proving the following.

Theorem 17 *If K is g_2-irreducible, then*

$$f_1(K) - \tfrac{9}{2} f_0(K) > \tfrac{1}{2}. \tag{49}$$

Walkup originally proved that for g_2-irreducible K, $f_1(K) - \frac{9}{2} f_0(K) > 0$. All that is needed to get the slight improvement we require is a little more care. Walkup's original result plus Theorem 16 are already enough to prove that for a fixed Γ there are only finitely many 3-manifolds such that $\Gamma(M) \leq \Gamma$ [43] (see also the next section). With the exception of Theorem 17, all of the remaining results in this section first appeared in [46] and we refer the reader there for the proofs.

Theorem 18 *If K is g_2-irreducible, u a vertex of degree less than 10 and v a vertex in the link of u, then the one-skeleton of the link of u with v and its incident edges removed is exactly one of those in Figures 1-6(4)–1-9d(4).*

From here on we write "$L_v u$ is of type" to mean that v is in the link of u in a g_2-irreducible triangulation, and the one-skeleton of the link of u with v and its incident edges removed is the referenced figure.

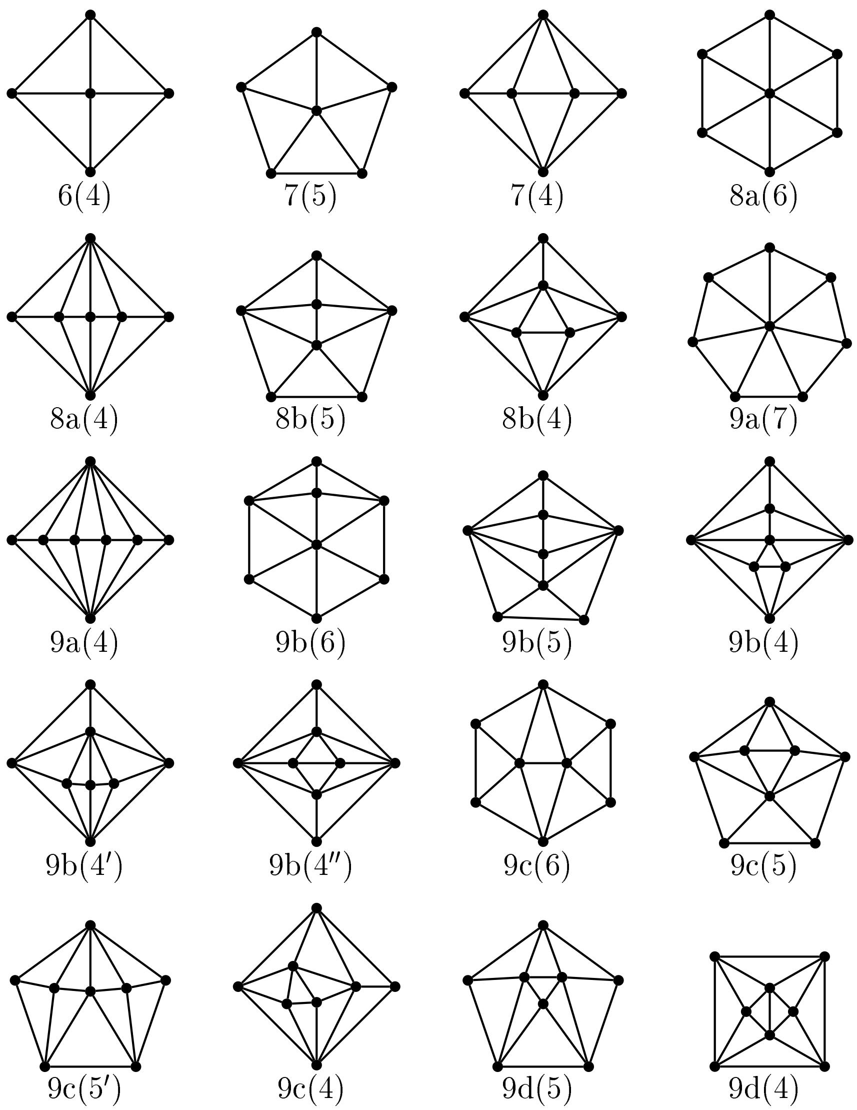

Figure 1: $L_v u$ when the degree of $u = 6, 7, 8, 9$

Theorem 19 *Let K be a g_2-irreducible triangulation. Then there exists a triangulation K' which is homeomorphic to K, has the same f-vector as K, and whose links satisfy the following:*

- *If $L_v u$ is of type 6(4), then* $\deg(v) \geq 10$.
- *If $L_v u$ is of type 7(5), then* $\deg(v) \geq 12$.
- *If $L_v u$ is of type 8a(6), then* $\deg(v) \geq 14$.
- *If $L_v u$ is of type 8b(5), then* $\deg(v) \geq 11$.
- *If all of the vertices of K' have degree at least* 9, *then there exists at least two vertices of degree at least* 10, *or there exists at least one vertex of degree at least* 11.

Proof of Theorem 17: Let K be g_2-irreducible. We can assume that K satisfies the conclusions of the previous theorem. Let (u, v) be an ordered pair of vertices of K which form an edge. Define $\lambda(u, v)$ as follows:

- $\lambda(u, v) = \frac{3}{4}$ if $L_v u$ is of type $6(4)$.
- $\lambda(u, v) = 1$ if $L_v u$ is of type $7(5)$.
- $\lambda(u, v) = \frac{3}{4}$ if $L_v u$ is of type $8a(6)$.
- $\lambda(u, v) = \frac{5}{8}$ if $L_v u$ is of type $8b(5)$.
- $\lambda(u, v) = \frac{1}{2}$ if $L_v u$ is of type $7(4), 8a(4), 8b(4)$, or if the degree of u is 9.
- $\lambda(u, v) = 1 - \lambda(v, u)$ if the degree of u is at least 10 and the degree of v is 9 or less.
- $\lambda(u, v) = \frac{1}{2}$ otherwise.

Define
$$\mu(u) = \sum_{v \in \mathrm{Lk}\, u} \lambda(u, v) - \tfrac{9}{2}.$$
By construction,
$$\sum_{u \in K} \mu(u) = f_1(K) - \tfrac{9}{2} f_0(K).$$

Suppose that u is a vertex of degree m. If v is in the link of u and $L_u v$ is of type $6(4)$, $7(5)$, $8a(6)$, or $8b(5)$, then Theorem 19 implies that the degree of u is at least ten. Therefore, in the link of u each triangle has at most one vertex v such that $L_u v$ is one of these four types. Let $n_{6(4)}, n_{7(4)}, n_{7(5)}$, etc., be the number of vertices v in the link of u of such that $L_u v$ is of type $6(4), 7(4), 7(5)$, etc. Since the link of u has $2m - 4$ triangles, the link of u must satisfy the integer constraint

$$4n_{6(4)} + 5n_{7(5)} + 6n_{8a(6)} + 5n_{8b(5)} \leq 2m - 4. \tag{50}$$

The minimum potential value of $\mu(u)$ is the minimum of

$$-\frac{1}{4}n_{6(4)} - \frac{1}{2}n_{7(5)} - \frac{1}{4}n_{8a(6)} - \frac{1}{8}n_{8b(5)} + \frac{m-9}{2} \tag{51}$$

under the above constraint and a few others discussed below. Now we determine lower bounds for $\mu(u)$ for a variety of values of m.

- $m < 10$. Then by definition $\mu(u) = 0$.
- $m = 10$. $\mu(u) \geq \frac{1}{4}$ [46, Lemma 11.9].
- $m = 11$. $\mu(u) \geq \frac{1}{2}$ [46, Lemma 11.9].
- $m = 12$. $\mu(u) \geq \frac{1}{2}$ [46, Lemma 11.9].
- $m = 13$. Walkup proved that $n_{7(5)} \leq 3$ in this case. With this additional restriction, the minimal value of (51) subject to the integral constraint (50) is $\frac{1}{4}$. This value occurs when $n_{6(4)} = 3$ and $n_{7(5)} = 2$, or $n_{6(4)} = 1$ and $n_{7(5)} = 3$. In all other cases $\mu(u) > \frac{1}{4}$.
- $m = 14$. The minimal value of (51) subject to the integral constraint (50) is $\frac{1}{4}$ and this only occurs if $n_{6(4)} = 1$ and $n_{7(5)} = 4$. In all other cases, $\mu(u) > \frac{1}{4}$.
- $m \geq 15$. Even without integer considerations, $\mu(u) > \frac{1}{4}$.

For notational purposes, define

$$\mu(K) = \sum_{u \in K} \mu(u) = f_1(K) - \tfrac{9}{2} f_0(K).$$

From above we know that $\mu(u) \geq 0$ for all vertices u. If K has no vertices of degree less than 9, then by the last line of Theorem 19 there exists at least two vertices which contribute at least $1/2$ to $\mu(K)$ or one vertex which contributes at least 1, so $\mu(K) \geq 1$. So suppose K has a vertex of degree less than nine. There are four possibilities.

1. K has a vertex of degree six. Then the six vertices of the link of this vertex all contribute at least $1/4$ to $\mu(K)$.
2. K has a vertex of degree seven. Consider the two vertices of type $7(5)$ whose existence is now guaranteed. Each of these either adds more than $1/4$ to $\mu(K)$ or imply the existence of a vertex of degree six.
3. K has a vertex whose link is of type $8a$. The same argument as the case of a vertex of degree seven applies.
4. K has a vertex whose link is of type $8b$. Then K has at least four vertices of type $8b(5)$ each of which either satisfy $\mu(u) > 1/4$ or imply the existence of a vertex of degree six. □

5 Enumeration of g_2-Irreducible Triangulations

A 3-manifold M is *irreducible* if every embedded 2-sphere in M bounds a 3-ball in M. In particular, if a triangulation K of an irreducible 3-manifold M has a missing facet, then $K = K_1 \# K_2$ with one part homeomorphic to M and the other homeomorphic to S^3. As a consequence, every g_2-minimal triangulation K of an irreducible 3-manifold M, different from S^3, is either g_2-irreducible or is obtained from a g_2-irreducible triangulation K' of M by successive stacking operations S.

As already mentioned in the previous section, for every fixed Γ there are only finitely many 3-manifolds M such that $\Gamma(M) \leq \Gamma$ [43]: If M is a 3-manifold with $\Gamma(M) \leq \Gamma$, then M has a triangulation K with f-vector $f = (f_0, f_1, f_2, f_3)$ such that $f_1 - 4f_0 + 10 \leq \Gamma$. If K is g_2-irreducible, then the additional restriction (49) holds, $f_1 > \frac{9}{2}f_0 + \frac{1}{2}$. These two inequalities together with the trivial inequality $f_1 \leq \binom{f_0}{2}$ allow for only finitely many tuples (f_0, f_1). Hence, there are only finitely many g_2-irreducible triangulations K with $g_2(K) \leq \Gamma$. This directly implies that there are only finitely many irreducible 3-manifolds M with $\Gamma(M) \leq \Gamma$. If K is a g_2-minimal triangulation of a non-irreducible 3-manifold M with $\Gamma(M) \leq \Gamma$, then K is either g_2-irreducible, in which case there are only finitely many such triangulations, or K is of the form $K_1 \# K_2$ or HK', where the component triangulations realize their minimum g_2. These components are either g_2-irreducible or can further be split up or reduced by deleting a handle. Since $g_2(K_1 \# K_2) = g_2(K_1) + g_2(K_2)$ and $g_2(HK') = g_2(K') + 10$, it follows that there are at most finitely many non-irreducible 3-manifolds M with $\Gamma(M) \leq \Gamma$ and therefore only finitely many 3-manifolds M with $\Gamma(M) \leq \Gamma$

Figure 2 displays in grey the admissible range for tuples (f_0, f_1) that can occur for g_2-irreducible triangulations of 3-manifolds with $\Gamma \leq 20$. These are precisely the tuples $(11, 51)$, $(11, 52)$, $(11, 53)$, $(11, 54)$, $(12, 55)$, $(12, 56)$, $(12, 57)$, $(12, 58)$, $(13, 60)$, $(13, 61)$, $(13, 62)$, $(14, 64)$, $(14, 65)$, $(14, 66)$, $(15, 69)$, $(15, 70)$, $(16, 73)$, $(16, 74)$, $(17, 78)$, and $(18, 82)$.

We conducted exhaustive computer searches to find all the g_2-irreducible triangulations of 3-manifolds with $g_2 \leq 20$ and candidates for all g_2-irreducible triangulations of 3-manifolds with $f_0 \leq 15$.

The 3-manifolds were constructed using the lexicographic enumeration technique described in [42]. This technique constructs 3-manifolds one facet at a time which allows local properties to be tested before the 3-manifolds are completely constructed. By checking local properties provided by Walkup [46] the searches can be pruned sufficiently to make them feasible.

Theorem 20 (Walkup [46, 10.1]) *Let K be a g_2-irreducible triangulation and (u, v) be an edge of K. Then* $\operatorname{Lk}(u, v)$ *contains at least* 4 *vertices.*

Theorem 21 (Walkup [46, 10.2]) *Let K be a g_2-irreducible triangulation and (u, v) be an edge of K. Then* $\operatorname{Lk} u \cap \operatorname{Lk} v - \operatorname{Lk}(u, v)$ *is nonempty.*

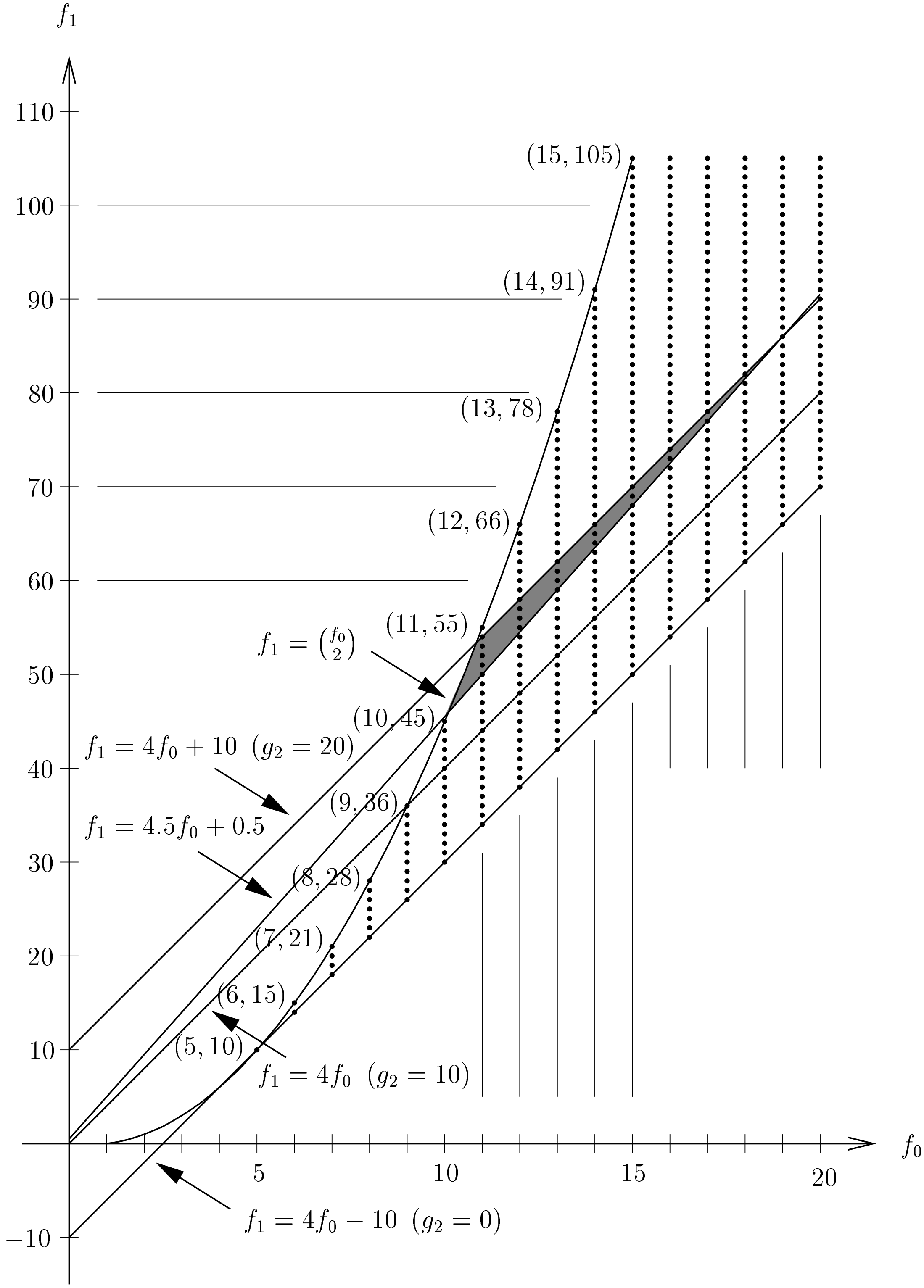

Figure 2: Range (in grey) for g_2-irreducible triangulations of 3-manifolds with $g_2 \leq 20$. The range is bounded by the inequalities $f_1 > \frac{9}{2}f_0 + \frac{1}{2}$, $f_1 - 4f_0 + 10 \leq 20$, and $f_1 \leq \binom{f_0}{2}$.

Theorem 22 (Walkup [46, 10.4]) *Let K be a g_2-irreducible triangulation and u be a vertex of K. Suppose* $\mathrm{Lk}\, u$ *contains the boundary complex of a 2-simplex (a, b, c) as a subcomplex. Then* $\mathrm{Lk}\, u$ *must also contain the 2-simplex (a, b, c).*

Theorem 23 (Walkup [46, 11.1]) *Let K be a g_2-irreducible triangulation and (u, v) be an edge of K. Suppose* $\mathrm{Lk}\, u \cap \mathrm{Lk}\, v - \mathrm{Lk}\,(u, v) = \{w\}$. *Then* $\mathrm{Lk}\,(u, w)$ *contains at least as many vertices as* $\mathrm{Lk}\,(u, v)$.

To find all the candidates for g_2-irreducible triangulations of 3-manifolds with $f_0 \leq 15$ Theorems 20, 21, 22, 23 were used to prune the searches. Run times for $11, 12, 13, 14, 15$ vertices were 2 seconds, 100 seconds, 3 hours, 60 days, and 7 years, respectively. Results from these runs are given in Table 2.

To find all the (candidates for) g_2-irreducible triangulations with $f_0 \geq 16$ and $g_2 \leq 20$ a lower bound for f_1 was maintained during the construction of the 3-manifolds. When this lower bound became too large the search backtracked. The lower bound for f_1 was computed from the degrees of the finished vertices and lower bounds for the degrees of the other vertices. Theorem 24 provides lower bounds for vertices which are neighbors of certain finished vertices. Examples have been found which show that the lower bounds in Theorem 24 are the best that can be obtained using just the necessary conditions of Theorems 20, 21, 22, and 23.

Theorem 24 (Walkup [46, 10.7]) *Let K be a g_2-irreducible triangulation and (u, v) be an edge of K.*

- *If $L_v u$ is of type 9b(4'), 9b(4"), 9c(4) or 9c(4), then* $\deg(v) \geq 7$.
- *If $L_v u$ is of type 8a(4), 8b(4), 9a(4), 9b(4) or 9d(4), then* $\deg(v) \geq 8$.
- *If $L_v u$ is of type 7(4) or 9d(5), then* $\deg(v) \geq 9$.
- *If $L_v u$ is of type 6(4), 8b(5), 9b(5), 9c(5) or 9c(5'), then* $\deg(v) \geq 10$.
- *If $L_v u$ is of type 9b(6) or 9c(6) then* $\deg(v) \geq 11$.
- *If $L_v u$ is of type 7(5), then* $\deg(v) \geq 12$.
- *If $L_v u$ is of type 8a(6), then* $\deg(v) \geq 14$.
- *If $L_v u$ is of type 9a(7), then* $\deg(v) \geq 16$.
- *If the degree of u is 10 or 11 and the degree of (u, v) is 5, then* $\deg(v) \geq 8$.
- *If the degree of u is 10 and the degree of (u, v) is 6, then* $\deg(v) \geq 10$.
- *If the degree of u is 11, 12 or 13 and the degree of (u, v) is 6, then* $\deg(v) \geq 9$.
- *If the degree of u is 11, 12, 13, 14 or 15 and the degree of (u, v) is 7, then the degree of v is at least 10.*
- *If the degree of u is 11 and the degree of (u, v) is 6, then* $\deg(v) \geq 9$.
- *If the degree of u is 11 and the degree of (u, v) is 7, then* $\deg(v) \geq 10$.
- *If the degree of (u, v) is d, then the degree of v is at least $d + 2$.*

Table 2: Triangulations found with $5 \leq f_0 \leq 15$ vertices after implementing Walkup's Lemmas 10.1, 10.2, 10.4, and 11.1.

Manifold	f_0	Range for f_1	Range for g_2	Count
$\mathbb{R}\mathbf{P}^3$	11	$51 \leq f_1 \leq 52$	$17 \leq g_2 \leq 18$	2
$S^2 \veebar S^1$	12	$60 \leq f_1 \leq 60$	$22 \leq g_2 \leq 22$	2
$\mathbb{R}\mathbf{P}^3$	12	$60 \leq f_1 \leq 60$	$22 \leq g_2 \leq 22$	4
$L(3,1)$	12	$66 \leq f_1 \leq 66$	$28 \leq g_2 \leq 28$	1
total	12			7
$S^2 \veebar S^1$	13	$70 \leq f_1 \leq 75$	$28 \leq g_2 \leq 33$	8
$\mathbb{R}\mathbf{P}^3$	13	$63 \leq f_1 \leq 63$	$21 \leq g_2 \leq 21$	1
$L(3,1)$	13	$70 \leq f_1 \leq 74$	$28 \leq g_2 \leq 32$	72
total	13			81
$S^2 \veebar S^1$	14	$75 \leq f_1 \leq 91$	$29 \leq g_2 \leq 45$	6860
$\mathbb{R}\mathbf{P}^3$	14	$69 \leq f_1 \leq 70$	$23 \leq g_2 \leq 24$	2
$L(3,1)$	14	$75 \leq f_1 \leq 84$	$29 \leq g_2 \leq 38$	7092
$\mathbb{R}\mathbf{P}^2 \times S^1$	14	$84 \leq f_1 \leq 91$	$38 \leq g_2 \leq 45$	1011
$L(4,1)$	14	$84 \leq f_1 \leq 90$	$38 \leq g_2 \leq 44$	738
$L(5,2)$	14	$86 \leq f_1 \leq 91$	$40 \leq g_2 \leq 45$	121
total	14			15824
S^3	15	$85 \leq f_1 \leq 94$	$35 \leq g_2 \leq 44$	28
$S^2 \veebar S^1$	15	$79 \leq f_1 \leq 104$	$29 \leq g_2 \leq 54$	1500836
$S^2 \times S^1$	15	$81 \leq f_1 \leq 97$	$31 \leq g_2 \leq 47$	73
$\mathbb{R}\mathbf{P}^3$	15	$72 \leq f_1 \leq 94$	$22 \leq g_2 \leq 44$	13
$L(3,1)$	15	$81 \leq f_1 \leq 97$	$31 \leq g_2 \leq 47$	240587
$(S^2 \times S^1)^{\#2}$	15	$91 \leq f_1 \leq 101$	$41 \leq g_2 \leq 51$	84
$(S^2 \veebar S^1)^{\#2}$	15	$95 \leq f_1 \leq 101$	$45 \leq g_2 \leq 51$	144
$\mathbb{R}\mathbf{P}^2 \times S^1$	15	$88 \leq f_1 \leq 105$	$38 \leq g_2 \leq 55$	3798307
$L(4,1)$	15	$89 \leq f_1 \leq 101$	$39 \leq g_2 \leq 51$	1968160
$L(5,2)$	15	$90 \leq f_1 \leq 101$	$40 \leq g_2 \leq 51$	504785
$(S^2 \times S^1)\#\mathbb{R}\mathbf{P}^3$	15	$91 \leq f_1 \leq 97$	$41 \leq g_2 \leq 47$	238
$(S^2 \veebar S^1)\#\mathbb{R}\mathbf{P}^3$	15	$90 \leq f_1 \leq 99$	$40 \leq g_2 \leq 49$	1913
T^3	15	$105 \leq f_1 \leq 105$	$55 \leq g_2 \leq 55$	1
$\mathbb{R}\mathbf{P}^3 \# \mathbb{R}\mathbf{P}^3$	15	$86 \leq f_1 \leq 102$	$36 \leq g_2 \leq 52$	570885
$L(5,1)$	15	$97 \leq f_1 \leq 102$	$47 \leq g_2 \leq 52$	1314
$P_2 = S^3/Q$	15	$90 \leq f_1 \leq 102$	$40 \leq g_2 \leq 52$	64475
P_3	15	$97 \leq f_1 \leq 105$	$47 \leq g_2 \leq 55$	1612
P_4	15	$104 \leq f_1 \leq 105$	$54 \leq g_2 \leq 55$	20
S^3/T^*	15	$102 \leq f_1 \leq 102$	$52 \leq g_2 \leq 52$	5
total	15			8653480

Proof: The last seven statements follow from the first eight. The first eight statements are from [46, 10.7], [46, 11.2], or [46, 11.4] or can be proved using the technique in the proof of [46, 10.7]. Two of these results differ from [46, 10.7].

To show that if $L_v u$ is of type 8b(5) then the degree of v is at least 10 assume the degree of v is less than 10. By [46, 11.1] w (the bottom interior vertex in Figure 1-8b(5)) is in $W(u, v)$. w is adjacent to four boundary vertices of 8b(5). For every type with the degree of v less than 10 and five boundary vertices every interior vertex is adjacent to at least two boundary vertices. This contradicts [46, 10.6].

If $L_v u$ is of type 8a(4) then the degree of v may also be 8. Let $L_u v$ also be of type 8a(4), let $W(u, v)$ be just the center vertex of Figure 1-8a(4), and identify the boundaries of $L_v u$ and $L_u v$ after rotating one copy of Figure 1-8a(4) a quarter of a turn. $\square$

The runs for $(f_0, f_1) = (16, 73)$, $(16, 74)$, $(17, 78)$, and $(18, 82)$ to search for potential g_2-irreducible triangulations produced no examples. The run times were 1, 4, 64, and 1000 cpu-days, respectively.

Theorem 25 *The unique g_2-irreducible triangulation of $\mathbb{R}\mathbf{P}^3$ with $f = (11, 51, 80, 40)$ and $g_2 = 17$ is the only g_2-irreducible triangulation of a 3-manifold with $g_2 \leq 20$.*

Proof: According to our enumeration, there are only two candidates for g_2-irreducible triangulations with $g_2 \leq 20$; see Table 2. Both candidates are triangulations of $\mathbb{R}\mathbf{P}^3$ with 11 vertices. One of the triangulations has f-vector $f = (11, 51, 80, 40)$ and is the unique g_2-irreducible triangulation of $\mathbb{R}\mathbf{P}^3$ by Theorem 1. The other candidate triangulation has f-vector $f = (11, 52, 82, 41)$ and is therefore not g_2-minimal. $\square$

6 Examples of Triangulations and Upper Bounds for Γ and Γ^*

Since $\Gamma(M) = \min\{\, g_2(K) \mid K \text{ is a triangulation of } M \,\}$ for any given 3-manifold M, we get an upper bound $\Gamma(M) \leq g_2(K)$ for each triangulation K of M. Therefore, we are interested in "small" triangulations of the given manifold M. A standard procedure (cf. [3, 24]) to obtain such small triangulations is to first construct any triangulation of M of "reasonable size" and then to apply bistellar flips until a small or perhaps even vertex-minimal triangulation is reached. The Tables 3–12 list the f-vectors of obtained triangulations along with resulting upper bounds on the respective Γ's and Γ^*'s. The triangulations that we found are available online at [28].

According to Perelman's proof [38] of Thurston's Geometrization Conjecture [45], every compact 3-manifold can be decomposed canonically into geometric pieces, which are modeled on one of eight model geometries. Six of the geometries, S^3 (spherical), $S^2 \times \mathbb{R}^1$, E^3 (Euclidean), Nil, $H^2 \times \mathbb{R}^1$, and $\widetilde{SL}(2, \mathbb{R})$, yield Seifert manifolds, the other two geometries are Sol and H^3 (hyperbolic); see [40] for a detailed discussion.

There are exactly four 3-manifolds of geometry $S^2 \times \mathbb{R}^1$ and ten flat 3-manifolds of geometry E^3, all other six geometries give each rise to infinitely many 3-manifolds.

The topological types of the 3-manifolds modeled on the Seifert geometries are completely classified up to homeomorphism (cf. [36, 41]). Moreover, it is possible to systematically construct triangulations of all Seifert manifolds; see [5, 24], as well as [27] for an implementation. For hyperbolic 3-manifolds it is unclear whether a complete classification can be obtained. In 1982 Thurston [45] proved that almost every prime 3-manifold is hyperbolic. Hyperbolic 3-manifolds can be ordered with respect to their hyperbolic volume. A census of 11,031 hyperbolic 3-manifolds, triangulated as pseudo-simplicial complexes with up to 30 tetrahedra, was obtained by Hodgson and Weeks [16] by enumeration. For a census of all pseudo-simplicial triangulations of orientable and non-orientable 3-manifolds with up to 10 tetrahedra as well as for further references on pseudo-triangulation results, see Burton [8].

In the Tables 3–10 we list manifolds of the six Seifert geometries, with the lens spaces of Table 3, the prism manifolds of Table 4, and the three examples of Table 5 of spherical geometry.

The lens spaces $L(p,q)$, the prism spaces $P(r)$, the Nil manifolds $\{Oo,1\mid b\}$, and the products $M^2_{(+,g)}\times S^1$ and $M^2_{(-,g)}\times S^1$ have homology groups

$$\begin{array}{rcl}
H_*(L(p,q)) & = & (\mathbb{Z},\mathbb{Z}_p,0,\mathbb{Z}),\\
H_*(P(r)) & = & \left\{\begin{array}{ll}(\mathbb{Z},\mathbb{Z}_2^2,0,\mathbb{Z}), & r \text{ even},\\ (\mathbb{Z},\mathbb{Z}_4,0,\mathbb{Z}), & r \text{ odd},\end{array}\right.\\
H_*(\{Oo,1\mid b\}) & = & (\mathbb{Z},\mathbb{Z}^2\oplus\mathbb{Z}_b,\mathbb{Z}^2,\mathbb{Z}),\\
H_*(M^2_{(+,g)}\times S^1) & = & (\mathbb{Z},\mathbb{Z}^{2g+1},\mathbb{Z}^{2g+1},\mathbb{Z}),\\
H_*(M^2_{(-,g)}\times S^1) & = & (\mathbb{Z},\mathbb{Z}^{g}\oplus\mathbb{Z}_2,\mathbb{Z}^{g-1}\oplus\mathbb{Z}_2,0),
\end{array}$$

respectively. Homology groups for the other examples are listed in the respective Tables.

Starting triangulations for the listed Seifert manifolds were either obtained by direct construction, as described in [24], or were produced with the program SEIFERT [27] (cf. also [5]). Small triangulations of these manifolds were already listed in [24]. For a substantial number of the examples from [24] we were able to find yet smaller triangulations due to refinements of the bistellar flip technique and an increase of the number of "rounds" for the search.

The refined simulated annealing process consisted of three stages. In the *heating stage* we started with the best known triangulation of the 3-manifold of interest. The number of vertices was increased by half the number of vertices in the starting triangulation using only random 0-moves; i.e., moves were randomly chosen with 0-moves, 1-moves, 2-moves, and 3-moves weighted by 1, 0, 0, and 0, respectively. In the *mixing stage* the heated triangulation was randomized without changing the number of vertices; $10,000$ random i-moves were made with the four types of moves weighted by 0, 1, 5, and 0. In the subsequent *cooling stage* the number of vertices was decreased whenever possible and the number of edges was kept low; $100,000,000$ i-moves were made with the types of moves weighted by 0, 1, 250, and ∞. The sequence of the mixing stage and the cooling stage was repeated ten times. Any triangulation which had a smaller f-vector than the starting triangulation was recorded.

Table 3: Lens spaces $L(p,q)$

Manifold	Smallest known f-vector	Upper Bound for Γ^*	Γ
$L(1,1)=S^3$	(5,10,10,5)	0	0
$L(2,1)=\mathbb{R}\mathbf{P}^3$	(11,51,80,40)	17	17
$L(3,1)$	(12,66,108,54)	28	28
$L(4,1)$	(14,84,140,70)	38	38
$L(5,1)$	(15,97,164,82)	47	47
$L(6,1)$	(16,110,188,94)	56	56
$L(7,1)$	(17,123,212,106)	67	65
$L(8,1)$	(17,130,226,113)	72	72
$L(9,1)$	(18,143,252,126)	81	81
$L(10,1)$	(19,155,272,136)	92	89
$L(5,2)$	(14,86,144,72)	40	40
$L(7,2)$	(16,104,176,88)	56	50
$L(8,3)$	(16,106,180,90)	56	
	[(17,109,184,92)]		51
$L(9,2)$	(16,114,196,98)	60	
	[(17,116,198,99)]		58
$L(10,3)$	(17,118,202,101)	67	
	[(18,121,206,103)]		59

Table 4: Prism manifolds

Manifold	Smallest known f-vector	Upper Bound for Γ^*	Γ
$P_2=S^3/Q$, cube space	(15,90,150,75)	46	40
P_3	(15,97,164,82)	47	47
P_4	(15,104,178,89)	54	54
P_5	(17,122,210,105)	67	64
P_6	(17,130,226,113)	72	72
P_7	(18,143,250,125)	81	81
P_8	(19,155,272,136)	92	89
P_9	(19,163,288,144)	97	97
P_{10}	(20,175,310,155)	106	105

Table 5: The spherical octahedral, truncated cube, and dodecahedral space

Manifold	Homology	Smallest known	Upper Bound for	
		f-vector	Γ^*	Γ
S^3/T^*	$(\mathbb{Z},\mathbb{Z}_3,0,\mathbb{Z})$	(15,102,174,87)	52	
		[(16,104,176,88)]		50
S^3/O^*	$(\mathbb{Z},\mathbb{Z}_2,0,\mathbb{Z})$	(16,109,186,93)	56	55
$S^3/I^*=\Sigma(2,3,5)$, Poincaré 3-sphere	$(\mathbb{Z},0,0,\mathbb{Z})$	(16,106,180,90)	56	52

Table 6: $(S^2\times\mathbb{R})$-spaces

Manifold	Homology	Smallest known	Upper Bound for	
		f-vector	Γ^*	Γ
$S^2\underline{\times}S^1$	$(\mathbb{Z},\mathbb{Z},\mathbb{Z}_2,0)$	(9,36,54,27)	10	10
$S^2\times S^1$	$(\mathbb{Z},\mathbb{Z},\mathbb{Z},\mathbb{Z})$	(10,40,60,30)	11	10
$\mathbb{R}\mathbf{P}^2\times S^1$	$(\mathbb{Z},\mathbb{Z}\oplus\mathbb{Z}_2,\mathbb{Z}_2,0)$	(14,84,140,70)	38	38
$\mathbb{R}\mathbf{P}^3\#\mathbb{R}\mathbf{P}^3$	$(\mathbb{Z},\mathbb{Z}_2^2,0,\mathbb{Z})$	(15,86,142,71)	46	
		[(16,89,146,73)]		
		[(18,96,156,78)]		34

Table 7: Flat manifolds

Manifold	Homology	Smallest known	Upper Bound for	
		f-vector	Γ^*	Γ
T^3	$(\mathbb{Z},\mathbb{Z}^3,\mathbb{Z}^3,\mathbb{Z})$	(15,105,180,90)	55	
		[(16,108,184,92)]		54
G_2	$(\mathbb{Z},\mathbb{Z}\oplus\mathbb{Z}_2^2,\mathbb{Z},\mathbb{Z})$	(16,116,200,100)	62	
		[(17,118,202,101)]		60
G_3	$(\mathbb{Z},\mathbb{Z}\oplus\mathbb{Z}_3,\mathbb{Z},\mathbb{Z})$	(17,117,200,100)	67	59
G_4	$(\mathbb{Z},\mathbb{Z}\oplus\mathbb{Z}_2,\mathbb{Z},\mathbb{Z})$	(16,115,198,99)	61	61
G_5	$(\mathbb{Z},\mathbb{Z},\mathbb{Z},\mathbb{Z})$	(16,112,192,96)	58	
		[(17,115,196,98)]		57
G_6	$(\mathbb{Z},\mathbb{Z}_4^2,0,\mathbb{Z})$	(17,124,214,107)	67	66
$K\times S^1$	$(\mathbb{Z},\mathbb{Z}^2\oplus\mathbb{Z}_2,\mathbb{Z}\oplus\mathbb{Z}_2,0)$	(16,115,198,99)	61	
		[(17,118,202,101)]		60
B_2	$(\mathbb{Z},\mathbb{Z}^2,\mathbb{Z}\oplus\mathbb{Z}_2,0)$	(16,110,188,94)	56	56
B_3	$(\mathbb{Z},\mathbb{Z}\oplus\mathbb{Z}_2^2,\mathbb{Z}_2,0)$	(17,119,204,102)	67	61
B_4	$(\mathbb{Z},\mathbb{Z}\oplus\mathbb{Z}_4,\mathbb{Z}_2,0)$	(17,117,200,100)	67	59

Table 8: $(H^2 \times \mathbb{R})$-spaces

Manifold	Smallest known f-vector	Upper Bound for Γ^*	Upper Bound for Γ
$M^2_{(+,2)} \times S^1$	(20,168,296,148)	106	98
$M^2_{(+,3)} \times S^1$	(22,210,376,188)	137	132
$M^2_{(+,4)} \times S^1$	(24,256,464,232)	172	170
$M^2_{(+,5)} \times S^1$	(26,299,546,273)	211	205
$M^2_{(-,3)} \times S^1$	(18,141,246,123)	79	79
$M^2_{(-,4)} \times S^1$	(19,163,288,144)	97	
	[(20,166,292,146)]		96
$M^2_{(-,5)} \times S^1$	(21,190,338,169)	121	116
$M^2_{(-,6)} \times S^1$	(22,212,380,190)	137	134
$M^2_{(-,7)} \times S^1$	(23,234,422,211)	154	152
$M^2_{(-,8)} \times S^1$	(24,256,464,232)	172	
	[(25,259,468,234)]		169
$M^2_{(-,9)} \times S^1$	(25,277,504,252)	191	187
$M^2_{(-,10)} \times S^1$	(26,296,540,270)	211	202

Table 9: Seifert homology spheres of geometry $SL(2,\mathbb{Z})$

Manifold	Smallest known f-vector	Upper Bound for Γ^*	Upper Bound for Γ
$\Sigma(2,3,7)$	(16,117,202,101)	63	63
$\Sigma(2,5,7)$	(18,138,240,120)	79	76
$\Sigma(3,4,5)$	(18,139,242,121)	79	77
$\Sigma(3,4,7)$	(18,151,266,133)	89	
	[(19,153,268,134)]		
	[(20,156,272,136)]		86
$\Sigma(3,5,7)$	(20,171,302,151)	106	101
$\Sigma(4,5,7)$	(20,177,314,157)	107	
	[(21,179,316,158)]		105

Table 10: Nil manifolds

Manifold	Smallest known f-vector	Upper Bound for Γ^*	Γ
$\{Oo, 1 \mid 1\}$	(16,113,194,97)	59	59
$\{Oo, 1 \mid 2\}$	(17,120,206,103)	67	62
$\{Oo, 1 \mid 3\}$	(17,125,216,108)	67	67
$\{Oo, 1 \mid 4\}$	(17,130,226,113)	72	72
$\{Oo, 1 \mid 5\}$	(18,142,248,124)	80	80

The examples of 3-manifolds of Sol geometry are either torus or Klein bottle bundles over S^1 or are composed of two twisted I-bundles over the torus or the Klein bottle; cf. Hempel and Jaco [15] and Scott [40]. However, a topological classification of the individual examples seems not to be known, which kept us from providing explicit examples of this geometry.

The Hodgson–Weeks census from 1994 [16] is still the main source for (pseudo-simplicial triangulations of) closed hyperbolic 3-manifold. Most of the census examples are orientable. The orientable example `or_0.94270736` of smallest listed hyperbolic volume 0.94270736 is called the Weeks manifold. It has recently been proved by Gabai, Meyerhoff, and Milley [12] that the Weeks manifold with homology $(\mathbb{Z}, \mathbb{Z}_5^2, 0, \mathbb{Z})$ has smallest possible volume among all orientable hyperbolic 3-manifolds.

The hyperbolic census data is accessible via the `SnapPea`-package [47] of Weeks, via the `Regina`-package [9] of Burton (cf. also [7]), or online via `http://regina.sourceforge.net/data.html`. For our purposes it was necessary to first turn the pseudo-simplicial complexes, consisting of a set of tetrahedra with gluing information for the boundaries, into proper simplicial complexes. The listed pseudo-triangulations all have only one vertex and between 9–30 tetrahedra. If the number of starting tetrahedra is ntet, the second barycentric subdivision is a proper simplicial complex with $24^2 \cdot \text{ntet}$ tetrahedra. The desired small triangulations are then obtained via bistellar flips. For example, the second barycentric subdivision of the Weeks manifold with ntet $= 9$ has f-vector $f = (940, 6124, 10368, 5184)$. In this case, the smallest triangulation of the Weeks manifold that we found has $f = (18, 141, 246, 123)$; see Table 11.

A well-known example of a hyperbolic 3-manifold is the Weber–Seifert hyperbolic dodecahedral space with homology $(\mathbb{Z}, \mathbb{Z}_5^3, 0, \mathbb{Z})$. Our smallest triangulation of this manifold has $f = (21, 193, 344, 172)$, which is close to the 18 vertices of the Weeks manifold. Nevertheless, the Weber–Seifert hyperbolic dodecahedral space does not appear in the Hodgson–Weeks census (there is no manifold with this homology in the census). In fact, sixteen of the first twenty examples from the census have triangulations as simplicial complexes with 18 vertices, the remaining four with 19 vertices; see Table 11. This seems to indicate that perhaps most of the 11,031 census examples have triangulations as proper simplicial complexes with 18–21 vertices.

Table 11: The first twenty hyperbolic 3-manifolds from the Hodgson–Weeks census and the Weber–Seifert hyperbolic dodecahedral space.

Manifold	Homology	Smallest known f-vector	Upper Bound for Γ^*	Γ
`or_0.94270736`	$(\mathbb{Z}, \mathbb{Z}_5^2, 0, \mathbb{Z})$	(18,139,242,121)	79	
		[(19,142,246,123)]		76
`or_0.98136883`	$(\mathbb{Z}, \mathbb{Z}_5, 0, \mathbb{Z})$	(18,135,234,117)	79	73
`or_1.01494161`	$(\mathbb{Z}, \mathbb{Z}_3 + \mathbb{Z}_6, 0, \mathbb{Z})$	(18,135,234,117)	79	73
`or_1.26370924`	$(\mathbb{Z}, \mathbb{Z}_5^2, 0, \mathbb{Z})$	(18,149,262,131)	87	
		[(19,150,262,131)]		84
`or_1.28448530`	$(\mathbb{Z}, \mathbb{Z}_6, 0, \mathbb{Z})$	(18,139,242,121)	79	77
`or_1.39850888`	$(\mathbb{Z}, 0, 0, \mathbb{Z})$	(18,140,244,122)	79	
		[(19,143,248,124)]		77
`or_1.41406104_a`	$(\mathbb{Z}, \mathbb{Z}_6, 0, \mathbb{Z})$	(18,136,236,118)	79	74
`or_1.41406104_b`	$(\mathbb{Z}, \mathbb{Z}_{10}, 0, \mathbb{Z})$	(18,145,254,127)	83	
		[(19,147,256,128)]		81
`or_1.42361190`	$(\mathbb{Z}, \mathbb{Z}_{35}, 0, \mathbb{Z})$	(19,153,268,134)	92	87
`or_1.44069901`	$(\mathbb{Z}, \mathbb{Z}_3, 0, \mathbb{Z})$	(18,141,246,123)	79	79
`or_1.46377664`	$(\mathbb{Z}, \mathbb{Z}_7, 0, \mathbb{Z})$	(18,148,260,130)	86	
		[(19,150,262,131)]		84
`or_1.52947733`	$(\mathbb{Z}, \mathbb{Z}_5, 0, \mathbb{Z})$	(18,144,252,126)	82	
		[(19,146,254,127)]		80
`or_1.54356891_a`	$(\mathbb{Z}, \mathbb{Z}_{35}, 0, \mathbb{Z})$	(19,152,266,133)	92	
		[(21,159,276,138)]		85
`or_1.54356891_b`	$(\mathbb{Z}, \mathbb{Z}_{21}, 0, \mathbb{Z})$	(18,144,252,126)	82	
		[(19,147,256,128)]		81
`or_1.58316666_a`	$(\mathbb{Z}, \mathbb{Z}_{21}, 0, \mathbb{Z})$	(18,140,244,122)	79	78
`or_1.58316666_b`	$(\mathbb{Z}, \mathbb{Z}_3 + \mathbb{Z}_9, 0, \mathbb{Z})$	(18,144,252,126)	82	
		[(19,147,256,128)]		
		[(20,150,260,130)]		80
`or_1.58864664_a`	$(\mathbb{Z}, \mathbb{Z}_{30}, 0, \mathbb{Z})$	(19,151,264,132)	92	85
`or_1.58864664_b`	$(\mathbb{Z}, \mathbb{Z}_{30}, 0, \mathbb{Z})$	(19,159,280,140)	93	
		[(20,162,284,142)]		92
`or_1.64960972`	$(\mathbb{Z}, \mathbb{Z}_{15}, 0, \mathbb{Z})$	(18,147,258,129)	85	
		[(19,150,262,131)]		84
`or_1.75712603`	$(\mathbb{Z}, \mathbb{Z}_7, 0, \mathbb{Z})$	(18,140,244,122)	79	78
hyperb. dodec. space	$(\mathbb{Z}, \mathbb{Z}_5^3, 0, \mathbb{Z})$	(21,190,338,169)	121	116

Conjecture 26 *At least* 18 *vertices are needed to triangulate a hyperbolic* 3*-manifold as a simplicial complex.*

It was proved by Brehm and Swiatkowski [6] that the number of non-homeomorphic lens spaces that can be triangulated with n vertices grows exponentially with n.

In contrast to the many triangulations that we expect from 18 vertices on, the list of 3-manifolds that can be triangulated with at most 17 vertices will be comparably short. In [25], 27 different 3-manifolds were described that can be triangulated with up to 15 vertices. With our improved bistellar flip techniques we were able to find further 6.

Theorem 27 *There are at least* 33 *different* 3*-manifolds that can be triangulated with up to* 15 *vertices. These examples are:*

$n=5$: S^3,
$n=9$: $S^2 \underline{\times} S^1$,
$n=10$: $S^2 \times S^1$,
$n=11$: $\mathbb{R}\mathbf{P}^3$,
$n=12$: $L(3,1)$, $(S^2\times S^1)^{\#2}$, $(S^2\underline{\times} S^1)^{\#2}$,
$n=13$: $(S^2\times S^1)^{\#3}$, $(S^2\underline{\times} S^1)^{\#3}$,
$n=14$: $\mathbb{R}\mathbf{P}^2\times S^1$, $L(4,1)$, $L(5,2)$, $(S^2\times S^1)^{\#4}$, $(S^2\underline{\times} S^1)^{\#4}$, $(S^2\times S^1)\#\mathbb{R}\mathbf{P}^3$, $(S^2\underline{\times} S^1)\#\mathbb{R}\mathbf{P}^3$, $(S^2\times S^1)^{\#2}\#\mathbb{R}\mathbf{P}^3$, $(S^2\underline{\times} S^1)^{\#2}\#\mathbb{R}\mathbf{P}^3$,
$n=15$: $\mathbb{R}\mathbf{P}^3\#\mathbb{R}\mathbf{P}^3$, $L(5,1)$, S^3/Q, P_3, P_4, S^3/T^*, T^3, $(S^2\times S^1)^{\#5}$, $(S^2\underline{\times} S^1)^{\#5}$, $(S^2\times S^1)^{\#3}\#\mathbb{R}\mathbf{P}^3$, $(S^2\underline{\times} S^1)^{\#3}\#\mathbb{R}\mathbf{P}^3$, $(S^2\times S^1)\#L(3,1)$, $(S^2\underline{\times} S^1)\#L(3,1)$, $(S^2\times S^1)^{\#2}\#L(3,1)$, $(S^2\underline{\times} S^1)^{\#2}\#L(3,1)$.

It is conjectured in [25] that this list is complete up to 13 vertices. The particular examples are listed in Tables 3–10 (Seifert manifolds) and in Table 12 (connected sums of Seifert manifolds).

By our bistellar flip search it turned out that not always the triangulations with fewest vertices have the smallest g_2 and therefore provide the best upper bound on Γ.

Theorem 28 *The* 3*-manifold* $\mathbb{R}\mathbf{P}^3\#\mathbb{R}\mathbf{P}^3$ *has (at least) two different minimal* g*-vectors.*

Proof: The range of f-vectors for the projective space $\mathbb{R}\mathbf{P}^3$ is described by Walkup's Theorem 1 by $\Gamma^* = \Gamma = 17$. The unique minimal triangulation of $\mathbb{R}\mathbf{P}^3$ has face vector $f=(11,51,80,40)$ and $g=(6,17)$. If we use this triangulation K to form a triangulation $K\#K$ of $\mathbb{R}\mathbf{P}^3\#\mathbb{R}\mathbf{P}^3$, then $f(K\#K)=(18,96,156,78)$ and $g(K\#K)=(13,34)$. In particular, $\Gamma(\mathbb{R}\mathbf{P}^3\#\mathbb{R}\mathbf{P}^3)\leq 34$.

On the other hand, by using bistellar flips, we obtained a triangulation of $\mathbb{R}\mathbf{P}^3\#\mathbb{R}\mathbf{P}^3$ with $f=(15,86,142,71)$ and $g=(10,36)$. This triangulation showed up in our enumeration of potential g_2-irreducible triangulations with up to 15 vertices. Since there are no g_2-irreducible 15-vertex triangulations of $\mathbb{R}\mathbf{P}^3\#\mathbb{R}\mathbf{P}^3$ with $g_2<36$ and no g_2-irreducible triangulations of $\mathbb{R}\mathbf{P}^3\#\mathbb{R}\mathbf{P}^3$ with fewer vertices, the theorem follows. $\square$

Table 12: Connected sums

Manifold	Smallest known	Upper Bound for	
	f-vector	Γ^*	Γ
$(S^2\times S^1)^{\#0} = S^3$	(5,10,10,5)	0	0
$(S^2\underline{\times} S^1)^{\#1}$	(9,36,54,27)	10	10
$(S^2\times S^1)^{\#1}$	(10,40,60,30)	11	10
$(S^2\times S^1)^{\#2},\ (S^2\underline{\times} S^1)^{\#2}$	(12,58,92,46)	22	20
$(S^2\times S^1)^{\#3},\ (S^2\underline{\times} S^1)^{\#3}$	(13,72,118,59)	30	30
$(S^2\times S^1)^{\#4},\ (S^2\underline{\times} S^1)^{\#4}$	(14,86,144,72)	40	40
$(S^2\times S^1)^{\#5},\ (S^2\underline{\times} S^1)^{\#5}$	(15,100,170,85)	50	50
$(S^2\times S^1)^{\#6},\ (S^2\underline{\times} S^1)^{\#6}$	(16,114,196,98)	60	60
$(S^2\times S^1)^{\#7},\ (S^2\underline{\times} S^1)^{\#7}$	(17,128,222,111)	70	70
$(S^2\times S^1)^{\#8},\ (S^2\underline{\times} S^1)^{\#8}$	(18,142,248,124)	80	80
$(S^2\times S^1)^{\#9},\ (S^2\underline{\times} S^1)^{\#9}$	(19,156,274,137)	92	90
$(S^2\times S^1)^{\#10},\ (S^2\underline{\times} S^1)^{\#10}$	(19,166,294,147)	100	100
$(S^2\times S^1)^{\#11},\ (S^2\underline{\times} S^1)^{\#11}$	(20,180,320,160)	110	110
$(S^2\times S^1)^{\#12},\ (S^2\underline{\times} S^1)^{\#12}$	(21,194,346,173)	121	120
$(S^2\times S^1)^{\#13},\ (S^2\underline{\times} S^1)^{\#13}$	(22,208,372,186)	137	130
$(S^2\times S^1)^{\#14},\ (S^2\underline{\times} S^1)^{\#14}$	(22,218,392,196)	140	140
$(S^2\times S^1)^{\#15},\ (S^2\underline{\times} S^1)^{\#15}$	(23,232,418,209)	154	150
$(S^2\times S^1)\#\mathbb{R}\mathbf{P}^3,\ (S^2\underline{\times} S^1)\#\mathbb{R}\mathbf{P}^3$	(14,73,118,59)	37	27
$(S^2\times S^1)^{\#2}\#\mathbb{R}\mathbf{P}^3,\ (S^2\underline{\times} S^1)^{\#2}\#\mathbb{R}\mathbf{P}^3$	(14,84,140,70)	38	
	[(15,87,144,72)]		37
$(S^2\times S^1)^{\#3}\#\mathbb{R}\mathbf{P}^3,\ (S^2\underline{\times} S^1)^{\#3}\#\mathbb{R}\mathbf{P}^3$	(15,97,164,82)	47	47
$(S^2\times S^1)^{\#4}\#\mathbb{R}\mathbf{P}^3,\ (S^2\underline{\times} S^1)^{\#4}\#\mathbb{R}\mathbf{P}^3$	(16,111,190,95)	57	57
$(S^2\times S^1)^{\#5}\#\mathbb{R}\mathbf{P}^3,\ (S^2\underline{\times} S^1)^{\#5}\#\mathbb{R}\mathbf{P}^3$	(17,125,216,108)	67	67
$(S^2\times S^1)\#L(3,1),\ (S^2\underline{\times} S^1)\#L(3,1)$	(15,89,148,74)	46	
	[(16,92,152,76)]		38
$(S^2\times S^1)^{\#2}\#L(3,1),\ (S^2\underline{\times} S^1)^{\#2}\#L(3,1)$	(15,100,170,85)	50	
	[(16,102,172,86)]		48
$L(3,1)\#L(3,1)$	(16,113,194,97)	59	
	[(17,115,196,98)]		
	[(20,126,212,106)]		56
$L(3,1)\#-L(3,1)$	(16,115,198,99)	61	
	[(17,115,196,98)]		
	[(20,126,212,106)]		56

Various other 3-manifolds also seem to have non-unique minimal g-vectors. Candidates for such manifolds are listed in the Tables 3–12 with additionally found f-vectors in brackets $[(f_0, f_1, f_2, f_3)]$. We use the refined simulated annealing process described earlier with a restriction on f_0 during the cooling stage that it be at least r greater than the smallest known f_0 for the given 3-manifold and a fixed $r = 1$, 2, 3, or 4.

For example, Kühnel and Lassmann [21] described a neighborly 15-vertex triangulation of the 3-torus T^3 with $f = (15, 105, 180, 90)$ and $g_2 = 55$. For this triangulation it is conjectured [25] that it is the unique vertex-minimal triangulation of T^3. By our search we found a triangulation of T^3 with $f = (16, 108, 184, 92)$ and $g_2 = 54$. Therefore, $\Gamma(T^3) \leq 54$, which disproves $\Gamma(T^3) = 55$ as was conjectured in [25].

We finally have to describe how to obtain upper bounds on Γ^*.

Lemma 29 *Let K be a neighborly triangulation of a 3-manifold M such that for some vertex v of K the link $\operatorname{Lk} v$ admits an Hamiltonian cycle going through all vertices of $\operatorname{Lk} v$. Then for every pair (f_0, f_1) with $f_0 \geq 0$ and*

$$\binom{f_0}{2} \geq f_1 \geq 4f_0 - 10 + g_2(K) \tag{52}$$

there is a triangulation of M with f_0 vertices and f_1 edges.

Proof: The lemma is a reformulation of Walkup's Lemmas 7.3 from [46]. The condition of the existence of an Hamiltonian cycle in the link of some vertex v is equivalent to Walkup's condition of the existence of a "spanning simple 3-tree T" whose 3-simplices have the vertex v in common. $\square$

Let K be a triangulation of some given 3-manifold M. If K is neighborly and fulfills the requirement of Lemma 29, then $\Gamma^*(M) \leq g_2(K)$. If K is not neighborly we can try with bistellar flips to reach a neighborly triangulation K' with the same number of vertices f_0 that fulfills the requirement of Lemma 29. Obviously, we then have $\Gamma^*(M) \leq g_2(K')$.

Lemma 30 *Let K be a triangulation with f-vector $f = (f_0, f_1, f_2, f_3)$ of a 3-manifold M that can be connected to a neighborly triangulation K' of M with the same number of vertices f_0 via bistellar flips (with all intermediate triangulations also with f_0 vertices). If K' fulfills the requirement of Lemma 29, then*

$$\Gamma^*(M) \leq \max\{\, g_2(K), \tbinom{f_0-1}{2} - 4(f_0 - 1) + 10 + 1 \,\}. \tag{53}$$

Proof: A neighborly triangulation K'' of M with $f_0 - 1$ vertices, if such a triangulation exists, would have $g_2(K'') = \binom{f_0-1}{2} - 4(f_0 - 1) + 10$. If $g_2(K) \geq \binom{f_0-1}{2} - 4(f_0 - 1) + 10 + 1$, then there is no triangulation K''' of M with fewer than f_0 vertices and $g_2(K''') \geq g_2(K)$. By the sequence of bistellar flips that connects K with K' we have triangulations of M with f_0 vertices for all integer values g_2 between $g_2(K)$ and $g_2(K')$. Applying subdivision operations S to these triangulations guarantees the existence of triangulations of M in the range $g_2(K) \leq g_2 \leq g_2(K')$ for all number of vertices $n \geq f_0$. In combination with $\Gamma^*(M) \leq g_2(K')$ it follows that $\Gamma^*(M) \leq g_2(K)$.

If $g_2(K) \leq \binom{f_0-1}{2} - 4(f_0-1) + 10$, then we can at least guarantee the upper bound $\Gamma^*(M) \leq \binom{f_0-1}{2} - 4(f_0-1) + 10 + 1$. $\square$

We used Lemmas 29 and 30 to establish the Γ^*-bounds in the Tables 3–12.

Theorem 31 *A complete description of the set of all possible f-vectors is given by*

(a) $\Gamma^* = \Gamma = 0$ *for* S^3,

(b) $\Gamma^* = \Gamma = 10$ *for* $S^2 \underline{\times} S^1$,

(c) $\Gamma^* = 11$ *and* $\Gamma = 10$ *for* $S^2 \times S^1$, *where, with the exception* $(9, 36)$, *all pairs* (f_0, f_1) *with* $f_0 \geq 0$ *and* $4f_0 \leq f_1 \leq \binom{f_0}{2}$ *occur,*

(d) $\Gamma^* = \Gamma = 17$ *for* $\mathbb{R}\mathbf{P}^3$,

(e) $\Gamma^* = 22$ *and* $\Gamma = 20$ *for* $(S^2 \times S^1)^{\#2}$ *and* $(S^2 \underline{\times} S^1)^{\#2}$, *where, with the exceptions* $(11, 44)$ *and* $(11, 45)$, *all pairs* (f_0, f_1) *with* $f_0 \geq 0$ *and* $4f_0 + 10 \leq f_1 \leq \binom{f_0}{2}$ *occur,*

(f) $\Gamma^* = \Gamma = 10k$ *for* $(S^2 \times S^1)^{\#k}$ *and* $(S^2 \underline{\times} S^1)^{\#k}$, $k = 3, 4, 5, 6, 7, 8, 10, 11, 14$,

while $\Gamma^*(M) \geq \Gamma(M) \geq 21$ *for all other* 3*-manifolds* M.

Proof: Parts (a)–(d) follow from Theorem 1 of Walkup. Since there are no 11-vertex triangulations of $(S^2 \times S^1)^{\#2}$ and $(S^2 \underline{\times} S^1)^{\#2}$ [42], the lower bound (21) and the triangulations that were used to compute the respective upper bounds on Γ and Γ^* in Table 12 together imply (e). The lower bound (21) and respective triangulations (cf. Table 12) imply (f).

Due to Theorem 25 there is only one g_2-irreducible triangulation of a 3-manifold with $\Gamma \leq 20$, which is the unique g_2-irreducible triangulation of $\mathbb{R}\mathbf{P}^3$. As observed in Section 4, a triangulation K which realizes the minimum g_2 for a particular 3-manifold M is either g_2-irreducible, or is of the form $K_1 \# K_2$ or HK', where the component triangulations realize their minimum g_2. In the case $K = K_1 \# K_2$ we have $g_2(K_1 \# K_2) = g_2(K_1) + g_2(K_2)$ according to (38), whereas $g_2(HK') = g_2(K') + 10$ in the case $K = HK'$ according to (36). The 3-manifolds obtained by adding a handle to one of the 3-manifolds listed in (a)–(c) are listed in (b), (c), and (e). If K' triangulates $\mathbb{R}\mathbf{P}^3$ or a 3-manifold M with $\Gamma(M) \geq 18$, then $g_2(HK') \geq 27$. Similarly, all manifolds with triangulations of the form $K = K_1 \# K_2$ (with one or both factors possibly triangulations of S^3) are either listed in (a)–(e) or have $g_2(K_1 \# K_2) \geq 27$. Thus $\Gamma^*(M) \geq \Gamma(M) \geq 21$ for a 3-manifold M other than those from (a)–(e). $\square$

Corollary 32 *The following 3-manifolds have unique componentwise minimal f-vectors:*

- S^3 *with* $f = (5, 10, 10, 5)$,
- $S^2 \underline{\times} S^1$ *with* $f = (9, 36, 54, 27)$,
- $S^2 \times S^1$ *with* $f = (10, 40, 60, 30)$,
- $\mathbb{RP}^3$ *with* $f = (11, 51, 80, 40)$,
- $(S^2 \times S^1)^{\#2}$ *and* $(S^2 \underline{\times} S^1)^{\#2}$ *with* $f = (12, 58, 92, 46)$,
- $(S^2 \times S^1)^{\#3}$ *and* $(S^2 \underline{\times} S^1)^{\#3}$ *with* $f = (13, 72, 118, 59)$,
- $(S^2 \times S^1)^{\#4}$ *and* $(S^2 \underline{\times} S^1)^{\#4}$ *with* $f = (14, 86, 144, 72)$,
- $(S^2 \times S^1)^{\#5}$ *and* $(S^2 \underline{\times} S^1)^{\#5}$ *with* $f = (15, 100, 170, 85)$,
- $(S^2 \times S^1)^{\#6}$ *and* $(S^2 \underline{\times} S^1)^{\#6}$ *with* $f = (16, 114, 196, 98)$,
- $(S^2 \times S^1)^{\#7}$ *and* $(S^2 \underline{\times} S^1)^{\#7}$ *with* $f = (17, 128, 222, 111)$,
- $(S^2 \times S^1)^{\#8}$ *and* $(S^2 \underline{\times} S^1)^{\#8}$ *with* $f = (18, 142, 248, 124)$,
- $(S^2 \times S^1)^{\#10}$ *and* $(S^2 \underline{\times} S^1)^{\#10}$ *with* $f = (19, 166, 294, 147)$,
- $(S^2 \times S^1)^{\#11}$ *and* $(S^2 \underline{\times} S^1)^{\#11}$ *with* $f = (20, 180, 320, 160)$,
- $(S^2 \times S^1)^{\#14}$ *and* $(S^2 \underline{\times} S^1)^{\#14}$ *with* $f = (22, 218, 392, 196)$.

Remark 33 *It is not known whether there is a 3- or higher-dimensional manifold that admits different minimal f-vectors.*

7 Γ-Values and Matveev Complexity

In [29] Matveev introduced a notion of complexity for three-manifolds. The main properties of $c(M)$, the *complexity* of M, are

- $c(M)$ is a nonnegative integer and for fixed n, the number of irreducible closed three manifolds with $c(M) \leq n$ is finite.
- $c(M_1 \# M_2) = c(M_1) + c(M_2)$.

Since then, a significant amount of research has gone into understanding $c(M)$ and determining all closed 3-manifolds with small complexity. See the recent text [30] for details. In view of the fact that $\Gamma(M)$ also has the first two properties and Conjecture 11 is the third property, it is natural to consider the relationship between $c(M)$ and $\Gamma(M)$.

For irreducible 3-manifolds with $c(M) > 0$ the complexity of M is the number of tetrahedra needed in a pseudo-simplicial triangulation. As noted earlier, two barycentric subdivisions produces a simplicial triangulation of M with $24^2 c(M)$ facets. As $g_2 \leq h_2$ and the number of facets for any K is $2 + 2h_1 + h_2$, we see that $\Gamma(M) \leq O(c(M))$. Conversely, if M is irreducible, then any triangulation K of M which realizes $\Gamma(M)$ must be a g_2-irreducible triangulation. By Theorem 17, the number of vertices in K is bounded linearly by $\Gamma(M)$. Hence h_1 and the number of facets are linearly bounded by $\Gamma(M)$. So, $c(M) \leq O(\Gamma(M))$. It is easy to see that these estimates are very crude.

Problem 34 *What is the exact relationship between $c(M)$ and $\Gamma(M)$?*

Table 13: Irreducible 3-manifolds sorted with respect to Γ-bounds up to 66.

Upper Bound for Γ	Manifolds
0	S^3
17	$\mathbb{R}\mathbf{P}^3$
28	$L(3,1)$
38	$L(4,1)$, $\mathbb{R}\mathbf{P}^2 \times S^1$
40	$L(5,2)$, $P_2 = S^3/Q$
47	$L(5,1)$, P_3
50	$L(7,2)$, S^3/T^*
51	$L(8,3)$
52	$S^3/I^* = \Sigma(2,3,5)$
54	P_4, T^3
55	S^3/O^*
56	$L(6,1)$, B_2
57	G_5
58	$L(9,2)$
59	$L(10,3)$, G_3, B_4, $\{Oo, 1 \mid 1\}$
60	$K \times S^1$, G_2
61	G_4, B_3
62	$\{Oo, 1 \mid 2\}$
63	$\Sigma(2,3,7)$
64	P_5
65	$L(7,1)$
66	G_6

Table 14: Irreducible orientable 3-manifolds of complexity up to 3.

Complexity	Manifolds
0	S^3, $\mathbb{R}\mathbf{P}^3$, $L(3,1)$
1	$L(4,1)$, $L(5,2)$
2	$L(5,1)$, $L(7,2)$, $L(8,3)$, $P_2 = S^3/Q$
3	$L(6,1)$, $L(9,2)$, $L(10,3)$, $L(11,3)$, $L(12,5)$, $L(13,5)$, P_3

Tables 13 and 14 (which is taken from [31]) show that, at least for manifolds of small complexity for which we have some data, the rank ordering of the two measures of complexity are very close. Indeed, we believe that the three irreducible 3-manifolds with smallest Γ are $\Gamma(S^3) = 0$, $\Gamma(\mathbb{R}P^3) = 17$ and $\Gamma(L(3,1)) = 28$.

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