# Superization and $(q,t)$-specialization in combinatorial Hopf algebras

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*Dedicated to Anders Björner on the occasion of his sixtieth birthday*

**Abstract**

We extend a classical construction on symmetric functions, the superization process, to several combinatorial Hopf algebras, and obtain analogs of the hook-content formula for the $(q,t)$-specializations of various bases. Exploiting the dendriform structures yields in particular $(q,t)$-analogs of the Björner-Wachs $q$-hook-length formulas for binary trees, and similar formulas for plane trees.

# Contents

# 1 Introduction

Combinatorial Hopf algebras are special graded and connected Hopf algebras based on certain classes of combinatorial objects. There is no general agreement of what their precise definition should be, but looking at their structure as well as to their existing applications, it is pretty clear that they are to be regarded as generalizations of the Hopf algebra $Sym$ of symmetric functions.

It is well-known that one can define symmetric functions $f(X-Y)$ of a formal difference of alphabets. This can be interpreted either as the image of the difference $\sum_i x_i - \sum_j y_j$ by the operator $f$ in the $\lambda$-ring generated by $X$ and $Y$, or, in Hopf-algebraic terms, as $(\mathrm{Id}\otimes\tilde{\omega})\circ\Delta(f)$, where $\Delta$ is the coproduct and $\tilde{\omega}$ the antipode. And in slightly less pedantic terms, this just amounts to replacing the power-sums $p_n(X)$ by $p_n(X)-p_n(Y)$, a process already discussed at length in Littlewood's book [26, p. 100].

This article deals with a class of combinatorial identities whose first examples involved Schur functions. As is well-known, the Schur functions $s_\lambda(X)$ are the characters of the irreducible tensor representations of the general Lie algebra $\mathfrak{g}l(n)$. Similarly, the $s_\lambda(X-Y)$ are the characters of the irreducible tensor representations of the general Lie superalgebras $\mathfrak{g}l(m|n)$ [3]. These symmetric functions are not positive sums of monomials, and for this reason, one often prefers to use as characters the so-called *supersymmetric functions* $s_\lambda(X|Y)$, which are defined by $p_n(X|Y) = p_n(X) + (-1)^{n-1}p_n(Y)$ (see [41]), and are indeed positive sums of monomials: their complete homogeneous functions are given by

$$\sigma_t(X|Y) = \sum_{n\geqslant 0} h_n(X|Y)t^n = \lambda_t(Y)\sigma_t(X) = \prod_{i,j}\frac{1+ty_j}{1-tx_i}\,. \tag{1}$$

Another (not unrelated) classical result on Schur functions is the *hook-content* formula [30, I.3 Ex. 3], which gives in closed form the specialization of a Schur function at the virtual alphabet

$$\frac{1-t}{1-q} = \frac{1}{1-q} - t\frac{1}{1-q} = 1+q+q^2+\cdots-(t+tq+tq^2+\cdots)\,. \tag{2}$$

This specialization was first considered by Littlewood [26, Ch. VII], who obtained a factorized form for the result, but with possible simplifications. The improved version known as the hook-content formula

$$s_\lambda\left(\frac{1-t}{1-q}\right) = q^{n(\lambda)}\prod_{x\in\lambda}\frac{1-tq^{c(x)}}{1-q^{h(x)}}\,, \tag{3}$$

which is a $(q,t)$-analog of the famous hook-length formula of Frame-Robinson-Thrall [11], is due to Stanley [44].

The first example of a combinatorial Hopf algebra generalizing symmetric functions is Gessel's algebra of quasi-symmetric functions [13]. Its Hopf algebra structure was further worked out in [31, 12], and later used in [24], where two *different* analogs of the hook-content formula for quasi-symmetric functions are given. Indeed, the notation

$$F_I\left(\frac{1-t}{1-q}\right) \tag{4}$$

is ambiguous. It can mean (at least) two different things:

$$\text{either } F_I\Big(\frac{1}{1-q}\,\hat{\times}\,(1-t)\Big) \quad\text{or}\quad F_I\Big((1-t)\,\hat{\times}\,\frac{1}{1-q}\Big)\,,$$

where $\hat{\times}$ denotes the ordered product of alphabets. The second one is of the form $F_I(X - Y)$ (in the sense of [24]), but the first one is not (cf. [24]).

In this article, we shall extend the notion of superization to several combinatorial Hopf algebras. We shall start with **FQSym** (Free quasi-symmetric functions, based on permutations), and our first result (Theorem 3.1) will allow us to give new expressions and combinatorial proofs of the $(q,t)$-specializations of quasi-symmetric functions. Next, we extend these results to **PBT**, the Loday-Ronco algebra of planar binary trees, and obtain a $(q,t)$-analog of the Knuth and Björner-Wachs hook-length formulas for binary trees. These results rely on the dendriform structure of **PBT**. Exploiting in a similar way the tridendriform structure of **WQSym** (Word quasi-symmetric functions, based on packed words, or set compositions), we arrive at a $(q,t)$ analog of the formula of [20] counting packed words according to the shape of their plane tree.

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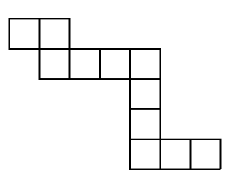
# 2 Background

## 2.1 Some conventions

The algebras considered in this paper are defined from infinite totally ordered sets of variables, referred to as *alphabets*. It is customary to reserve the letters $A$, $B, \ldots$ for noncommutative alphabets, and $X$, $Y, \ldots$ for commutative ones. If $A$ and $B$ are two alphabets, their ordinal sum is denoted by $A \hat{+} B$, or simply by $A + B$ when there is no ambiguity. Their cartesian product, endowed with the lexicographic order, is denoted by $AB$.

Multi-indices in upper position denote a product: if elements $Z_i$ are defined, $Z^{(i_1,\ldots,i_r)}$ means $Z_{i_1} \cdots Z_{i_r}$.

The symmetric group is denoted by $\mathfrak{S}_n$. A permutation $\sigma \in \mathfrak{S}_n$ is said to have a *descent* at $i \in [1, n-1]$ if $\sigma(i) > \sigma(i+1)$. The set of such $i$ is called the *descent set* of $\sigma$ and denoted by $\mathrm{Des}(\sigma)$. Descent sets of permutations of $\mathfrak{S}_n$ can be encoded by compositions of $n$, *i.e.*, finite sequences of positive integers $I = (i_1, \ldots, i_r)$ summing to $n$. To the descent set $D = \{d_1, \ldots, d_{r-1}\}$ we associate the composition $I$ such that $d_k = i_1 + i_2 + \cdots + i_k$, and $i_r = n - d_{r-1}$. We say that $I$ is the *descent composition* of $\sigma$. We also say that $D$ is the descent set of $I$ and write $I = C(\sigma) = C(D)$, and $D = \mathrm{Des}(\sigma) = \mathrm{Des}(I)$. These defintions extend to words $w = w_1 \cdots w_n$ over an arbitrary ordered alphabet: $w$ has a descent at $i$ if $w_i > w_{i+1}$. The notation $I \vDash n$ means that $I$ is a composition of $n$. Compositions are represented by ribbon diagrams:



(5)

where the number of cells of the $k$-th row is the $k$-th part of the composition. Thus, the above diagram corresponds to the composition $(2, 4, 1, 1, 3)$.

The conjugate composition, obtained by reading from right to left the heights of the columns of this diagram, is denoted by $I^{\sim}$.

The sum of the descents of a word $w$ or of a composition $I$ is called its *major index*, and is denoted by $\mathrm{maj}\,(w)$ or $\mathrm{maj}\,(I)$.

The *evaluation* $\mathrm{Ev}(w)$ of a word $w$ over a totally ordered alphabet $A$ is the sequence $(|w|_a)_{a\in A}$ where $|w|_a$ is the number of occurences of $a$ in $w$. The *packed evaluation* $I = \mathbf{pEv}(w)$ is the composition obtained by removing the zeros in $\mathrm{Ev}(w)$.

The *standardized word* $\mathrm{std}(w)$ of a word $w \in A^*$ is the permutation obtained by iteratively scanning $w$ from left to right, and labelling $1, 2, \ldots$ the occurrences of its smallest letter, then numbering the occurrences of the next one, and so on. For example, $\mathrm{std}(bbacab) = 341625$. For a word $w$ on the alphabet $\{1, 2, \ldots\}$, we denote by $w[k]$ the word obtained by replacing each letter $i$ by the integer $i + k$.

All algebras are over some field $\mathbb{K}$ of characteristic 0.

## 2.2 Noncommutative symmetric functions

### 2.2.1 The algebra Sym

The reader is referred to [12] for the basic theory of noncommutative symmetric functions. The encoding of Hopf-algebraic operations by means of sums, differences, and products of virtual alphabets is fully explained in [24]. Here is a brief reminder.

The algebra of noncommutative symmetric functions, denoted by $\mathbf{Sym}$, or by $\mathbf{Sym}(A)$ if we consider the realization in terms of an auxiliary alphabet, is defined as the free associative algebra over an infinite sequence of generators $S_i$, $i \geqslant 1$. We also set $S_0 = 1$. It is graded by $\deg S_i = i$. Its homogeneous component of degree $n$ is denoted by $\mathbf{Sym}_n$.

If $A$ is an infinite totally ordered alphabet, we can set

$$\sigma_t(A) := \sum_{n\geqslant 0} S_n(A)t^n = \overrightarrow{\prod_{i\geqslant 1}}(1 - ta_i)^{-1} \tag{6}$$

where $t$ is an auxiliary indeterminate commuting with $A$. Thus, $S_n(A)$ is the sum of all nondecreasing words of length $n$ over $A$. The inverse of the generating series $\sigma_t(A)$ is

$$\lambda_{-t}(A) := \sum_{n\geqslant 0} \Lambda_n(A)(-t)^n = \overleftarrow{\prod_{i\geqslant 1}}(1 - ta_i). \tag{7}$$

Thus, $\Lambda_n(A)$ is the sum of all decreasing words of length $n$ over $A$. Under the commutative image $A \to X$ (*i.e.*, sending the letters of $A$ to commuting variables), $S_n(A)$ and $\Lambda_n(A)$ go to $h_n(X)$ (complete homogeneous symmetric functions) and $e_n(X)$ (elementary symmetric functions) respectively.

### 2.2.2 Linear bases

Bases of the homogeneous component $\mathbf{Sym}_n$ are labelled by compositions $I$ of $n$. The ribbon basis may be defined by

$$R_I(A) = \sum_{C(w)=I} w\,, \tag{8}$$

that is, the sum of all words over $A$ whose descent composition is $I$. We have the relation

$$S^I = \sum_{J \leqslant I} R_J \tag{9}$$

where $\leqslant$ is the reverse refinement order.

### 2.2.3 Hopf algebra structure

If $A$ and $B$ are two totally ordered alphabets we can define $S_n(A\hat{+}B)$, which is clearly equal to

$$S_n(A\hat{+}B) = \sum_{i+j=n} S_i(A)S_j(B)\,. \tag{10}$$

If we assume that $A$ and $B$ commute with each other, this defines a coproduct on **Sym**, under the usual identification $f(A)g(B) \equiv f \otimes g$:

$$\Delta F = F(A\hat{+}B)\,. \tag{11}$$

Clearly, this is an algebra morphism, endowing **Sym** with the structure of a bialgebra. Being graded and connected, **Sym** is actually a Hopf algebra.

The graded dual of **Sym** is $QSym$ (quasi-symmetric functions). The dual basis of $(S^I)$ is $(M_I)$ (monomial), and that of $(R_I)$ is $(F_I)$ (Gessel's fundamental basis).

### 2.2.4 The $(1-q)$-transform

The Hopf structures on **Sym** and $QSym$ allows one to mimic, up to a certain extent, the $\lambda$-ring notation. In particular, the $(1-q)$-transform of ordinary symmetric functions (sending power-sums $p_n$ to $(1-q^n)p_n$) is easily generalized. Recall from [24] that the noncommutative symmetric functions of a difference of alphabets are defined by the change of generators

$$S_n \mapsto S_n(A-B) = \sum_{i+j=n} (-1)^i \Lambda_i(B) S_j(A) = p \circ (I \otimes \gamma) \circ \Delta S_n, \tag{12}$$

where $\Delta$ is the coproduct, $\gamma$ the antipode, $I$ the identity map, and

$$p(F \otimes G) := G(B)F(A). \tag{13}$$

Thus,

$$\sigma_t((1-q)A) := \lambda_{-qt}(A)\sigma_t(A), \tag{14}$$

Since we also know the inverse transform $(A/(1-q))$, we can introduce a second variable and define noncommutative symmetric functions and quasi-symmetric functions of the virtual alphabet $(1-t)/(1-q)$ (see [24] and below for details). In the sequel, we shall need to define this specialization in more complicated Hopf algebras. The best way to achieve this is to rely upon another description, involving the internal product of **Sym**.

### 2.2.5 Internal product

The Hopf algebra **Sym** is sometimes improperly called the Solomon descent algebra in the literature. This is because its homogeneous components $\mathbf{Sym}_n$ can be endowed with a new product, called the *internal product*, for which they are anti-isomorphic to the descent algebras of symmetric groups [12]. To cut the story short, if

$$D_I = \sum_{C(\sigma)=I} \sigma \tag{15}$$

denotes the sum of all permutations with descent composition $I$ in the group algebra of $\mathfrak{S}_n$, then the $D_I$ form the basis of a subalgebra $\Sigma_n$ of $\mathbb{Z}\mathfrak{S}_n$ (Solomon's descent algebra, [43]), and the map

$$\alpha : \quad D_I \mapsto R_I \tag{16}$$

is an anti-isomorphism from $\Sigma_n$ and $\mathbf{Sym}_n$ endowed with its internal product $*$. The internal product is extended to **Sym** by requiring that $F * G = 0$ if $F$ and $G$ are homogeneous of different degrees.

The fundamental property of the internal product is the *splitting formula*

$$(f_1 \dots f_r) * g = \mu_r \left[ (f_1 \otimes \cdots \otimes f_r) *_r \Delta^r g \right], \tag{17}$$

where $\mu_r$ denotes the $r$-fold multiplication, $*_r$ the internal product in $\mathbf{Sym}^{\otimes r}$, and $\Delta^r$ the iterated coproduct with values in $\mathbf{Sym}^{\otimes r}$.

This formula implies that for any (genuine or virtual) alphabet $X$, the algebra morphism $F \mapsto F(XA)$ is given by

$$F(XA) = F(A) * \sigma_1(XA)\,. \tag{18}$$

In particular,

$$F((1-q)A) = F(A) * \sigma_1((1-q)A) \tag{19}$$

and the inverse transform is given by

$$F(A) = F((1-q)A) * \sigma_1\left(\frac{A}{1-q}\right), \tag{20}$$

where

$$\sigma_t\left(\frac{A}{1-q}\right) := \cdots \sigma_{q^2 t}(A)\sigma_{qt}(A)\sigma_t(A)\,. \tag{21}$$

We usually consider that our auxiliary variable $t$ is of rank one, which means that $\sigma_t(A) = \sigma_1(tA)$.

## 2.3 Free quasi-symmetric functions

Recall from [9] that for an infinite totally ordered alphabet $A$, $\mathbf{FQSym}(A)$ is the subalgebra of $\mathbb{K}\langle A\rangle$ spanned by the polynomials

$$\mathbf{G}_\sigma(A) = \sum_{\mathrm{std}(w)=\sigma} w \tag{22}$$

the sum of all words in $A^n$ whose standardization is the permutation $\sigma \in \mathfrak{S}_n$. The multiplication rule is, for $\alpha \in \mathfrak{S}_k$ and $\beta \in \mathfrak{S}_\ell$,

$$\mathbf{G}_\alpha \mathbf{G}_\beta = \sum_{\substack{\gamma\in\mathfrak{S}_{k+l};\,\gamma=u\cdot v\\ \mathrm{std}(u)=\alpha,\mathrm{std}(v)=\beta}} \mathbf{G}_\gamma\,. \tag{23}$$

The noncommutative ribbon Schur function $R_I \in \mathbf{Sym}$ is then

$$R_I = \sum_{C(\sigma)=I} \mathbf{G}_\sigma\,. \tag{24}$$

This defines a Hopf embedding $\mathbf{Sym} \to \mathbf{FQSym}$. Indeed, the coproduct of $\mathbf{FQSym}$ may also be defined by

$$\Delta \mathbf{G}_\sigma = \mathbf{G}_\sigma(A \hat{+} B), \tag{25}$$

where $A\hat{+}B$ denotes the ordinal sum. This clearly is an algebra morphism, which restricts to the coproduct of $\mathbf{Sym}$. As a Hopf algebra, $\mathbf{FQSym}$ is self-dual. It is isomorphic to the Hopf algebra of permutations considered in [31] and [2].

The scalar product materializing this duality is the one for which $(\mathbf{G}_\sigma\,,\,\mathbf{G}_\tau) = \delta_{\sigma,\tau^{-1}}$ (Kronecker symbol). Hence, $\mathbf{F}_\sigma := \mathbf{G}_{\sigma^{-1}}$ is the dual basis of $\mathbf{G}_\sigma$.

The *internal product* $*$ of $\mathbf{FQSym}$ is induced by composition $\circ$ in $\mathfrak{S}_n$ in the basis $\mathbf{F}_\sigma$, that is,

$$\mathbf{F}_\sigma * \mathbf{F}_\tau = \mathbf{F}_{\sigma\circ\tau} \quad \text{so that} \quad \mathbf{G}_\sigma * \mathbf{G}_\tau = \mathbf{G}_{\tau\circ\sigma}\,. \tag{26}$$

Its restriction to $\mathbf{Sym}_n$ coincides with the internal product already defined.

The transpose of the Hopf embedding $\mathbf{Sym} \to \mathbf{FQSym}$ is the *commutative image* $\mathbf{F}_\sigma \mapsto \mathbf{F}_\sigma(X) = F_I(X)$, where $I$ is the descent composition of $\sigma$, and $F_I$ is Gessel's fundamental basis of $QSym$. Note that this implies that if $X$ is a commutative alphabet, $\mathbf{F}_\sigma(X)$ depends only on the descent composition $I = C(\sigma)$.

## 2.4 Word quasi-symmetric functions

A word $u$ over $\mathbb{N}^*$ is said to be *packed* if the set of letters appearing in $u$ is an interval of $\mathbb{N}^*$ containing 1. The algebra $\mathbf{WQSym}(A)$ (Word Quasi-Symmetric functions) is defined as the subalgebra of $\mathbb{K}\langle A\rangle$ based on *packed words* and spanned by the elements

$$\mathbf{M}_u(A) := \sum_{\mathrm{pack}(w)=u} w, \tag{27}$$

where $\operatorname{pack}(w)$ is the *packed word* of $w$, that is, the word obtained by replacing all occurrences of the $k$-th smallest letter of $w$ by $k$. For example,

$$\operatorname{pack}(871883319) = 431442215. \tag{28}$$

Let $\mathbf{N}_u = \mathbf{M}_u^*$ be the dual basis of $(\mathbf{M}_u)$. It is known that $\mathbf{WQSym}$ is a self-dual Hopf algebra [17, 37] and that on the graded dual $\mathbf{WQSym}^*$, an internal product $*$ may be defined by

$$\mathbf{N}_u * \mathbf{N}_v = \mathbf{N}_{\operatorname{pack}(u,v)}, \tag{29}$$

where the packing of biwords is defined with respect to the lexicographic order on biletters, so that, for example,

$$\operatorname{pack}\begin{pmatrix} 42412253 \\ 53154323 \end{pmatrix} = 62513274. \tag{30}$$

This product is induced from the internal product of parking functions [38, 33, 39] and allows one to identify the homogeneous components $\mathbf{WQSym}_n$ with the (opposite) Solomon-Tits algebras, in the sense of [40].

The (opposite) Solomon descent algebra, realized as $\mathbf{Sym}_n$, is embedded in the (opposite) Solomon-Tits algebra realized as $\mathbf{WQSym}_n^*$ by

$$S^I = \sum_{\operatorname{Ev}(u)=I} \mathbf{N}_u, \tag{31}$$

where $\operatorname{Ev}(u)$ is the evaluation of $u$ defined in Section 2.

# 3 Free super-quasi-symmetric functions

## 3.1 Supersymmetric functions

As already mentioned in the introduction, in the $\lambda$-ring notation, the definition of *supersymmetric functions* is transparent. If $X$ and $\bar{X}$ are two independent infinite alphabets, the *superization* $f^\#$ of $f \in Sym$ is

$$f^\# := f(X \mid \bar{X}) = f(X - q\bar{X})|_{q=-1}, \tag{32}$$

where $f(X - q\bar{X})$ is interpreted in the $\lambda$-ring sense ($p_n(X - q\bar{X}) := p_n(X) - q^n p_n(\bar{X})$), $q$ being of rank one, so that $p_n(X|\bar{X}) = p_n(X) - (-1)^n p_n(\bar{X})$. This can also be written as an internal product

$$f^\# = f * \sigma_1^\#, \tag{33}$$

where $\sigma_1^\# = \sigma_1(X - q\bar{X})|_{q=-1} = \lambda_1(\bar{X})\sigma_1(X)$, and the internal product is extended to the algebra generated by $Sym(X)$ and $Sym(\bar{X})$ by means of the splitting formula (17) and the rules

$$\sigma_1 * f = f * \sigma_1, \qquad \overline{\sigma}_1 * \overline{\sigma}_1 = \sigma_1. \tag{34}$$

Here, the bar means $\overline{f(X, \overline{X})} = f(\overline{X}, X)$. In particular, $\overline{\sigma}_1 = \overline{\sigma_1(X)} = \sigma_1(\overline{X})$.

## 3.2 Noncommutative supersymmetric functions

### 3.2.1 A basis of $\mathbf{Sym}^{(2)}$

The superization map can be lifted to noncommutative symmetric functions. We need two independent infinite totally ordered alphabets $A$ and $\bar{A}$. Let $\mathbf{Sym}^{(2)} := \mathbf{Sym}(A)\star\mathbf{Sym}(\bar{A})$ be the free product of two copies of $\mathbf{Sym}$, *i.e.*, the free algebra generated by the $S_n(A)$ and $S_n(\overline{A})$. We define $\mathbf{Sym}(A|\bar{A})$ as the subalgebra of $\mathbf{Sym}^{(2)}$ generated by the $S_n^{\#}$ where

$$\sigma_1^{\#} = \bar{\lambda}_1\sigma_1 = \sum_{I=(i_1,\ldots,i_{r+1})} (-1)^{i_1+\cdots+i_r-r}\,\overline{S^{i_1\ldots i_r}}S^{i_{r+1}}. \tag{35}$$

For example,

$$S_1^{\#} = S^1 + S^{\bar{1}}, \qquad S_2^{\#} = S^2 + S^{\bar{1}1} - S^{\bar{2}} + S^{\overline{11}}, \tag{36}$$

$$S_3^{\#} = S^3 + S^{\bar{1}2} + S^{\overline{11}1} - S^{\bar{2}1} + S^{\overline{111}} - S^{\overline{21}} - S^{\overline{12}} + S^{\bar{3}}. \tag{37}$$

We shall denote the generators of $\mathbf{Sym}^{(2)}$ by $S_{(k,\epsilon)}$ where $\epsilon = \{\pm 1\}$, so that $S_{(k,1)} = S_i$ and $S_{(k,-1)} = \bar{S}_k$.

The corresponding basis of $\mathbf{Sym}^{(2)}$ is then written

$$S^{(I,\epsilon)} = S^{(i_1,\ldots,i_r),(\epsilon_1,\ldots,\epsilon_r)} := S_{(i_1,\epsilon_1)}S_{(i_2,\epsilon_2)}\ldots S_{(i_r,\epsilon_r)}, \tag{38}$$

where $I = (i_1,\ldots,i_r)$ is a composition and $\epsilon = (\epsilon_1,\ldots,\epsilon_r) \in \{\pm 1\}^r$ is a vector of signs.

The superization $f^\sharp$ of $f \in \mathbf{Sym}$ is defined as its image by the algebra morphism $S_n \mapsto S_n^\sharp$.

### 3.2.2 Signed ribbons

Following [19], we define an order on signed compositions as follows: let

$$(I,\epsilon) = ((i_1,\ldots,i_m),(\epsilon_1,\ldots,\epsilon_m)) \quad \text{and} \quad (J,\eta) = ((j_1,\ldots,j_p),(\eta_1,\ldots,\eta_p))$$

two signed compositions. Then $(I,\epsilon)$ is *coarser* than $(J,\eta)$, and we write $(I,\epsilon) \leqslant (J,\eta)$, if there exists a sequence $(l_0 = 0, l_1, \ldots, l_p = m)$ such that for any integer $k$,

$$j_k = i_{l_{k-1}+1} + \cdots + i_{l_k} \quad \text{and} \quad \eta_k = \epsilon_{l_{k-1}+1} = \cdots = \epsilon_{l_k}. \tag{39}$$

For example, the signed compositions coarser than $((1,1,3,2),(-1,1,1,-1))$ are

$$((1,1,3,2),(-1,1,1,-1)) \quad \text{and} \quad ((1,4,2),(-1,1,-1)). \tag{40}$$

The *signed ribbons* $R_{(J,\eta)}$ are defined by the following formula [19]:

$$S^{(I,\epsilon)} =: \sum_{(J,\eta)\leqslant(I,\epsilon)} R_{(J,\eta)}. \tag{41}$$

### 3.2.3 Internal product of $\mathbf{Sym}^{(2)}$

Again, we extend the internal product by formulas (17) and (34) where, now, $f_1, \ldots, f_r$, $g \in \mathbf{Sym}^{(2)}$, and $\sigma_1 = \sigma_1(A)$, $\overline{\sigma_1} = \sigma_1(\bar{A})$. The resulting algebra is isomorphic to the Mantaci-Reutenauer algebra of type $B$ ([32], see [1] for this version). The superization of $f \in \mathbf{Sym}$ can now be written as

$$f^{\#} = f * \sigma_1^{\#} = f(A - q\bar{A})|_{q=-1} = f * (\overline{\lambda}_1 \sigma_1). \tag{42}$$

## 3.3 Super-quasi-symmetric functions

There are two natural and nonequivalent choices for defining super-quasi-symmetric functions. The first one is to set $F(X|\bar{X}) = F(X - q\bar{X})|_{q=-1}$ as in [15]. The second one is obtained by commutative image from the free super-quasi-symmetric functions to be defined below. Let us note that super-quasi-symmetric functions have been recently interpreted as characters of certain abstract crystals of the Lie superalgebras $\mathfrak{gl}(m|n)$ [25].

## 3.4 Free super-quasi-symmetric functions

### 3.4.1 The superization map

The expressions (42) are still well-defined for an arbitrary $f \in \mathbf{FQSym}$. We can define $\mathbf{FQSym}(A|\bar{A})$ as a subalgebra of the free product $\mathbf{FQSym}^{(2)} = \mathbf{FQSym}(A) \star \mathbf{FQSym}(\bar{A})$, *i.e.*, the algebra freely generated by the $\mathbf{G}_\alpha(A)$, $\mathbf{G}_\beta(\overline{A})$, where $\alpha$ and $\beta$ run over connected permutations.

Again, the internal product $*$ is extended to the free product by conditions (17) (valid only if $g \in \mathbf{Sym}^{(2)}$, which is enough for our purpose, *cf.* [8]), and (34). Note that $\mathbf{FQSym}^{(2)}$ is the algebra of free quasi-symmetric functions of level 2, as defined in [34].

### 3.4.2 Conventions for signed words

Let us set

$$A^{(0)} = A = \{a_1 < a_2 < \ldots < a_n < \ldots\}\,, \tag{43}$$

$$A^{(1)} = \bar{A} = \{\ldots < \bar{a}_n < \ldots < \bar{a}_2 < \bar{a}_1\}\,, \tag{44}$$

order $\mathbf{A} = \bar{A} \cup A$ by $\bar{a}_i < a_j$ for all $i, j$, and denote by std the standardization of signed words with respect to this order.

We denote by $\operatorname{maj}(w, \epsilon)$ the major index of $\mathbf{w}$ with respect to this order on the signed alphabet.

We also need the *signed standardization* $\mathbf{Std}$, defined as follows. Represent a signed word $\mathbf{w} \in \mathbf{A}^n$ by a pair $(w, \epsilon)$, where $w \in A^n$ is the underlying unsigned word, and $\epsilon \in \{\pm 1\}^n$ is the vector of signs. Then $\mathbf{Std}(w, \epsilon) = (\operatorname{std}(w), \epsilon)$.

We denote by $m(\epsilon)$ the number of entries $-1$ in $\epsilon$.

### 3.4.3 A combinatorial expression of the superization map

A basis of $\mathbf{FQSym}^{(2)}$ is given by

$$\mathbf{G}_{\sigma,\epsilon} := \sum_{\mathbf{Std}(\mathbf{w})=(\sigma,\epsilon)} \mathbf{w} \quad \in \mathbb{Z}\langle \mathbf{A}\rangle. \tag{45}$$

and the internal product obtained from (17) and (34) coincides with the one of [34], so that it is in fact always well-defined. In particular, viewing signed permutations as elements of the group $\{\pm 1\} \wr \mathfrak{S}_n$,

$$\mathbf{G}_{\alpha,\epsilon} * \mathbf{G}_{\beta,\eta} = \mathbf{G}_{(\beta,\eta)\circ(\alpha,\epsilon)} = \mathbf{G}_{\beta\circ\alpha,(\eta\alpha)\cdot\epsilon}, \tag{46}$$

with $\eta\alpha = (\eta_{\alpha(1)}, \ldots, \eta_{\alpha(n)})$ and $\epsilon \cdot \eta = (\epsilon_1\eta_1, \ldots, \epsilon_n\eta_n)$.

We now embed $\mathbf{FQSym}$ in $\mathbf{FQSym}^{(2)}$ by

$$\mathbf{G}_\sigma \mapsto \mathbf{G}_{(\sigma,1^n)}, \tag{47}$$

which allows us to define

$$\mathbf{G}^\#_\sigma := \mathbf{G}_\sigma(A|\bar{A}) = \mathbf{G}_\sigma * \sigma^\#_1, \tag{48}$$

and $\mathbf{FQSym}(A|\bar{A})$ as the algebra spanned by the $\mathbf{G}_\sigma(A|\bar{A})$.

**Theorem 3.1** *The expansion of $\mathbf{G}_\sigma(A|\bar{A})$ on the basis $\mathbf{G}_{\tau,\epsilon}$ is*

$$\mathbf{G}_\sigma(A|\bar{A}) = \sum_{\mathrm{std}(\tau,\epsilon)=\sigma} \mathbf{G}_{\tau,\epsilon}\,. \tag{49}$$

*Proof* – This is clear for $\sigma = 12\ldots n$:

$$\sum_n \mathbf{G}_{12\ldots n}(A|\bar{A}) = \bar{\lambda}_1 \cdot \sigma_1 = \sum_{\substack{i_1<i_2<\cdots<i_k \\ j_1\leqslant j_2\leqslant\cdots\leqslant j_\ell}} \bar{a}_{i_1}\bar{a}_{i_2}\cdots\bar{a}_{i_k} a_{j_1}a_{j_2}\cdots a_{j_\ell}\,, \tag{50}$$

and writing

$$\mathbf{G}_\sigma(A|\bar{A}) = \mathbf{G}_\sigma * (\bar{\lambda}_1 \cdot \sigma_1) = \sum_{\mathrm{std}(\tau,\epsilon)=12\cdots n} \mathbf{G}_{\tau\sigma,\epsilon\sigma} = \sum_{\mathrm{std}(\tau,\epsilon)=\sigma} \mathbf{G}_{\tau,\epsilon}\,, \tag{51}$$

we obtain (49). ∎

# 4 An application: the $(1-t)$-transform

Our analogs of the $(q,t)$-hook-content formulas will be obtained by lifting the definition of the vitual alphabet $(1-t)/(1-q)$ to various combinatorial Hopf algebras, and then evaluating on it a special basis. Since $1/(1-q)$ is a genuine alphabet, what we have to do is to understand the transformation $A \to (1-t)A$.

## 4.1 The canonical projection

We have an obvious projection

$$\mathbf{FQSym}(A|\bar{A}) \to \mathbf{FQSym}(A) \tag{52}$$

consisting in setting $\bar{A} = A$. We need the refined map

$$\eta_t(\mathbf{G}_\sigma^{\#}) = \mathbf{G}_\sigma(A|tA) = \mathbf{G}_\sigma((1-t)A)|_{t=-t}. \tag{53}$$

**Corollary 4.1** *In the special case $\bar{A} = tA$, one gets*

$$\mathbf{G}_\sigma(A|tA) = \sum_{\mathrm{std}(\tau,\epsilon)=\sigma} t^{m(\epsilon)} \mathbf{G}_\tau(A). \tag{54}$$

*Proof* – This follows from (49). ∎

**Example 4.2** We have

$$\mathbf{G}_{12}(A|tA) = (1+t)(\mathbf{G}_{12} + t\,\mathbf{G}_{21}), \qquad \mathbf{G}_{21}(A|tA) = (1+t)(\mathbf{G}_{21} + t\,\mathbf{G}_{12}). \tag{55}$$

$$\begin{aligned}
\mathbf{G}_{123}(A|tA) &= (1+t)(\mathbf{G}_{123} + t\,\mathbf{G}_{213} + t\,\mathbf{G}_{312} + t^2\,\mathbf{G}_{321}),\\
\mathbf{G}_{132}(A|tA) &= (1+t)(\mathbf{G}_{132} + t\,\mathbf{G}_{231} + t\,\mathbf{G}_{321} + t^2\,\mathbf{G}_{312}),\\
\mathbf{G}_{213}(A|tA) &= (1+t)(\mathbf{G}_{213} + t\,\mathbf{G}_{123} + t\,\mathbf{G}_{132} + t^2\,\mathbf{G}_{231}),\\
\mathbf{G}_{231}(A|tA) &= (1+t)(\mathbf{G}_{231} + t\,\mathbf{G}_{132} + t\,\mathbf{G}_{123} + t^2\,\mathbf{G}_{213}),\\
\mathbf{G}_{312}(A|tA) &= (1+t)(\mathbf{G}_{312} + t\,\mathbf{G}_{231} + t\,\mathbf{G}_{321} + t^2\,\mathbf{G}_{132}),\\
\mathbf{G}_{321}(A|tA) &= (1+t)(\mathbf{G}_{321} + t\,\mathbf{G}_{312} + t\,\mathbf{G}_{213} + t^2\,\mathbf{G}_{123}).
\end{aligned} \tag{56}$$

$$\begin{aligned}
\mathbf{G}_{4132}(A|tA) = (1+t)(&\mathbf{G}_{4132} + t\,\mathbf{G}_{3421} + t\,\mathbf{G}_{4231} + t\,\mathbf{G}_{4321}\\
&+ t^2\,\mathbf{G}_{2413} + t^2\,\mathbf{G}_{3412} + t^2\,\mathbf{G}_{4312} + t^3\,\mathbf{G}_{1423}).
\end{aligned} \tag{57}$$

Indeed, (57) is obtained from the 16 signed permutations whose standardized word is 4132:

$$\begin{array}{l}
4132,\ 4\bar{1}32,\quad 3\bar{4}21,\ 3\bar{4}2\bar{1},\quad 4\bar{2}31,\ 4\bar{2}3\bar{1},\quad 4\bar{3}21,\ 4\bar{3}2\bar{1},\\
2413,\ 24\bar{1}3,\quad 3\bar{4}1\bar{2},\ 3\overline{412},\quad 4\bar{3}1\bar{2},\ 4\overline{312},\quad 1\overline{423},\ \overline{1423}.
\end{array} \tag{58}$$

Summing over a descent class, we obtain the natural embedding of $\mathbf{Sym}(A|\bar{A})$ into $\mathbf{FQSym}(A|\bar{A})$:

$$R_I(A|\bar{A}) = \sum_{C(\sigma)=I} \mathbf{G}_\sigma(A|\bar{A}). \tag{59}$$

We then have

**Corollary 4.3**

$$R_I(A|\bar{A}) = \sum_{C(J,\epsilon)=I} R_{(J,\epsilon)}, \tag{60}$$

*where $R_{(J,\epsilon)}$ is defined in (41) and $C(J,\epsilon)$ is the composition whose descents are the descents of any signed permutation $(\sigma,\epsilon)$ where $\sigma$ is of shape $J$.*

Substituting $\bar{A} = tA$ yields

$$R_I(A|tA) = \sum_{C(J,\epsilon)=I} t^{m(\epsilon)} R_J(A), \tag{61}$$

which allows us to recover directly a formula of [24] (note that in [24], the recursive description of $b(I,J)$ is incorrect). Recall that a *peak* of a composition is a cell of its ribbon diagram having no cell to its right nor on its top and that a *valley* is a cell having no cell to its left nor at its bottom.

For example, Equation (56) gives

$$\begin{aligned}
R_3(A|tA) &= (1+t)(R_3 + t\,R_{12} + t^2\,R_{111}),\\
R_{21}(A|tA) &= (1+t)(t\,R_3 + (1+t)\,R_{21} + t^2\,R_{12} + t\,R_{111}),\\
R_{12}(A|tA) &= (1+t)(t\,R_3 + t(1+t)\,R_{21} + R_{12} + t\,R_{111}),\\
R_{111}(A|tA) &= (1+t)(t^2\,R_3 + t\,R_{12} + R_{111}).
\end{aligned} \tag{62}$$

On this example, one can check the following result for all pairs of compositions of 3.

**Corollary 4.4 ([24], (121))**

$$R_I(A|tA) = \sum_J (1+t)^{v(J)} t^{b(I,J)} R_J(A), \tag{63}$$

*where the sum is over all compositions $J$ such that $I$ has either a peak or a valley at each peak of $J$. Here $v(J)$ is the number of valleys of $J$ and $b(I,J)$ is the number of values $d$ such that, either $d$ is a descent of $J$ and not a descent of $I$, or $d-1$ is a descent of $I$ and not a descent of $J$.*

*Proof* – This is best understood at the level of permutations. First, the coefficient of $R_J(A)$ is, by definition, the $t$-number of signed permutations of shape $I$ whose underlying (unsigned) permutation is of shape $J$, the power of $t$ being the number of negative signs. Now, to insert signs in the ribbon diagram of a permutation of shape $J$ in order to obtain a signed permutation of shape $I$, we distinguish three kinds of cells: those which must have a plus sign, those which must have a minus sign, and those which can have both signs. The valleys of $J$ can get any sign without changing their final shape whereas all other cells have a fixed plus or minus sign, depending on $I$ and $J$, thus explaining the fact that the coefficient in front of $R_J$ is a polynomial of the form $(1+t)^{v(J)}\,t^b$. The power of $t$ corresponds to the number of cells that must have a minus sign. It is enough to determine for all pairs of compositions of 3 if the middle cell has to be negative or not since this depends only on the relative positions of their adjacent cells in $I$ and $J$. Looking more carefully at the positions that must have a negative sign, one finds the following sets, given $I$ and $J$:

| $I\backslash J$ | 3 | 21 | 12 | 111 |
|---|---|---|---|---|
| 3 | $\emptyset$ | $-$ | $\{1\}$ | $\{1,2\}$ |
| 21 | $\{1\}$ | $\emptyset$ | $\{1,3\}$ | $\{1\}$ |
| 12 | $\{1\}$ | $\{2\}$ | $\emptyset$ | $\{2\}$ |
| 111 | $\{2\}$ | $-$ | $\{3\}$ | $\emptyset$ |

(64)

where $-$ means that $b(I,J)$ is not defined since in that case $R_J$ does not appear is the expansion of $R_I(A|tA)$. This is is accordance with (63), and establish its validity in general. ∎

## 4.2 The dual transformation

Corollary 4.1 is equivalent, up to substituting $-t$ to $t$, to a combinatorial description of

$$\mathbf{G}_\sigma((1-t)A) = \mathbf{G}_\sigma(A) * \sigma_1((1-t)A). \tag{65}$$

Let $\eta_t^*$ be the adjoint of $\eta_t$. We can consistently define the virtual alphabet $A\cdot(1-t)$ by

$$\mathbf{F}_\sigma(A\cdot(1-t)) := \eta_{-t}^*(\mathbf{F}_\sigma(A)), \tag{66}$$

since, for two mutually commuting alphabets, the noncommutative Cauchy formula [8] states

$$\sigma_1(AB) = \sum_\sigma \mathbf{F}_\sigma(A)\mathbf{G}_\sigma(B), \tag{67}$$

so that we should have

$$\begin{aligned}\sigma_1(A\cdot(1-t)\cdot B) &= \sum_\alpha \mathbf{F}_\alpha(A\cdot(1-t))\mathbf{G}_\alpha(B)\\ &= \sum_\beta \mathbf{F}_\beta(A)\mathbf{G}_\beta((1-t)B).\end{aligned} \tag{68}$$

Writing

$$\mathbf{G}_\beta((1-t)B) = \mathbf{G}_\beta(B) * S_n((1-t)B), \tag{69}$$

and using the expression [24, Prop. 5.2],

$$S_n((1-t)A) = \sum_{k=0}^{n-1}(1-t)(-t)^k R_{(1^k,n-k)}, \tag{70}$$

we get

$$\sum_{k=0}^{n-1}(1-t)(-t)^k \mathbf{G}_\beta * R_{(1^k,n-k)} = \sum_{k=0}^{n-1}(1-t)(-t)^k \sum_{\mathrm{Des}(\tau)=\{1,\dots,k\}} \mathbf{G}_{\tau\circ\beta}(B), \tag{71}$$

since Equation (24) yields

$$R_{(1^k,n-k)} = \sum_{\mathrm{Des}(\sigma)=\{1,\ldots,k\}} \mathbf{G}_\sigma. \tag{72}$$

We have, setting $\gamma = \tau \circ \beta$ in Equation (71), and plugging it into (68),

$$\sigma_1(A \cdot (1-t) \cdot B) = 1 + \sum_{\substack{|\gamma|=n\geqslant 1\\ 0\leqslant k\leqslant n-1}} \left( \sum_{\mathrm{Des}(\tau)=\{1,\ldots,k\}} (1-t)(-t)^k \mathbf{F}_{\tau^{-1}\circ\gamma}(A) \right) \mathbf{G}_\sigma(B), \tag{73}$$

so that

$$\mathbf{F}_\gamma(A \cdot (1-t)) = \sum_{k=0}^{n-1} (1-t)(-t)^k \sum_{\mathrm{Des}(\tau)=\{1,\ldots,k\}} \mathbf{F}_{\tau^{-1}\circ\gamma}(A). \tag{74}$$

**Theorem 4.5** *In terms of signed permutations,*

$$\mathbf{F}_\gamma(A \cdot (1-t)) = \sum_{\epsilon\in\{\pm 1\}^n} (-t)^{m(\epsilon)} \mathbf{F}_{\mathrm{std}(\gamma,\epsilon)}(A). \tag{75}$$

*Proof* – An equivalent formulation of Equation (74) is

$$\mathbf{F}_\gamma(A \cdot (1-t)) = \sigma_1((1-t)A) * \mathbf{F}_\gamma. \tag{76}$$

Since

$$\sigma_1((1-t)A) = 1 + \sum_{\substack{|\tau|=n\geqslant 1\\ 0\leqslant k\leqslant n}} \sum_{\mathrm{Des}(\tau)=\{1,\ldots,k\}} \mathbf{F}_{\tau^{-1}}, \tag{77}$$

by Corollary 4.1, we also have

$$\sigma_1((1-t)A) = 1 + \sum_{n\geqslant 1} \sum_{\epsilon\in\{\pm 1\}^n} (-t)^{m(\epsilon)} \mathbf{F}_{\mathrm{std}(1\ldots n,\epsilon)}(A), \tag{78}$$

whence the statement. ∎

For example, let us compute $\mathbf{F}_\sigma(A \cdot (1-t))$ for all permutations of 3.

$$\begin{aligned}
\mathbf{F}_{123}(A \cdot (1-t)) &= (1-t)(\mathbf{F}_{123} - t\,\mathbf{F}_{213} - t\,\mathbf{F}_{231} + t^2\,\mathbf{F}_{321}),\\
\mathbf{F}_{132}(A \cdot (1-t)) &= (1-t)(\mathbf{F}_{132} - t\,\mathbf{F}_{231} - t\,\mathbf{F}_{213} + t^2\,\mathbf{F}_{312}),\\
\mathbf{F}_{213}(A \cdot (1-t)) &= (1-t)(\mathbf{F}_{213} - t\,\mathbf{F}_{123} - t\,\mathbf{F}_{321} + t^2\,\mathbf{F}_{231}),\\
\mathbf{F}_{231}(A \cdot (1-t)) &= (1-t)(\mathbf{F}_{231} - t\,\mathbf{F}_{132} - t\,\mathbf{F}_{312} + t^2\,\mathbf{F}_{213}),\\
\mathbf{F}_{312}(A \cdot (1-t)) &= (1-t)(\mathbf{F}_{312} - t\,\mathbf{F}_{321} - t\,\mathbf{F}_{123} + t^2\,\mathbf{F}_{132}),\\
\mathbf{F}_{321}(A \cdot (1-t)) &= (1-t)(\mathbf{F}_{321} - t\,\mathbf{F}_{312} - t\,\mathbf{F}_{132} + t^2\,\mathbf{F}_{123}),
\end{aligned} \tag{79}$$

## 4.3 The $(q,t)$-specialization

Recall from [9] that when $\mathbb{X}$ is a commutative alphabet, the specialization

$$\mathbf{F}_\sigma(\mathbb{X}) = F_{C(\sigma)}(\mathbb{X}) \tag{80}$$

depends only on the descent composition of $\sigma$, and is equal to a fundamental quasi-symmetric function. Recall also that, for a composition $I$ of $n$ (see [14]), one has

$$F_I\left(\frac{1}{1-q}\right) = \frac{q^{\mathrm{maj}\,(I)}}{(q)_n}. \tag{81}$$

From now on, we denote by $\mathbb{X}$ the alphabet $\frac{|1-t}{1-q|} := \frac{1}{1-q} \hat{\times} (1-t)$. That is,

$$F(\mathbb{X}) := F(A \cdot (1-t))|_{A=\frac{1}{1-q}}\ . \tag{82}$$

**Corollary 4.6** *Specializing $A = \frac{1}{1-q}$ in Theorem 4.5, we obtain*

$$\begin{aligned} \mathbf{F}_\sigma(\mathbb{X}) &= F_{C(\sigma)}(\mathbb{X}) \quad \text{in the notation of [24]} \\ &= \frac{1}{(q)_n} \sum_{\epsilon\in\{\pm1\}^n} (-t)^{m(\epsilon)} q^{\mathrm{maj}\,(\sigma,\epsilon)}, \end{aligned} \tag{83}$$

*where the major index of a signed word $(w,\epsilon)$ is as in 3.4.2.*

For example,

$$(q)_4\mathbf{F}_{1324}(\mathbb{X}) = -q^5t^3 + 2q^5t^2 - q^5t + q^4t^4 - 2q^4t^3 + q^4t^2 + q^2t^2 - 2q^2t + q^2 - qt^3 + 2qt^2 - qt. \tag{84}$$

as can be checked on the 16 signed words

| $(\sigma,\epsilon)$ | $m(\epsilon)$ | $\mathrm{maj}\,(\sigma,\epsilon)$ | $(\sigma,\epsilon)$ | $m(\epsilon)$ | $\mathrm{maj}\,(\sigma,\epsilon)$ |
|---|---|---|---|---|---|
| $1324$ | 0 | 2 | $132\overline{4}$ | 1 | 5 |
| $\overline{1}324$ | 1 | 2 | $\overline{1}32\overline{4}$ | 2 | 5 |
| $1\overline{3}24$ | 1 | 1 | $1\overline{3}2\overline{4}$ | 2 | 4 |
| $\overline{13}24$ | 2 | 1 | $\overline{13}2\overline{4}$ | 3 | 4 |
| $13\overline{2}4$ | 1 | 2 | $13\overline{24}$ | 2 | 5 |
| $\overline{1}3\overline{2}4$ | 2 | 2 | $\overline{1}3\overline{24}$ | 3 | 5 |
| $1\overline{32}4$ | 2 | 1 | $1\overline{324}$ | 3 | 4 |
| $\overline{132}4$ | 3 | 1 | $\overline{1324}$ | 4 | 4 |

(85)

## 4.4 Hook-content formulas in FQSym

Let us denote by $\mathrm{SP}_i$ the set of words $\epsilon \in \{\pm1\}^n$ where $\epsilon_i = 1$ and by $\mathrm{SM}_i$ the set of words $\epsilon \in \{\pm1\}^n$ where $\epsilon_i = -1$.

Let $\phi_i$ be the involution on signed permutations $(\sigma,\epsilon)$ which changes the sign of $\epsilon_i$ and leaves the rest unchanged.

**Lemma 4.7** *Let $(\sigma, \epsilon)$ be a signed permutation such that $\epsilon_i = 1$ and let $(\sigma, \epsilon') = \phi_i(\sigma, \epsilon)$. Then*

$$(-t)^{m(\epsilon')} q^{\mathrm{maj}\,(\sigma,\epsilon')} = (-t) \frac{q^{(i-1)x_i}}{q^{iy_i}} (-t)^{m(\epsilon)} q^{\mathrm{maj}\,(\sigma,\epsilon)}, \tag{86}$$

*where $x_i = 0$ if $\sigma_{i-1} > \sigma_i$ and $x_i = 1$ otherwise, and $y_i = 0$ if $\sigma_i < \sigma_{i+1}$ and $y_i = 1$ otherwise. By convention, $x_1 = 0$ and $y_n = 0$, which is equivalent to fixing $\sigma_0 = \sigma_{n+1} = +\infty$. Note that $x_{i+1} = 1 - y_i$.*

*Proof* – The factor $(-t)$ is obvious. The difference between the $q$-statistics of both words depends only on the descents at position $i-1$ and position $i$. Let us discuss position $i-1$ (value of $x_i$). If $\sigma_{i-1} > \sigma_i$, we have

$$-\sigma_{i-1} < -\sigma_i < \sigma_i < \sigma_{i-1}, \tag{87}$$

so that there is a descent at position $i-1$ in $(\sigma, \epsilon)$ iff there is a descent at the same position in $(\sigma, \epsilon')$. This proves the case $x_i = 0$.

Now, if $\sigma_{i-1} > \sigma_i$, we have

$$-\sigma_i < -\sigma_{i-1} < \sigma_{i-1} < \sigma_i, \tag{88}$$

so that there is no descent at position $i-1$ in $(\sigma, \epsilon)$ and there is a descent at the same position in $(\sigma, \epsilon')$. This proves the case $x_i = 1$. The discussion of position $i$ is similar. ∎

**Lemma 4.8** *For $i \in [1, n]$ and $\sigma \in \mathfrak{S}_n$,*

$$(q)_n \mathbf{F}_\sigma(\mathbb{X}) = \sum_{\epsilon \in \{\pm 1\}^n} (-t)^{m(\epsilon)} q^{\mathrm{maj}\,(\sigma,\epsilon)} = \left(1 - t\frac{q^{(i-1)x_i}}{q^{iy_i}}\right) \sum_{\epsilon \in \mathrm{SP}_i} (-t)^{m(\epsilon)} q^{\mathrm{maj}\,(\sigma,\epsilon)}. \tag{89}$$

*Proof* – From Lemma 4.7, we see that each signed permutation $(\sigma, \epsilon')$ with $\epsilon'_i = -1$ gives the same contribution as $\phi_i(\sigma, \epsilon')$ up to the factor $-t\, q^{(i-1)x_i - iy_i}$. ∎

The following theorem may be regarded as an analog of the hook-content formula, where the hook-length of cell $\#i$ is its "ribbon length" $i$, and its "content" is $c_i = (i-1)x_i - iy_i$. It gives in particular a factorized expression of the r.h.s of (83) in Corollary 4.6.

**Theorem 4.9** *Let $\sigma \in \mathfrak{S}_n$. Then*

$$\mathbf{F}_\sigma(\mathbb{X}) = q^{\mathrm{maj}\,(\sigma)} \prod_{i=1}^{n} \frac{1 - q^{(i-1)x_i - iy_i}\, t}{1 - q^i} = \prod_{i=1}^{n} \frac{q^{iy_i} - q^{(i-1)x_i}\, t}{1 - q^i}, \tag{90}$$

*where $x_i$ and $y_i$ are as in Lemma 4.7.*

*Proof* – The argument of the proof of Lemma 4.8 applies similarly to signed permutations of the set $\mathrm{SP}_i$ such that $\epsilon'_j = -1$, and so on, so that the whole expression factors and gives the first formula of (90). The second expression is clearly equivalent. ∎

For example, with $\sigma = (1, 3, 2, 4)$, (4.9) gives

$$(q)_4 \mathbf{F}_{1324}(\mathbb{X}) = (1-t)(q^2 - qt)(1-t)(1-q^3 t). \tag{91}$$

Expanding this expression gives back the r.h.s. of Equation (84).

## 4.5 Other approaches

We shall now see two alternative hook-content formulas for $\mathbf{F}_\sigma(\mathbb{X})$. All versions end up with the same factorizations, but the factors do not arise in the same order, and simplifications can occur beween numerators and denominators. The first one is obtained from an induction formula expressing $\mathbf{F}_\sigma(\mathbb{X})$ from $\mathbf{F}_{\mathrm{std}(\sigma_1\ldots\sigma_{n-1})}(\mathbb{X})$, and follows directly from Theorem 4.9.

**Corollary 4.10** *Let $\partial\mathbf{F}_\sigma(\mathbb{X}) := \mathbf{F}_{\mathrm{std}(\sigma_1\ldots\sigma_{n-1})}(\mathbb{X})$ as in [20]. Then,*

$$\mathbf{F}_\sigma(\mathbb{X}) = \partial\mathbf{F}_\sigma(\mathbb{X}) \times \begin{cases} \dfrac{1-q^{n-1}t}{1-q^n} & \text{if } \sigma_{n-1} < \sigma_n, \\ \dfrac{q^{n-1}-t}{1-q^n} & \text{if } \sigma_{n-2} > \sigma_{n-1} > \sigma_n, \\ \dfrac{(q^{n-1}-q^{n-2}t)(1-t)}{(1-q^{n-2}t)(1-q^n)} & \text{if } \sigma_{n-2} < \sigma_{n-1} > \sigma_n, \end{cases} \tag{92}$$

*or, equivalently*

$$\mathbf{F}_\sigma(\mathbb{X}) = \partial\mathbf{F}_\sigma(\mathbb{X}) \cdot \frac{q^{(n-1)a} - q^{(n-2)b}t}{1-q^{(n-2)b}t}\ \frac{1-q^{(n-1)(1-a)}t}{1-q^n}, \tag{93}$$

*where $a=1$ if $\sigma_{n-1} > \sigma_n$ and $a=0$ otherwise, and $b=1$ if $\sigma_{n-2} < \sigma_{n-1} > \sigma_n$ and $b=0$ otherwise.*

For example, with $\sigma = (1,3,2,4)$, one obtains

$$\begin{aligned} \mathbf{F}_{1324}(\mathbb{X}) &= \mathbf{F}_{132}(\mathbb{X}) \frac{1-tq^3}{1-q^4} \\ &= \mathbf{F}_{12}(\mathbb{X}) \frac{(q^2-qt)(1-t)}{(1-qt)(1-q^3)}\ \frac{1-tq^3}{1-q^4} \\ &= \mathbf{F}_{1}(\mathbb{X}) \frac{1-qt}{1-q^2}\ \frac{(q^2-qt)(1-t)}{(1-qt)(1-q^3)}\ \frac{1-tq^3}{1-q^4} \\ &= \frac{1-t}{1-q}\ \frac{1-qt}{1-q^2}\ \frac{(q^2-qt)(1-t)}{(1-qt)(1-q^3)}\ \frac{1-tq^3}{1-q^4} \end{aligned} \tag{94}$$

Simplifying the last expression gives back the r.h.s. of (91). Expanding it gives back the r.h.s. of (84).

The "hook-content" factors of Corollary 4.10 can have either two or four terms. But one easily checks that, if a factor has four terms, those terms simplify with the factors associated to the preceding letter in the permutation. We recover in this way the partial factors of [24] and obtain a third version of the hook-content formula:

**Corollary 4.11 ([24], (152))**

$$\mathbf{F}_\sigma(\mathbb{X}) = \prod_{i=1}^{n} \frac{1}{1-q^i} \begin{cases} 1-q^{i-1}t & \text{if } \sigma_{i-1} < \sigma_i < \sigma_{i+1}, \\ 1-t & \text{if } \sigma_{i-1} < \sigma_i > \sigma_{i+1}, \\ q^{i-1}-t & \text{if } \sigma_{i-2} > \sigma_{i-1} > \sigma_i, \\ q^{i-1}-q^{i-2}t & \text{if } \sigma_{i-2} < \sigma_{i-1} > \sigma_i, \end{cases} \tag{95}$$

*with the conventions* $\sigma_0 = 0$ *and* $\sigma_{n+1} = +\infty$.

For example, with $\sigma = (1,3,2,4)$, one finds

$$(q)_4 \mathbf{F}_{1324}(\mathbb{X}) = (1-t)(1-t)(q^2-qt)(1-q^3t). \tag{96}$$

Simplifying (94) gives this expression, which is the same as (91). Hence, its expansion gives back the r.h.s. of (84).

## 4.6 Graphical representations

We shall see later that (90) is the special case of Formula (152) for binary trees, when the tree is a zig-zag line. This is why we choose to represent graphically $\mathbf{F}_\sigma(\mathbb{X})$ with hook-content type factors in the following way: let the *mirror shape* of a permutation $\sigma$ be the shape of the mirror image $\overline{I} = (i_r, \ldots, i_1)$ of its descent composition $I = (i_1, \ldots, i_r)$. We represent it as the binary tree in which each internal node has only one subtree, depending on whether the corresponding cell of the composition is followed by a cell to its right or to its bottom. For example, with $\sigma = (5,6,7,4,3,2,8,9,10,1,11)$, the shape is $(3,1,1,4,2)$, the mirror shape is $(2,4,1,1,3)$ and its binary tree is shown on Figure 1.

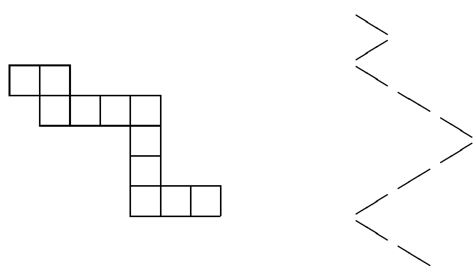


Figure 1: The mirror shape of $\sigma = (5,6,7,4,3,2,8,9,10,1,11)$ and its representation as a binary tree.

Theorem 4.9 can be visualized by placing into the $i$-th node (from bottom to top) of the tree of $\sigma$ the $i$-th factor of $\mathbf{F}_\sigma(\mathbb{X})$ in Equation (90). For example, the first tree of Figure 2 shows the expansion of $\mathbf{F}_\sigma(\mathbb{X})$ with the hook-content factors of $\sigma = (5,6,7,4,3,2,8,9,10,1,11)$.

Similarly, Formula (92) of Corollary 4.10 can be represented graphically with analogs of the hook-content factors, by placing into node $i$ (from bottom to top) the $i$-th factor

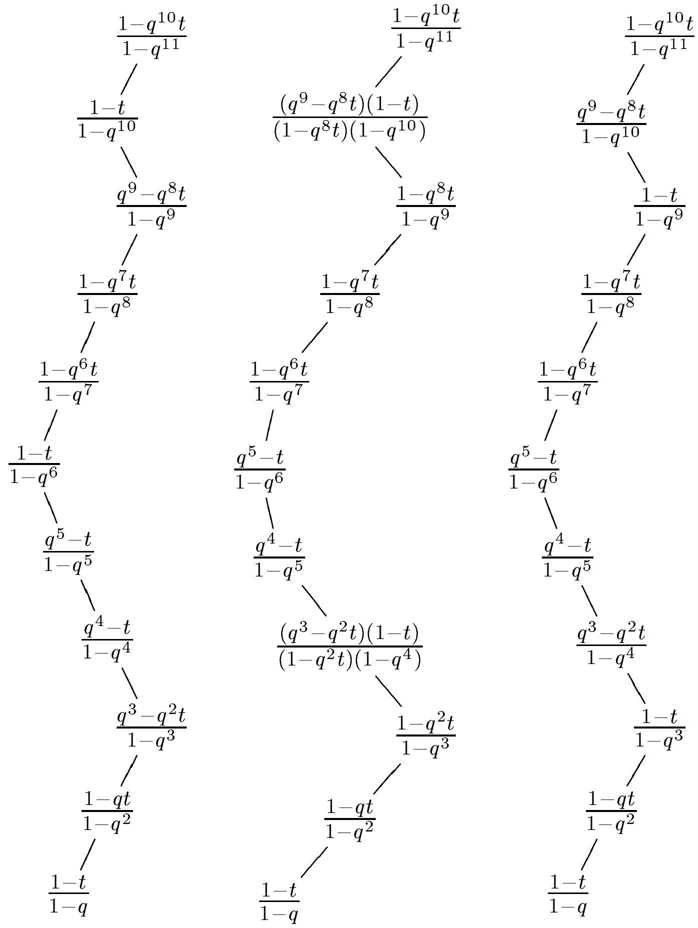



Figure 2: The three hook-content formulas for the permutation $(5,6,7,4,3,2,8,9,10,1,11)$: signed permutations (left diagram), induction (middle diagram), and simplification of the induction (right diagram).

of $\mathbf{F}_\sigma(\mathbb{X})$. For example, the second tree of Figure 2 shows the expansion of $\mathbf{F}_\sigma(\mathbb{X})$ with our second hook-contents.

Finally, the third tree of Figure 2 shows the expansion of $\mathbf{F}_\sigma(\mathbb{X})$ given by Formula (95) of Corollary 4.11. Note that it is obtained by permuting cyclically the numerators of the first formula among right branches.

# 5 Dendriform operations and $(q,t)$-specialization

## 5.1 Dendriform algebras

A dendriform algebra [27] is an associative algebra whose multiplication $\cdot$ is the sum of two operations

$$a \cdot b = a \prec b + a \succ b \tag{97}$$

satisfying

$$\begin{cases} (x \prec y) \prec z &= x \prec (y \cdot z)\,, \\ (x \succ y) \prec z &= x \succ (y \prec z)\,, \\ (x \cdot y) \succ z &= x \succ (y \succ z)\,. \end{cases} \tag{98}$$

For example, **FQSym** is dendriform with the following rules

$$\mathbf{G}_\alpha \prec \mathbf{G}_\beta = \sum_{\substack{\gamma=uv\,|\,\mathrm{std}(u)=\alpha;\,\mathrm{std}(v)=\beta \\ \max(v)<\max(u)}} \mathbf{G}_\gamma\,, \tag{99}$$

$$\mathbf{G}_\alpha \succ \mathbf{G}_\beta = \sum_{\substack{\gamma=uv\,|\,\mathrm{std}(u)=\alpha;\,\mathrm{std}(v)=\beta \\ \max(v)\geqslant\max(u)}} \mathbf{G}_\gamma\,. \tag{100}$$

Note that $x = \mathbf{G}_1 = \mathbf{F}_1$ generates a free dendriform algebra in **FQSym**, isomorphic to **PBT**, the Loday-Ronco algebra of planar binary trees [28].

## 5.2 The half-products

### 5.2.1 Descent statistics on half-shuffles

On the basis $\mathbf{F}_\sigma$, the half-products are shifted half-shuffles. Recall that the half-shuffles are the two terms of the recursive definition of the shuffle product. For an alphabet $A$, and two words $u = u'a$, $v = v'b$, $a, b \in A$, one has

$$u \sqcup\!\sqcup v = u \prec v + u \succ v\,, \tag{101}$$

where the half-products are now

$$u \prec v = (u' \sqcup\!\sqcup v)a \text{ and } u \succ v = (u \sqcup\!\sqcup v')b\,. \tag{102}$$

Assuming now that $A$ is totally ordered, we want to investigate the distribution of descents on half-shuffles. To this aim we introduce a linear map

$$\langle w \rangle = F_{C(w)}(X) = \langle w | \sigma_1(XA) \rangle \tag{103}$$

from $\mathbb{K}\langle A \rangle$ to $QSym(X)$, the scalar product on $\mathbb{K}\langle A \rangle$ being defined by $\langle u|v \rangle = \delta_{u,v}$.

For $w \in A^*$, let $\mathrm{alph}(w) \subseteq A$ be the set of letters occuring in $w$.

**Lemma 5.1** *If* $\mathrm{alph}(u) \cap \mathrm{alph}(v) = \emptyset$*, then*

$$\langle u \sqcup\!\sqcup v \rangle = \langle u \rangle \langle v \rangle\,. \tag{104}$$

*In particular, the descents of the words occuring in the shuffle of two words $u$ and $v$ on disjoint alphabets depend only on the descents of $u$ and $v$.*

*Proof* – Denote by $\Delta$ the canonical (unshuffle) coproduct of $\mathbb{K}\langle A\rangle$, and write $u'v''$ for $u\otimes v$, so that $\Delta(a)=1\otimes a+a\otimes 1=a'+a''$ for $a\in A$. Then,

$$\begin{aligned}\langle u\sqcup\!\sqcup v\rangle=\langle u\sqcup\!\sqcup v|\sigma_1(XA)\rangle&=\left\langle u'v''|\prod_{x\in X}^{\rightarrow}\Delta\sigma_x(A)\right\rangle\\&=\left\langle u'v''|\prod_{x\in X}^{\rightarrow}\prod_{a\in A}^{\rightarrow}(1-x(a'+a''))^{-1}\right\rangle\\&=\left\langle u'v''|\prod_{x\in X}^{\rightarrow}\prod_{a'\in\mathrm{alph}(u')}^{\rightarrow}(1-xa')^{-1}\prod_{a''\in\mathrm{alph}(v'')}^{\rightarrow}(1-xa'')^{-1}\right\rangle\\&=\langle u|\sigma_1(XA)\rangle\langle v|\sigma_1(XA)\rangle=\langle u\rangle\langle v\rangle\,.\end{aligned}\tag{105}$$

■

There is a refined statement for the dendriform half-products.

**Theorem 5.2** *Let $u=u_1\cdots u_k$ and $v=v_1\cdots v_\ell$ of respective lengths $k$ and $\ell$. If* $\mathrm{alph}(u)\cap\mathrm{alph}(v)=\emptyset$, *then*

$$\langle u\prec v\rangle=\langle\sigma\prec\tau\rangle\tag{106}$$

*where $\sigma=\mathrm{std}(u)$ and $\tau=\mathrm{std}(v)[k]$ if $u_k<v_\ell$, and $\sigma=\mathrm{std}(u)[\ell]$ and $\tau=\mathrm{std}(v)$ if $u_k>v_\ell$.*

*Proof* – It is enough to check the first case, so we assume $u_k<v_\ell$. The proof proceeds by induction on $n=k+\ell$. Let us set $u=u'a'a$ and $\mathrm{std}(u)=u'_1a'_1a_1$.

If $a'>a$, since $u\prec v=(u'a'\sqcup\!\sqcup v)a$, we have

$$\langle u\prec v\rangle=\sum_{w\in u'a'\sqcup\!\sqcup v}F_{C(w)\cdot 1}=\langle(u'_1a'_1\sqcup\!\sqcup\tau)\cdot a_1\rangle\tag{107}$$

with $\tau=\mathrm{std}(v)[k]$, according to Lemma 5.1.

If $a'<a$, write $u\prec v=(u'a'\prec v)\cdot a+(u'a'\succ v)\cdot a$. From the induction hypothesis, we have, with $\tau$ as above, $\langle u'a'\prec v\rangle=\langle u'_1a'_1\prec\tau\rangle$ and $\langle u'a'\succ v\rangle=\langle u'_1a'_1\succ\tau\rangle$, so that

$$\langle u\prec v\rangle=\sum_{w\in u'_1a'_1\prec\tau}F_{C(w)\triangleright 1}+\sum_{w\in u'_1a'_1\succ\tau}F_{C(w)\cdot 1}\,,\tag{108}$$

as required. ■

For example,

$$\begin{aligned}\langle 634\prec 125\rangle=&\langle 631254+613254+612354+612534+163254\\&+162354+162534+126354+126534+125634\rangle,\end{aligned}\tag{109}$$

$$\begin{aligned}\langle 312\prec 456\rangle=&\langle 314562+341562+345162+345612+431562\\&+435162+435612+453162+453612+456312\rangle,\end{aligned}\tag{110}$$

and one can check that both expressions are equal to

$$F_{132}+F_{141}+F_{1131}+F_{1221}+F_{222}+F_{231}+F_{2121}+F_{312}+F_{321}+F_{42}.\tag{111}$$

#### 5.2.2 Special case: the major index

**Corollary 5.3** *Let $u$ and $v$ be two words of respective lengths $k$ and $\ell$. Then, if* $\mathrm{alph}(u) \cap \mathrm{alph}(v) = \emptyset$,

$$\sum_{x \in u \prec v} q^{\mathrm{maj}\,(x)} = \sum_{y \in \sigma \prec \tau} q^{\mathrm{maj}\,(y)}. \tag{112}$$

*where $\sigma = \mathrm{std}(u)$ and $\tau = \mathrm{std}(v)[k]$ if $u_k < v_\ell$, and $\sigma = \mathrm{std}(u)[\ell]$ and $\tau = \mathrm{std}(v)$ if $u_k > v_\ell$.*

### 5.3 $(q,t)$-specialization

We shall now see that Corollary 5.3 implies a hook-content formula for half-products evaluated over $\mathbb{X}$. Let $\sigma \in \mathfrak{S}_k$ and $\tau \in \mathfrak{S}_\ell$. Recall that $\tau[k]$ denotes the word $\tau_1 + k, \tau_2 + k, \dots, \tau + \ell + k$. We have

$$\begin{aligned}(q)_{k+\ell}\left(\mathbf{F}_\sigma \prec \mathbf{F}_\tau\right)(\mathbb{X}) &= \sum_{\epsilon \in \{\pm 1\}^{k+\ell}} \sum_{\mu \in \sigma \prec \tau[k]} (-t)^{m(\epsilon)} q^{\mathrm{maj}\,(\mu,\epsilon)} \\ &= \sum_{\substack{\epsilon' \in \{\pm 1\}^{k} \\ \epsilon'' \in \{\pm 1\}^{\ell}}} (-t)^{m(\epsilon')} (-t)^{m(\epsilon'')} \sum_{(\nu,\eta) \in u \prec v} q^{\mathrm{maj}\,(\nu,\eta)},\end{aligned} \tag{113}$$

where $u := (\sigma, \epsilon')$, $v := (\tau[k], \epsilon'')$, and $(\nu, \eta)$ are regarded as words over the alphabet $\{\dots < -2 < -1 < 1 < 2 < \dots\}$. The inner sum is of the form

$$(q)_{k+\ell} \langle u \prec v \rangle |_{X = \frac{1}{1-q}}. \tag{114}$$

According to Theorem 5.2, $\langle u \prec v \rangle$ can be replaced by $\langle \alpha \prec \beta \rangle$, where $\alpha$ and $\beta$ are the two permutations

$$\begin{cases} \alpha = \mathrm{std}(u),\ \beta = \mathrm{std}(v)[k] & \text{if } \epsilon''_\ell = +1, \\ \alpha = \mathrm{std}(u)[\ell],\ \beta = \mathrm{std}(v) & \text{if } \epsilon''_\ell = -1. \end{cases} \tag{115}$$

Translating Formulas (34) and (35) of [20] in the language of the present paper (in [20], $\alpha \prec \beta$ (resp. $\alpha \succ \beta$) represents what we would denote here by $\alpha^{-1} \prec \beta^{-1}[k]$ (resp. $\beta^{-1}[k] \prec \alpha^{-1}$)), we have

$$\begin{aligned}\sum_{(\nu,\eta) \in u \prec v} q^{\mathrm{maj}\,(\nu,\eta)} &= q^{\mathrm{maj}\,(\alpha)} q^{\mathrm{maj}\,(\beta)} q^\ell \begin{bmatrix} k + \ell - 1 \\ \ell \end{bmatrix}_q \\ &= q^{\mathrm{maj}\,(u)} q^{\mathrm{maj}\,(v)} q^\ell \begin{bmatrix} k + \ell - 1 \\ \ell \end{bmatrix}_q\end{aligned} \tag{116}$$

if $\epsilon''_\ell = +1$, and

$$\sum_{(\nu,\eta) \in u \prec v} q^{\mathrm{maj}\,((\nu,\eta))} = q^{\mathrm{maj}\,(u)} q^{\mathrm{maj}\,(v)} \begin{bmatrix} k + \ell - 1 \\ \ell - 1 \end{bmatrix}_q \tag{117}$$

if $\epsilon''_\ell = -1$. We then deduce

$$\sum_{\epsilon\in\{\pm1\}^{k+\ell}} \sum_{\mu\in\sigma\prec\tau[k]} (-t)^{m(\epsilon)} q^{\mathrm{maj}\,(\mu,\epsilon)} = \left( \sum_{\epsilon'\in\{\pm1\}^{k}} (-t)^{m(\epsilon')} q^{\mathrm{maj}\,(\sigma,\epsilon')} \right)$$
$$\times \left( \sum_{\substack{\epsilon''\in\{\pm1\}^{\ell} \\ \epsilon''_\ell=+1}} \frac{(q)_{k+\ell-1}}{(q)_{k-1}(q)_\ell} (-t)^{m(\epsilon'')} q^{\ell} q^{\mathrm{maj}\,(\tau,\epsilon'')} + \sum_{\substack{\epsilon''\in\{\pm1\}^{\ell} \\ \epsilon''_\ell=-1}} \frac{(q)_{k+\ell-1}}{(q)_{k-1}(q)_\ell} (-t)^{m(\epsilon'')} q^{\mathrm{maj}\,(\tau,\epsilon'')} \right). \tag{118}$$

This equality will now be rewritten in two different ways, leading to two interpretations of the specialization of the left dendriform product $\mathbf{F}_\sigma \prec \mathbf{F}_\tau$.

#### 5.3.1 Regrouping signed words into blocks

First, instead of summing over all signed words, one can sum over the subset where *one value* has been assigned a plus sign. This yields identities similar to Lemma 4.8 and Theorem 4.9:

**Lemma 5.4** *Let $\sigma \in \mathfrak{S}_k$, $\tau \in \mathfrak{S}_\ell$, and $i \in [1, k+\ell]$. Then*

$$\sum_{\mu\in\sigma\prec\tau[k]} \sum_{\epsilon\in\{\pm1\}^{k+\ell}} (-t)^{m(\epsilon)} q^{\mathrm{maj}\,(\mu,\epsilon)} = \left(1 - t\frac{q^{(i-1)x_i}}{q^{iy_i}}\right) \sum_{\mu\in\sigma\prec\tau[k]} \sum_{\substack{\epsilon\in\{\pm1\}^{k+\ell} \\ \epsilon_{z(\mu)}=+1}} (-t)^{m(\epsilon)} q^{\mathrm{maj}\,(\mu,\epsilon)}, \tag{119}$$

*where $z(\mu)$ is either $\mu^{-1}(\sigma_i)$ if $i \leqslant k$ or $\mu^{-1}(k+\tau_{i-k})$ otherwise, and where $x_i$ and $y_i$ are defined, for $i \leqslant k$, as the $x_i$ and $y_i$ associated to $\sigma$ as in Lemma 4.7, and, for $i > k$, as $x_{i-k}$ and $y_{i-k}$ associated to $\tau$, except the (new) convention $y_{k+\ell} = 1$.*

*Proof* – Equation (118) is equivalent to

$$\sum_{\epsilon\in\{\pm1\}^{k+\ell}} \sum_{\mu\in\sigma\prec\tau[k]} (-t)^{m(\epsilon)} q^{\mathrm{maj}\,(\mu,\epsilon)} = \left( \sum_{\epsilon'\in\{\pm1\}^{k}} (-t)^{m(\epsilon')} q^{\mathrm{maj}\,(\sigma,\epsilon')} \right)$$
$$\times \begin{bmatrix} k+\ell-1 \\ k-1 \end{bmatrix}_q \left(q^{\ell} - t\, q^{(\ell-1)x_{k+\ell}}\right) \left( \sum_{\substack{\epsilon''\in\{\pm1\}^{\ell} \\ \epsilon''_\ell=+1}} (-t)^{m(\epsilon'')} q^{\mathrm{maj}\,(\tau,\epsilon'')} \right). \tag{120}$$

The required factor will then come from the first, third or second factor, according to whether $i \leqslant k$, $k < i < k+\ell$ or $i = k+\ell$. ■

For example, let us take $\sigma = 12$ and $\tau = 1$. We have $\mathbf{F}_{12} \prec \mathbf{F}_1 = \mathbf{F}_{132} + \mathbf{F}_{312}$ so that

$$\begin{aligned}
(q)_3\ (\mathbf{F}_{12} \prec \mathbf{F}_1)(\mathbb{X}) &= q^2 - (q+2q^2)\,t + (2q+q^2)\,t^2 - q\,t^3 + \\
&\quad\ q - (1+q+q^3)\,t + (1+q^2+q^3)\,t^2 - q^2\,t^3 \\
&= q + q^2 - (1{+}2q{+}2q^2{+}q^3)\,t + (1{+}2q{+}2q^2{+}q^3)\,t^2 - (q+q^2)\,t^3 \\
&= (1+q)(q-t)(1-t)(1-qt). \qquad (121)
\end{aligned}$$

Now, summing the $(q,t)$ statistic over the signed words obtained by signing the *values* 1 and 2 in 132 or 312 in all possible ways,

$$-q^3\,t + q^3 + q^2\,t^2 - 2q^2\,t + q^2 + q\,t^2 - qt = q(1-t)(1+q)(q-t). \qquad (122)$$

Dividing the whole sum by this expression, one gets the factor $1 - t/q$, which is indeed the result predicted by the lemma for $i = 3$.

Let us now consider the case $\sigma = 25134$, $\tau = 3421$, and $i = 4$. Then one easily checks with a computer that the whole $(q,t)$ polynomial obtained by summing over all signed words divided by the polynomial obtained by summing over all signed words obtained by putting plus or minus signs on all *values* except $\sigma_i = 3$, is $1 - q^3t$. The same example with $i = 7$, which assigns a plus sign to $5 + \tau_2 = 9$, gives the factor $1 - tq^{-1}$.

**Note 5.5** Lemma 5.4 means that one can split the set of signed words occuring in a left (or right) dendriform product according to the sign of a given value and that the quotient of the $(q,t)$ polynomials of those sets is $1 - r$ where $r$ is a monomial in $q$ and $t$ or $t^{-1}$. We shall refer to this factor as the *contribution* of a letter $a$ to the whole sum. The value of $i$ in Lemma 5.4 is $i = \sigma^{-1}(a)$ if $a \leqslant k$ or $i = k + \tau^{-1}(a-k)$ otherwise.

### 5.3.2 Extraction of $\mathbf{F}_\sigma(\mathbb{X})$ and $\mathbf{F}_\tau(\mathbb{X})$

One can also rewrite the r.h.s. of Equation (118) as

$$(q)_k \mathbf{F}_\sigma(\mathbb{X}) \frac{(q)_{k+\ell-1}}{(q)_{k-1}(q)_\ell} \left( q^\ell (q)_\ell \frac{1}{(1 - tq^{(\ell-1)x_\ell})} + (q)_\ell \frac{-tq^{(\ell-1)x_\ell}}{(1 - tq^{(\ell-1)x_\ell})} \right) \mathbf{F}_\tau(\mathbb{X}), \qquad (123)$$

thanks to Equation (89) and to its analog for $\epsilon''_\ell = -1$, where $x_\ell$ is defined on $\tau$ as in Lemma 4.7. This yields

$$(\mathbf{F}_\sigma \prec \mathbf{F}_\tau)(\mathbb{X}) = \frac{1-q^k}{1-q^{k+\ell}} \, \frac{q^\ell - q^{(\ell-1)x_\ell}\,t}{1 - q^{(\ell-1)x_\ell}\,t} \mathbf{F}_\sigma(\mathbb{X}) \mathbf{F}_\tau(\mathbb{X}). \qquad (124)$$

Hence, we have, deducing (126) from (125), since their sum is $\mathbf{F}_\sigma(\mathbb{X})\mathbf{F}_\tau(\mathbb{X})$:

**Corollary 5.6** *Let $\sigma \in \mathfrak{S}_k$ and $\tau \in \mathfrak{S}_\ell$. Then*

$$(\mathbf{F}_\sigma \prec \mathbf{F}_\tau)(\mathbb{X}) = \frac{1-q^k}{1-q^{k+\ell}} \, \frac{q^\ell - q^{(\ell-1)d}t}{1 - q^{(\ell-1)d}t} \, \mathbf{F}_\sigma(\mathbb{X}) \mathbf{F}_\tau(\mathbb{X}), \qquad (125)$$

*and*

$$\left(\mathbf{F}_\sigma \succ \mathbf{F}_\tau\right)(\mathbb{X}) = \frac{1-q^\ell}{1-q^{k+\ell}}\ \frac{1-q^{k+(\ell-1)d}t}{1-q^{(\ell-1)d}t}\ \mathbf{F}_\sigma(\mathbb{X})\mathbf{F}_\tau(\mathbb{X}), \tag{126}$$

*where* $d$ *is* $1$ *if* $\tau_{\ell-1} < \tau_\ell$ *and* $0$ *otherwise.*

**Example 5.7** Let us present all possible cases on the left dendriform product.

$$(\mathbf{F}_{3241} \prec \mathbf{F}_{213})(\mathbb{X}) = \frac{1-q^4}{1-q^7}\ \frac{q^3-q^2t}{1-q^2t}\,\mathbf{F}_{3241}(\mathbb{X})\mathbf{F}_{213}(\mathbb{X}). \tag{127}$$

$$(\mathbf{F}_{25134} \prec \mathbf{F}_{3421})(\mathbb{X}) = \frac{1-q^5}{1-q^9}\ \frac{q^4-t}{1-t}\,\mathbf{F}_{25134}(\mathbb{X})\mathbf{F}_{3421}(\mathbb{X}). \tag{128}$$

# 6 A hook-content formula for binary trees

## 6.1 Classical constructions on trees

Let us first fix some notations and recall the construction of some binary trees associated with a permutation.

The tree with a root and no other vertex is represented by $\bullet$. The *size* of a tree is its number of nodes.

The *decreasing tree* $T(w)$ of a permutation, or more generally, of a word $w$ with distinct letters, is defined recursively as follows: if $w$ consists of one letter $x$, then the tree has a single node, labelled $x$. Otherwise, write $w = unv$, where $n$ is its maximal letter. Then, $T(w)$ is the binary tree with root $n$, left subtree $T(u)$ and right subtree $T(v)$.

The classical hook-length formula for binary trees [23] gives the number of permutations whose decreasing tree has a given shape. Its $q$-analog, due to Björner and Wachs [5, 6], counts these permutations by inverse major index or inversion number. New proofs of these identities, relying on combinatorial Hopf algebras, are given in [20].

The decreasing tree can be interpreted as the $Q$-symbol of an analog of the Robinson-Schensted correspondence [18]. Here, it will be easier to work with the corresponding analog of the $P$-symbol. To define it, we need a simple classical algorithm: the *binary search tree insertion*, such as presented, for example, by Knuth in [23].

Recall that a *right strict binary search tree* $T$ is a labeled binary tree such that for each internal node $n$, its label is greater than or equal to the labels of its left subtree and strictly smaller than the labels of its right subtree.

Let $\sigma$ be a permutation. Its *binary search tree* $\mathcal{P}(\sigma)$ is obtained as follows: reading $\sigma$ from *right to left*, one inserts each letter in a binary search tree in the following way: if the tree is empty, one creates a node labeled by the letter ; otherwise, this letter is recursively inserted in the left (resp. right) subtree if it is smaller than or equal to (resp. strictly greater than) the root. Figure 3 shows the iterative construction of the binary search tree of $(2, 10, 6, 5, 11, 7, 4, 8, 1, 9, 3)$.

The binary search tree of a permutation has the same shape as the decreasing tree of its inverse.

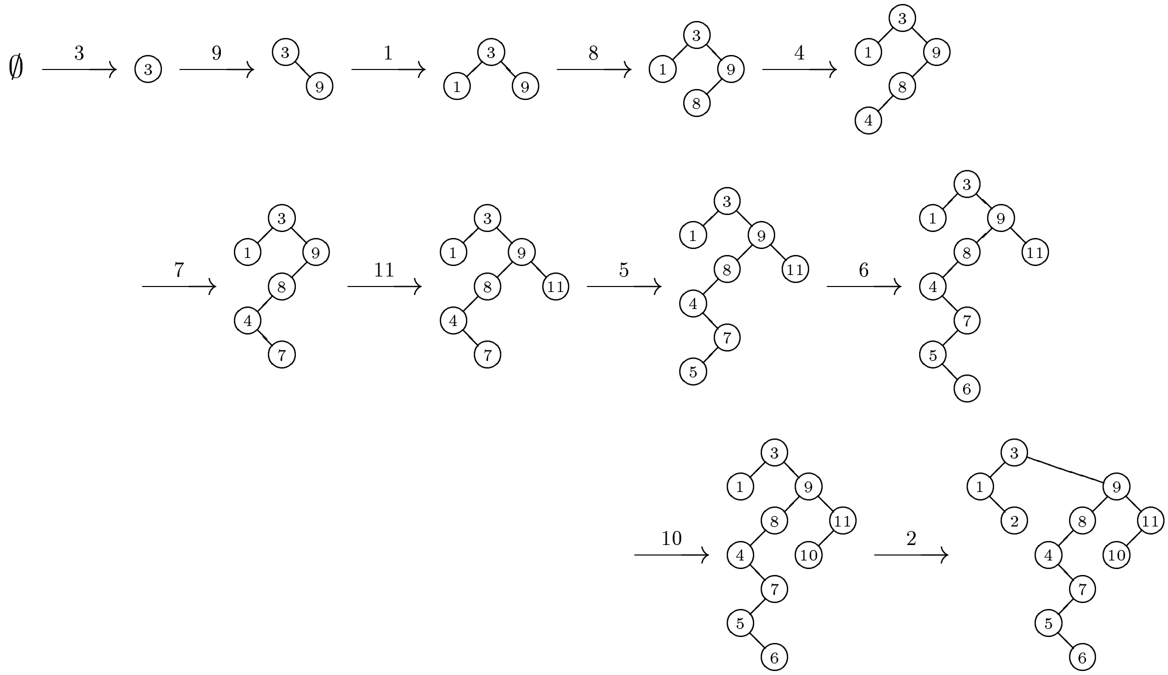


Figure 3: The binary search tree of $(2, 10, 6, 5, 11, 7, 4, 8, 1, 9, 3)$.

## 6.2 PBT: a subalgebra of FQSym

Recall that **PBT**, the Loday-Ronco algebra of planar binary trees [28], is naturally a subalgebra of **FQSym**, the embedding being

$$\mathbf{P}_T(A) = \sum_{\text{shape}\,(\mathcal{P}(\sigma))=T} \mathbf{F}_\sigma(A)\,, \tag{129}$$

where shape $(\mathcal{P}(\sigma))$ is the shape of the binary search tree associated with $\sigma$ [18]. Hence, $\mathbf{P}_T(\mathbb{X})$ is well-defined.

The algebra **PBT** was originally defined in [28] as the free dendriform algebra on one generator as follows: if $T$ is a binary tree $T_1$ (resp. $T_2$) be its left (resp. right) subtree, then

$$\mathbf{P}_T = \mathbf{P}_{T_1} \succ \mathbf{P}_1 \prec \mathbf{P}_{T_2}. \tag{130}$$

## 6.3 A hook-content formula in PBT

Thanks to Corollary 4.6, the $\mathbb{X}$-specialization of $\mathbf{P}_T$ can be expressed as a sum over signed permutations:

**Corollary 6.1** *Let $T$ be a binary tree of size $k$. Then*

$$\mathbf{P}_T(\mathbb{X}) = \frac{1}{(q)_k} \sum_{(\sigma,\epsilon)|\text{shape}\,(\mathcal{P}(\sigma))=T} (-t)^{m(\epsilon)} q^{\text{maj}\,(\sigma,\epsilon)}. \tag{131}$$

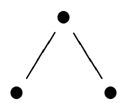
In particular, replacing $t$ by $-t$ gives the following combinatorial interpretation:

**Corollary 6.2** *Let $T$ be a binary tree of size $k$. The generating function by number of signs and major index of all signed permutations $(\sigma, \epsilon)$ such that the binary search tree of the underlying unsigned permutation $\sigma$ has shape $T$ is*

$$(q)_k \mathbf{P}_T(\mathbb{X})|_{t=-t} = \sum_{(\sigma,\epsilon)|\text{shape}\,(\mathcal{P}(\sigma))=T} t^{m(\epsilon)} q^{\text{maj}\,(\sigma,\epsilon)}. \tag{132}$$

The smallest interesting example is the tree $T$ of size 3

$$T = \tag{133}$$



Indeed, $\mathbf{P}_T = \mathbf{F}_{132} + \mathbf{F}_{312}$ and one gets

$$(q)_3 \mathbf{P}_T(\mathbb{X})_{t=-t} = (1+q)(q+t)(1+t)(1+qt), \tag{134}$$

thanks to (121). The factorization property of $\mathbf{P}_T(\mathbb{X})$ is general, as we shall see below, and is of the same nature as the factorizations of $\mathbf{F}_\sigma(\mathbb{X})$ and $(\mathbf{F}_\sigma \prec \mathbf{F}_\tau)(\mathbb{X})$, whence the denomination $(q,t)$-*hook-content* formulas for trees. Recall that any binary tree has a unique standard labelling that makes it a binary search tree.

Let $T$ be a tree of size $k$. Define

$$\Sigma_T := (q)_k \mathbf{P}_T(\mathbb{X}) = \sum_{(\sigma,\epsilon)|\text{shape}\,(\mathcal{P}(\sigma))=T} (-t)^{m(\epsilon)} q^{\text{maj}\,(\sigma,\epsilon)}, \tag{135}$$

and, for all $i \in [1,k]$,

$$\Sigma_T^{(i)} := \sum_{\substack{(\sigma,\epsilon)|\text{shape}\,(\mathcal{P}(\sigma))=T \\ \text{sign}(i)=+1}} (-t)^{m(\epsilon)} q^{\text{maj}\,(\sigma,\epsilon)}. \tag{136}$$

Imitating the argument of Lemma 5.4 and taking into account Equation (130), we have:

**Lemma 6.3** *Let $T$ be a binary tree of size $k$ and $i \in [1,k]$. Let $s$ be the node labelled $i$ in the binary search tree associated with $T$. Then*

$$\frac{\Sigma_T}{\Sigma_T^{(i)}} = 1 - t\, q^{n'-na}, \tag{137}$$

*where $n$ is the size of the subtree of root $s$, $n'$ is the size of the left subtree of the previous one, and $a = 1$ if $s$ is the right son of its father, and $0$ otherwise.* ∎

Applying now Corollary 5.6, we get a two-parameter version of the $q$-hook-length formula of Björner and Wachs [5, 6] (see also [20]):

**Theorem 6.4** *Let $T$ be a tree and $s$ a node of $T$. Let $n$ be the size of the subtree of root $s$ and let $n'$ be the size of the left subtree of the previous one. The $(q,t)$-hook-content factor of $s$ into $T$ is given by the following rules:*

$$h_s(q,t) := \frac{1}{1-q^n} \begin{cases} q^n - q^{n'}t & \text{if } s \text{ is the right son of its father,} \\ 1 - q^{n'}t & \text{otherwise.} \end{cases} \tag{138}$$

*We then have*

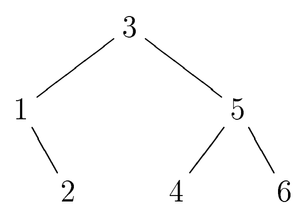
$$\mathbf{P}_T(\mathbb{X}) = \prod_{s \in T} h_s(q,t). \tag{139}$$

*Proof* – As in the cases of $\mathbf{F}_\sigma(\mathbb{X})$ and of the left dendriform product $(\mathbf{F}_\sigma \prec \mathbf{F}_\tau)(\mathbb{X})$, the formula follows from Lemma 6.3, since one can fix the signs of different values of $[1,k]$ independently, hence having a factorization of $(q)_k \mathbf{P}_T(\mathbb{X})$ as the product of the right-hand sides of (137) for all $i \in [1,k]$ multiplied by $(q)_k \mathbf{P}_T(1/(1-q))$. The complete factorization follows by induction. ∎

For example, let $T$ be the tree (133). Then, picking the $h_s(q,t)$ of all vertices by reading $T$ in postfix order (left subtree, then right subtree, then root), one obtains

$$\mathbf{P}_T(\mathbb{X}) = \frac{1-t}{1-q} \, \frac{q-t}{1-q} \, \frac{1-qt}{1-q^3}, \tag{140}$$

which gives back the factorization (134) of $\mathbf{P}_T(\mathbb{X})$. Let now $T$ be the following tree, labelled as a binary search tree:

$T =$  (141)

then, the product $h_1 \cdots h_6$ (in this order) is one gets:

$$\begin{aligned} \mathbf{P}_T(\mathbb{X}) &= \frac{q-t}{1-q} \, \frac{1-t}{1-q^2} \, \frac{1-t}{1-q} \, \frac{q-t}{1-q} \, \frac{q^3-qt}{1-q^3} \, \frac{1-q^2t}{1-q^6} \\ &= \frac{(q-t)^2(1-t)^2(1-q^2t)(q^3-qt)[4]_q[5]_q}{(1-q)(1-q^2)(1-q^3)(1-q^4)(1-q^5)(1-q^6)}. \end{aligned} \tag{142}$$

Larger examples with their graphical representation are given in the next section.

## 6.4 Other versions

Here are the analogs of the other two hook-content formulas of **FQSym**.

**Theorem 6.5** *Let $T$ be a binary tree and $T_1$ (resp. $T_2$) be its left (resp. right) subtree. Let $T_2'$ be the left subtree of $T_2$. We then have*

$$\mathbf{P}_T(\mathbb{X}) = \frac{(q^{\#T_2} - q^{\#T_2'}t)(1-q^{\#T_1}t)}{(1-q^{\#T_2'}t)(1-q^n)}\mathbf{P}_{T_1}(\mathbb{X})\mathbf{P}_{T_2}(\mathbb{X}), \tag{143}$$

*where $\#U$ denotes the size of $U$.*

*Proof* – Let $T'$ be the tree having $T_2$ as right subtree and no left subtree. Then, thanks to (130), we have

$$\mathbf{P}_T = \mathbf{P}_{T_1} \succ (\mathbf{P}_\bullet \prec \mathbf{P}_{T_2}) = \mathbf{P}_{T_1} \succ \mathbf{P}_{T'}. \tag{144}$$

Now, since all permutations whose inverse have $T'$ as decreasing tree end by 1, if one regards $\mathbf{P}_{T_1}$ and $\mathbf{P}_{T'}$ as elements of **FQSym**, all products $\mathbf{F}_\sigma \succ \mathbf{F}_\tau$ with $\mathcal{T}(\sigma^{-1}) = T_1$ and $\mathcal{T}(\tau^{-1}) = T'$ are given by Formula (126) with $d=0$. Hence,

$$\mathbf{P}_T = \frac{1-q^{\#T'}}{1-q^{\#T}}\frac{1-q^{\#T_1}t}{1-t}\mathbf{P}_{T_1}\mathbf{P}_{T'}. \tag{145}$$

This equation is equivalent to the claim in the special case where $T_1$ is the empty tree. Thus, we only have to prove the following identity to establish the general case:

$$\mathbf{P}_{T'} = \frac{q^{\#T_2} - q^{\#T_2'}t}{1-q^{\#T_2'}t}\frac{1-t}{1-q^{\#T'}}\mathbf{P}_{T_2}. \tag{146}$$

We have first

$$\mathbf{P}_{T'} = \mathbf{P}_\bullet \prec \mathbf{P}_{T_2} = \mathbf{P}_\bullet\mathbf{P}_{T_2} - \mathbf{P}_\bullet \succ \mathbf{P}_{T_2} = \mathbf{P}_\bullet\mathbf{P}_{T_2} - \mathbf{P}_\bullet \succ (\mathbf{P}_{T_2'} \succ \mathbf{P}_{T_3}), \tag{147}$$

where $T_3$ is the tree having the right subtree of $T_2$ as right subtree and no left subtree. Applying one of the basic dendriform relations, one gets

$$\mathbf{P}_{T'} = \mathbf{P}_\bullet\mathbf{P}_{T_2} - (\mathbf{P}_\bullet\mathbf{P}_{T_2'}) \succ \mathbf{P}_{T_3}, \tag{148}$$

and as above, since $T_3$ has no left subtree, the right dendriform product is given uniformly by (126) with $d=0$. We arrive at

$$\begin{aligned}\mathbf{P}_{T'} =& \mathbf{P}_\bullet\mathbf{P}_{T_2} - \frac{1-q^{\#T_3}}{1-q^{1+\#T_2}}\frac{1-q^{1+\#T_2'}t}{1-t}\mathbf{P}_\bullet\mathbf{P}_{T_2'}\mathbf{P}_{T_3}\\ =& \mathbf{P}_\bullet\mathbf{P}_{T_2} - \frac{1-q^{\#T_3}}{1-q^{1+\#T_2}}\frac{1-q^{1+\#T_2'}t}{1-t}\mathbf{P}_\bullet\frac{1-q^{\#T_2}}{1-q^{\#T_3}}\frac{1-t}{1-q^{\#T_2'}t}\mathbf{P}_{T_2},\end{aligned} \tag{149}$$

by application of (145) from left to right. Simplifying the fractions, we obtain

$$\begin{aligned}\mathbf{P}_{T'} =& \mathbf{P}_\bullet\mathbf{P}_{T_2}\frac{(1-q^{1+\#T_2})(1-q^{\#T_2'}t) - (1-q^{\#T_2})(1-q^{1+\#T_2'}t)}{(1-q^{1+\#T_2})(1-q^{\#T_2'}t)}\\ =& \mathbf{P}_\bullet\mathbf{P}_{T_2}\frac{-q^{1+\#T_2} + q^{\#T_2'}t + q^{\#T_2} - q^{1+\#T_2'}t}{(1-q^{1+\#T_2})(1-q^{\#T_2'}t)}\\ =& \mathbf{P}_{T_2}\frac{1-t}{1-q}\frac{(1-q)(q^{\#T_2} - q^{\#T_2'}t)}{(1-q^{1+\#T_2})(1-q^{\#T_2'}t)}\\ =& \frac{(1-t)(q^{\#T_2} - q^{\#T_2'}t)}{(1-q^{1+\#T_2})(1-q^{\#T_2'}t)}\mathbf{P}_{T_2},\end{aligned} \tag{150}$$

whence the result. ∎

For example, with the tree $T$ represented in (141), we get

$$\begin{aligned}\mathbf{P}_T(\mathbb{X}) &= \frac{(q^3-qt)(1-q^2t)}{(1-qt)(1-q^6)}\mathbf{P}_{T_1}\mathbf{P}_{T_2}\\ &= \frac{(q^3-qt)(1-q^2t)}{(1-qt)(1-q^6)}\frac{(q-t)(1-t)}{(1-t)(1-q^2)}\mathbf{P}_{\bullet}\frac{(q-t)(1-qt)}{(1-t)(1-q^3)}\mathbf{P}_{\bullet}^2\\ &= \frac{(q^3-qt)(1-q^2t)}{(1-qt)(1-q^6)}\frac{(q-t)}{(1-q^2)}\mathbf{P}_{\bullet}\frac{(q-t)(1-qt)}{(1-t)(1-q^3)}\mathbf{P}_{\bullet}^2\\ &= \frac{(q^3-qt)(1-q^2t)(q-t)^2(1-t)^2}{(1-q)^3(1-q^2)(1-q^3)(1-q^6)}\end{aligned} \tag{151}$$

which is the same as Formula (142).

As in the case of **FQSym**, the product can be simplified so as to get a single quotient at each node: as before, the simplification occurs among nodes belonging to the same right branch.

**Corollary 6.6** *Let $T$ be a tree. Then $\mathbf{P}_T(\mathbb{X})$ is given by the product of the $(q,t)$-hook-content factors of the nodes $s$ of $T$, given by the following rules:*

$$\frac{1}{1-q^n}\begin{cases} q^{n'}-q^{n''}t & \text{if } s \text{ has a right son,}\\ 1-q^{n-1}t & \text{if } s \text{ has no right son and is not the right son of its father,}\\ 1-q^{d}t & \text{if } s \text{ has no right son and is the right son of its father,}\end{cases} \tag{152}$$

*where $n$ is the size of the subtree of root $s$, $n'$ is the size of the right subtree of $s$, $n''$ is the size of the left subtree of the right subtree of $s$, and $d$ is the size of the left subtree of the topmost ancestor of $s$ leading to $s$ only by right branches.*

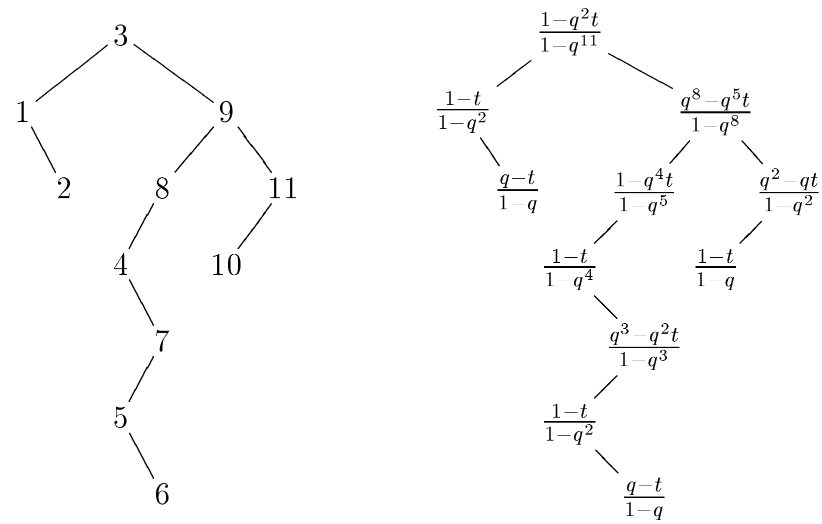
Again, with the tree $T$ represented in (141), we get

$$\begin{aligned}\mathbf{P}_T(\mathbb{X}) &= \frac{1-t}{1-q}\frac{q-t}{1-q^2}\frac{1-t}{1-q}\frac{1-q^2t}{1-q}\frac{q-t}{1-q^3}\frac{q^3-qt}{1-q^6}\\ &= \frac{(q-t)^2(1-t)^2(1-q^2t)(q^3-qt)[4]_q[5]_q}{(1-q)(1-q^2)(1-q^3)(1-q^4)(1-q^5)(1-q^6)},\end{aligned} \tag{153}$$

which is the same as (142) and (151).

## 6.5 Graphical representations

As in the case of **FQSym**, the first $(q-t)$-hook-length formula can be represented graphically by placing into each node the fraction appearing in Equation (138). For example, the first tree of Figure 2 shows the expansion of $\mathbf{F}_\sigma(\mathbb{X})$ with the first hook-content factors of a zig-zag tree. Figure 4 gives another example of this construction.




(154)

Figure 4: A binary tree (left diagram) labelled as a standard binary search tree and the first $(q,t)$-hook-content formula on trees (right diagram).

Let us now illustrate the second and third $(q,t)$-hook-content formulas for binary trees. For example, on Figure 5, the rightmost node of the second tree has coefficient $\frac{1-q^2t}{1-q^2}$: its topmost ancestor is the root of the tree and the left subtree of the root is of size 2 (see Corollary 6.6). Note that it is obtained by permuting cyclically the numerators of the second formula (see Theorem 6.5) among right branches, as was already the case in **FQSym**.

# 7 Word Super-quasi-symmetric functions

It has been explained in [20] that the use of **FQSym** to recover the Björner-Wachs $q$-hook-length formulas could be extended to other combinatorial Hopf algebras. This was illustrated on a construction associating a plane tree to a packed word, interpreted as a map from **WQSym** to the free tridendriform algebra on one generator. In this section, we will give a $(q,t)$-analog of the formula of [20] counting the number of packed words yielding a given tree according to the length of their evaluation. To this aim, we first need to extend the superization map to **WQSym**.

## 7.1 An algebra on signed packed words

Let us define $\mathbf{WQSym}^{(2)}$ as the space spanned by the $\mathbf{M}_{u,\epsilon}$, where for a signed packed word $(u,\epsilon)$,

$$\mathbf{M}_{u,\epsilon}(A) := \sum_{\substack{(w,\epsilon)\\ \mathrm{pack}(w)=u}} (w,\epsilon). \tag{155}$$

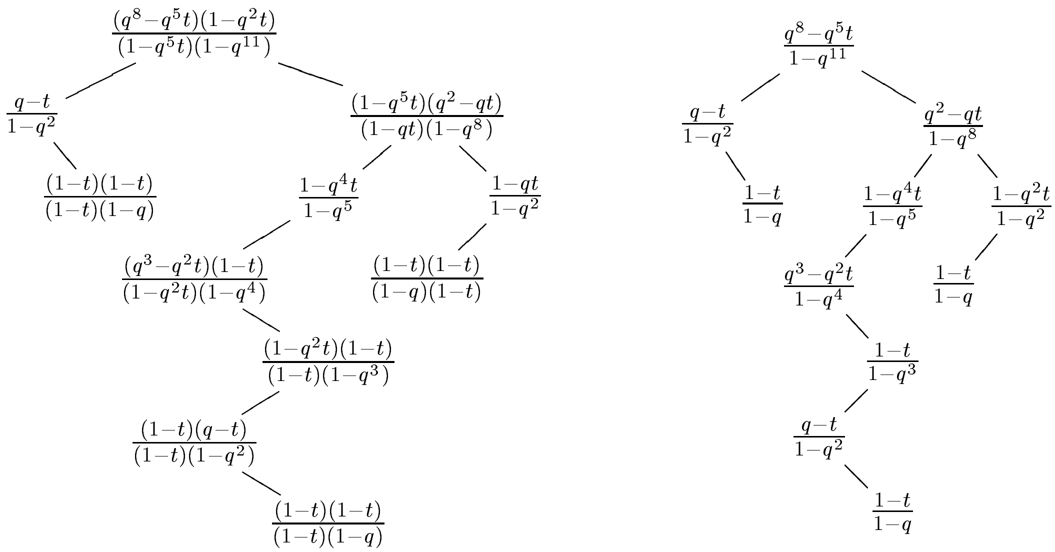



Figure 5: Second and third $(q,t)$-hook-content formulas of a binary tree: by induction (left diagram) and simplification of the induction (right diagram).

Again, it is self-dual as a graded Hopf algebra for the standard operations. We denote by $\mathbf{N}_{u,\epsilon}$ the dual basis of $\mathbf{M}_{u,\epsilon}$. This algebra contains $\mathbf{Sym}^{(2)}$, the Mantaci-Reutenauer algebra of type $B$. To show this, let us describe the embedding.

A signed word is said to be *regular* if all occurences of any letter have same sign. For example, $11\overline{22}31$ is regular, but $11\overline{1}$ and $1121\overline{2}$ are not.

The *signed evaluation* $\operatorname{sev}(w,\epsilon)$ of a regular word is the signed composition $(I,\mu)$ where $i_j$ is the number of occurrences of the (unsigned) letter $j$ and $\mu_j$ is the sign of $j$ in $(w,\epsilon)$.

Let $\phi$ be the morphism from $\mathbf{Sym}^{(2)}$ into $\mathbf{WQSym}^{(2)}$ defined by

$$\phi(S_n) = \mathbf{N}_{1^n}, \qquad \phi(S_{\overline{n}}) = \mathbf{N}_{\overline{1}^n}. \tag{156}$$

We then have :

**Lemma 7.1**

$$\phi(S^{(I,\epsilon)}) = \sum_{\substack{(u,\epsilon')\text{regular}\\ \operatorname{sev}(u,\epsilon')=(I,\epsilon)}} \mathbf{N}_{u,\epsilon'}. \tag{157}$$

*Proof* – This follows from the product formula of the $\mathbf{N}_{u,\epsilon}$, which is a special case of the multiplication of signed parking functions [34]. ∎

The image of $\mathbf{Sym}^{(2)}$ by this embedding is contained in the vector space $\mathcal{B}W$ of $\mathbf{WQSym}^{(2)}$ spanned by the $\mathbf{N}_{u,\epsilon}$ indexed by regular signed packed words. Although this property will not be used in the sequel, it is worth mentionning that $\mathcal{B}W$ is a Hopf subalgebra of $\mathbf{WQSym}^{(2)}$. The dimensions of its homogeneous components $\mathcal{B}W_n$ are given by Sequence A004123 of [42] whose first values are

$$1,\ 2,\ 10,\ 74,\ 730,\ 9002,\ 133210. \tag{158}$$

Note in particular that $\sigma_1^{\#}$ has a simple expression in terms of $\mathbf{N}_{u,\epsilon}$.

**Lemma 7.2** *Let* PW *denote the set of packed words, and* $\max(u)$ *the maximal letter of* $u$. *Then*

$$\sigma_1^{\#} = \sum_{u\in \mathrm{PW}} \left( (-1)^{n-\max(u)} \mathbf{N}_{u,(-1)^n} + (-1)^{m(\epsilon')-(\max(u)-1)} \mathbf{N}_{u,\epsilon'} \right), \tag{159}$$

*where* $(u,\epsilon')$ *is such that all letters of* $u$ *except the maximal one are signed.*

**Example 7.3**

$$S_2^{\#} = -\mathbf{N}_{\overline{11}} + \mathbf{N}_{11} + \mathbf{N}_{\overline{12}} + \mathbf{N}_{\bar{1}2} + \mathbf{N}_{\overline{21}} + \mathbf{N}_{2\bar{1}}. \tag{160}$$

$$\begin{aligned} S_3^{\#} = &+ \mathbf{N}_{\overline{111}} + \mathbf{N}_{111} \\ &- \mathbf{N}_{\overline{112}} - \mathbf{N}_{\overline{11}2} - \mathbf{N}_{\overline{121}} - \mathbf{N}_{\bar{1}2\bar{1}} - \mathbf{N}_{\overline{211}} - \mathbf{N}_{2\overline{11}} \\ &- \mathbf{N}_{\overline{221}} + \mathbf{N}_{22\bar{1}} - \mathbf{N}_{\overline{212}} + \mathbf{N}_{2\bar{1}2} - \mathbf{N}_{\overline{122}} + \mathbf{N}_{\bar{1}22} \\ &+ \mathbf{N}_{\overline{123}} + \mathbf{N}_{\overline{12}3} + \mathbf{N}_{\overline{132}} + \mathbf{N}_{\bar{1}3\bar{2}} + \mathbf{N}_{\overline{213}} + \mathbf{N}_{\overline{21}3} \\ &+ \mathbf{N}_{\overline{231}} + \mathbf{N}_{\bar{2}3\bar{1}} + \mathbf{N}_{\overline{312}} + \mathbf{N}_{3\overline{12}} + \mathbf{N}_{\overline{321}} + \mathbf{N}_{3\overline{21}}. \end{aligned} \tag{161}$$

## 7.2 An internal product on signed packed words

The internal product of $\mathbf{WQSym}^*$ (29) can be extended to $\mathbf{WQSym}^{(2)^*}$ by

$$\mathbf{N}_{u,\epsilon} * \mathbf{N}_{v,\rho} = \mathbf{N}_{\mathrm{pack}(u,v),\epsilon\rho}, \tag{162}$$

where $\epsilon\rho$ is the componentwise product. One obtains in this way the (opposite) Solomon-Tits algebra of type $B$. This product is induced from the internal product of signed parking functions [34] and can be shown to coincide with the one introduced by Hsiao [22].

From this definition, we have immediately:

**Proposition 7.4** $\mathcal{B}W$ *is a subalgebra of* $\mathbf{WQSym}^{(2)^*}$ *for the internal product.*

Since $\sigma_1^{\#}$ belongs to $\mathbf{WQSym}^{(2)^*}$, we can define

$$\mathbf{N}_u^{\#} := \mathbf{N}_u(A|\bar{A}) = \mathbf{N}_u * \sigma_1^{\#}. \tag{163}$$

**Example 7.5** Let us compute the first $\mathbf{N}_u(A|\bar{A})$.

$$\mathbf{N}_{11}^{\#} = -\mathbf{N}_{\overline{11}} + \mathbf{N}_{11} + \mathbf{N}_{\overline{12}} + \mathbf{N}_{\bar{1}2} + \mathbf{N}_{\overline{21}} + \mathbf{N}_{2\bar{1}}. \tag{164}$$

$$\mathbf{N}_{12}^{\#} = \mathbf{N}_{\overline{12}} + \mathbf{N}_{\bar{1}2} + \mathbf{N}_{1\bar{2}} + \mathbf{N}_{12}. \tag{165}$$

$$\mathbf{N}_{21}^{\#} = \mathbf{N}_{\overline{21}} + \mathbf{N}_{\bar{2}1} + \mathbf{N}_{2\bar{1}} + \mathbf{N}_{21}. \tag{166}$$

$$\begin{aligned} \mathbf{N}_{112}^{\#} = &- \mathbf{N}_{\overline{112}} - \mathbf{N}_{\overline{11}2} + \mathbf{N}_{11\bar{2}} + \mathbf{N}_{112} \\ &+ \mathbf{N}_{\overline{123}} + \mathbf{N}_{\overline{12}3} + \mathbf{N}_{\bar{1}2\bar{3}} + \mathbf{N}_{\bar{1}23} \\ &+ \mathbf{N}_{\overline{213}} + \mathbf{N}_{\overline{21}3} + \mathbf{N}_{2\overline{13}} + \mathbf{N}_{2\bar{1}3} \end{aligned} \tag{167}$$

$$\begin{aligned} \mathbf{N}_{121}^{\#} = &- \mathbf{N}_{\overline{121}} - \mathbf{N}_{\bar{1}2\bar{1}} + \mathbf{N}_{1\bar{2}1} + \mathbf{N}_{121} \\ &+ \mathbf{N}_{\overline{132}} + \mathbf{N}_{\bar{1}3\bar{2}} + \mathbf{N}_{\overline{13}2} + \mathbf{N}_{\bar{1}32} \\ &+ \mathbf{N}_{\overline{231}} + \mathbf{N}_{\bar{2}3\bar{1}} + \mathbf{N}_{2\overline{31}} + \mathbf{N}_{23\bar{1}} \end{aligned} \tag{168}$$

In the light of the previous examples, let us say that a packed word $v$ is *finer* than a packed word $u$, and write $v \geqslant u$ if $u$ can be obtained from $v$ by application of a nondecreasing map $\phi : \mathbb{N}^* \to \mathbb{N}^*$. If $u = \phi(v)$, that is, $u_i = \phi(v_i)$, we say that $v_i$ *goes to* $u_i$. Note that this definition is also easily described on set compositions: $u$ is then obtained by gluing together consecutive parts of $v$. For example, the words finer than 121 are 121, 132, and 231.

**Theorem 7.6** *Let $u$ be a packed word. Then*

$$\mathbf{N}^{\#}_{u} = \sum_{v \geqslant u} \sum_{\epsilon} (-1)^{m(\epsilon)+m'(v,\epsilon)} \mathbf{N}_{v,\epsilon} \tag{169}$$

*where $m'(v,\epsilon)$ is equal to the number of* different *signed letters of $(v,\epsilon)$ and where the sum over $\epsilon$ is such that the words $(v,\epsilon)$ are regular and such that if more than two letters of $v$ go to the same letter of $u$, all letters but the greatest are signed (the greatest can be either signed or not). In particular, the number of such $\epsilon$ for a given $v$ is equal to $2^{\max(u)}$, so is independent of $v$.*

*Proof* – From the definitions of $\sigma_1^{\#}$ and of the packing algorithm, it is clear that the words appearing on the expansion of $\mathbf{N}^{\#}_{u}$ are exactly the words given in the previous statement.

Moreover, the coefficient of a signed word $(w,\epsilon)$ in $\sigma_1^{\#}$ is equal to the coefficient of any of its rearrangements (where the signs follow their letter). Now, given a permutation $\sigma$ and two words $u$ and $u'$ having a word $v$ as packed word, the packed word of $u \cdot \sigma$ and $u' \cdot \sigma$ is $v \cdot \sigma$. So we can restrict ourselves to compute $\mathbf{N}^{\#}_{u}$ for all nondecreasing words $u$ since all the other ones are obtained by permutation of the entries.

Assume now that $u$ is a nondecreasing word, and let us show that the coefficient of $(v,\epsilon)$ in $\mathbf{N}^{\#}_{u}$ is either 1 or $-1$. The only terms $\mathbf{N}$ in $\sigma_1^{\#}$ that can yield $(v,\epsilon)$ when multiplied on the left by $\mathbf{N}_u$ are the signed words with negative entries exactly as in $\epsilon$. Let $T_\epsilon$ denote this set. Thanks to Lemma 7.2, the $\mathbf{N}$ appearing in the expansion of $\sigma_1^{\#}$ with negative signs at $k$ given slots are the following packed words: all the elements of $\mathrm{PW_k}$ at the negative slots and one letter greater than all the others at the remaining slots. In particular, the cardinality of $T_\epsilon$ depends only on $k$ and is equal to $|PW_k|$. Since there is only one positive value for each element, two words $w$ and $w'$ of $T_\epsilon$ give the same result by packing $(u,w)$ and $(u,w')$ if they coincide on the negative slots.

This means that we can restrict to the special case where $\epsilon = (-1)^n$ since the positive slot do not change the way of regrouping the elements of $T_\epsilon$ to obtain $(v,\epsilon)$. Now, the sign has been disposed of and we can concentrate on the packing algorithm. The previous discussion shows that we only need to prove that, given a word $v$ finer than a word $u$, the set $T$ of packed words $w$ such that $\mathrm{pack}(u,w) = v$ satisfies the following property: if $t_d$ is the number of elements of $T$ with maximum $d$, then

$$\sum_{d} (-1)^d t_d = \pm 1. \tag{170}$$

From the definition of the packing algorithm, we see that $T$ is the set of packed words with (in)equalities coming from the values of $v$ at the places where $u$ have equal letters.

So $T$ is a set of packed words with (in)equalities between adjacent places with no other relations. Hence, if $u$ has $l$ different letters, $T$ is obtained as the product of $l$ quasi-monomial functions $\mathbf{M}_w$. The conclusion of the proof comes from the following lemma. ■

**Lemma 7.7** *Let $w_1, \ldots, w_k$ be $k$ packed words with respective maximum letters $a_1, \ldots, a_k$. Let $T$ be the set of packed words appearing in the expansion of*

$$\mathbf{M}_{w_1} \ldots \mathbf{M}_{w_k}. \tag{171}$$

*Then, if $t_d$ is the number of elements of $T$ with maximum $d$, then*

$$\sum_d (-1)^d t_d = (-1)^{a_1+\cdots+a_k}. \tag{172}$$

*Proof* – We only need to prove the result for $k=2$ since the other cases follow by induction: compute $\mathbf{M}_{w_1} \ldots \mathbf{M}_{w_{k-1}}$ and multiply this by $\mathbf{M}_{w_k}$ to get the result.

Let us compute $\mathbf{M}_{w_1}\mathbf{M}_{w_2}$. The number of words with maximum $a_1+a_2-d$ in this product is equal to

$$\binom{a_1}{d}\binom{a_1+a_2-d}{a_1}. \tag{173}$$

Indeed, a word in $\mathbf{M}_{w_1}\mathbf{M}_{w_2}$ with maximum $a_1+a_2-d$ is completely characterized by the $d$ integers between 1 and $a_1+a_2-d$ common to the prefix of size $|w_1|$ and the suffix of size $|w_2|$ of $w$, by the $(a_1-d)$ integers only appearing in the prefix, and the $(a_2-d)$ integers only appearing in the suffix, which hence gives the enumeration formula

$$t_{a_1+a_2-d} = \frac{(a_1+a_2-d)!}{d!(a_1-d)!(a_2-d)!}, \tag{174}$$

equivalent to the previous one.

It remains to compute

$$\sum_d (-1)^{a_1+a_2-d}\binom{a_1}{d}\binom{a_1+a_2-d}{a_1}, \tag{175}$$

which is, with the usual notation for elementary and complete homogeneous symmetric functions, understood as operators of the $\lambda$-ring $\mathbb{Z}$,

$$\begin{aligned}
&(-1)^{a_1+a_2}\sum_d (-1)^d e_d(a_1) h_{a_2-d}(a_1+1)\\
&= (-1)^{a_1+a_2}\sum_d h_d(-a_1) h_{a_2-d}(a_1+1)\\
&= (-1)^{a_1+a_2} h_{a_2}(-a_1+a_1+1)\\
&= (-1)^{a_1+a_2} h_{a_2}(1) = (-1)^{a_1+a_2}.
\end{aligned} \tag{176}$$

■

This combinatorial interpretation of (173) gives back in particular one interpretation of the Delannoy numbers (sequence A001850 of [42]) and of their usual refinement (sequence A008288 of [42]).

## 7.3 Specializations

The internal product of **WQSym**$^*$ allows in particular to define

$$\mathbf{N}_u((1-t)A) := \mathbf{N}_u(A) * \sigma_1((1-t)A) = \eta_t(\mathbf{N}_u), \tag{177}$$

so that we have

$$\mathbf{S}^I((1-t)A) = \sum_{\mathrm{Ev}(u)=I} \mathbf{N}_u((1-t)A). \tag{178}$$

**Example 7.8** Taking the same five examples as in Example 7.5, we get

$$\mathbf{N}_{11}((1-t)A) = (1-t^2)\mathbf{N}_{11} - t(1-t)\mathbf{N}_{12} - t(1-t)\mathbf{N}_{21}. \tag{179}$$

$$\mathbf{N}_{12}((1-t)A) = (1-t)^2\mathbf{N}_{12} \qquad \text{and} \qquad \mathbf{N}_{21}((1-t)A) = (1-t)^2\mathbf{N}_{21}. \tag{180}$$

$$\mathbf{N}_{112}((1-t)A) = (1-t)(1-t^2)\mathbf{N}_{112} - t(1-t)^2\mathbf{N}_{123} - t(1-t)^2\mathbf{N}_{213}. \tag{181}$$

$$\mathbf{N}_{121}((1-t)A) = (1-t)(1-t^2)\mathbf{N}_{121} - t(1-t)^2\mathbf{N}_{132} - t(1-t)^2\mathbf{N}_{231}. \tag{182}$$

**Theorem 7.9** *Let $u$ be a packed word. Then*

$$\mathbf{N}_u((1-t)A) = \sum_{v \geqslant u} (-1)^{\max(v)-\max(u)} t^{f(u,v)} \prod_{k=1}^{\max(u)} (1 - t^{g(u,v,k)}) \ \ \mathbf{N}_v(A). \tag{183}$$

*where, if one writes*

$$\mathrm{Ev}(u) = (i_1, \ldots, i_p) \ \textit{and}\ \mathrm{Ev}(v) = ((i_1^{(1)}, \ldots, i_1^{(q_1)}), \ldots, (i_p^{(1)}, \ldots, i_p^{(q_p)})), \tag{184}$$

*then*

$$f(u,v) := \sum_{k=1}^{p} \sum_{j=1}^{q_k-1} i_k^{(j)} \qquad \textit{and} \qquad g(u,v,k) := i_k^{(q_k)}. \tag{185}$$

*Proof* – This is a direct consequence of Theorem 7.6. ∎

## 7.4 Duality

By duality, one defines

$$\mathbf{M}_u(A \cdot (1-t)) := \eta_t^*(\mathbf{M}_u(A)), \tag{186}$$

since

$$\sum_u \mathbf{M}_u(A \cdot (1-t)) \otimes \mathbf{N}_u(B) = \sum_u \mathbf{M}_u(A) \otimes \mathbf{N}_u((1-t)B). \tag{187}$$

**Example 7.10**

$$\mathbf{M}_{11}(A\cdot(1-t)) = (1-t^2)\mathbf{M}_{11}(A). \tag{188}$$

$$\mathbf{M}_{12}(A\cdot(1-t)) = -t(1-t)\mathbf{M}_{11}(A) + (1-t)^2\mathbf{M}_{12}(A). \tag{189}$$

$$\mathbf{M}_{21}(A\cdot(1-t)) = -t(1-t)\mathbf{M}_{11}(A) + (1-t)^2\mathbf{M}_{21}(A). \tag{190}$$

$$\mathbf{M}_{112}(A\cdot(1-t)) = (1-t)(1-t^2)\mathbf{M}_{112}(A) - t^2(1-t)\mathbf{M}_{111}(A). \tag{191}$$

$$\mathbf{M}_{121}(A\cdot(1-t)) = (1-t)(1-t^2)\mathbf{M}_{121}(A) - t^2(1-t)\mathbf{M}_{111}(A). \tag{192}$$

$$\begin{aligned}\mathbf{M}_{123}(A\cdot(1-t)) =&(1-t)^3\mathbf{M}_{123}(A) - t(1-t)^2\mathbf{M}_{112}(A)\\ &- t(1-t)^2\mathbf{M}_{122}(A) + t^2(1-t)\mathbf{M}_{111}(A).\end{aligned} \tag{193}$$

Since the transition matrix from $\mathbf{M}(A\cdot(1-t))$ to $\mathbf{M}(A)$ is the transpose of the transition matrix from $\mathbf{N}((1-t)A)$ to $\mathbf{N}(A)$, we can obtain a simple combinatorial interpretation of $\mathbf{M}(A\cdot(1-t))$.

First, let us define the *super-packed word* $v := \mathrm{spack}(u,\epsilon)$ associated with a regular signed word $(u,\epsilon)$. Let $f_\epsilon$ be the nondecreasing function sending 1 to 1 and each value $i$ either to $f_\epsilon(i-1)$ if the value $i-1$ is signed in $\epsilon$ or to $1+f_\epsilon(i-1)$ if not. Extend $f_\epsilon$ to a morphism of $A^*$. Then $v = f_\epsilon(u)$.

For example,

$$\mathrm{spack}(75\overline{1}\overline{2}\overline{1}53\overline{4}\overline{4}6\overline{1}) = 42111212231. \tag{194}$$

Let $[v,u]$ be the interval for the refinement order on words, that is, the set of packed words $w$ such that $u \geqslant w \geqslant v$.

**Proposition 7.11** *Let $u$ be a word. Then*

$$\mathbf{M}_u(A\cdot(1-t)) = \sum_{(u,\epsilon)\ \text{regular}} (-1)^{m'(u,\epsilon)} t^{m(\epsilon)} \sum_{w\in[\mathrm{spack}(u,\epsilon),u]} \mathbf{M}_w(A). \tag{195}$$

*Proof* – Observe that if a signed word $(u,\epsilon)$ appears in $\mathbf{N}^{\#}_w$ then it also appears in $\mathbf{N}^{\#}_v$ for all $v\in[u,w]$. The rest comes directly from Theorem 7.6 and from the fact that $\mathbf{N}_{(u,\epsilon)}$ is sent to $(-t)^{m(\epsilon)}\mathbf{N}_u$ when sending $\overline{A}$ to $-tA$. ■

**Example 7.12**

$$\mathbf{M}_{21}(A\cdot(1-t)) = (-t+t^2)(\mathbf{M}_{11}+\mathbf{M}_{21}) + (1-t)\mathbf{M}_{21}. \tag{196}$$

$$\mathbf{M}_{112}(A\cdot(1-t)) = (-t^2+t^3)(\mathbf{M}_{111}+\mathbf{M}_{112}) + (1-t)\mathbf{M}_{112}. \tag{197}$$

$$\begin{aligned}\mathbf{M}_{123}(A\cdot(1-t)) =&(t^2-t^3)(\mathbf{M}_{111}+\mathbf{M}_{112}+\mathbf{M}_{122}+\mathbf{M}_{123})\\ &+(-t+t^2)(\mathbf{M}_{112}+\mathbf{M}_{123})\\ &+(-t+t^2)(\mathbf{M}_{122}+\mathbf{M}_{123})\\ &+(1-t)\mathbf{M}_{123}.\end{aligned} \tag{198}$$

When $A$ is a commutative alphabet $X$, this specializes to $M_I(X(1-t))$ where $I = \mathrm{Ev}(u)$ and in particular, for $X = \frac{1}{1-q}$, we recover a result of [24]:

**Theorem 7.13 ([24])** *Let $u$ be a packed word of size $n$.*

$$\mathbf{M}_u(\mathbb{X}) = M_I(\mathbb{X}) = \frac{1-t^{i_p}}{1-q^n} \prod_{k=1}^{p-1} \frac{q^{i_1+\cdots+i_k} - t^{i_k}}{1-q^{i_1+\cdots+i_k}}. \tag{199}$$

*where the composition $I = (i_1, \ldots, i_p)$ is the evaluation of $u$.*

*Proof* – From Proposition 7.11 giving a combinatorial interpretation of $\mathbf{M}_u(A \cdot (1-t))$, we have:

$$\mathbf{M}_u(\mathbb{X}) = \sum_{(u,\epsilon) \text{ regular}} (-1)^{m'(u,\epsilon)} t^{m(\epsilon)} \sum_{w \in [\mathrm{spack}(u,\epsilon), u]} \mathbf{M}_w(1/(1-q)). \tag{200}$$

We now have to evaluate the sum of $\mathbf{M}_w(1/(1-q))$ over an interval of the composition lattice. Thanks to Lemma 7.14 below, it is equal to

$$\frac{q^{\mathrm{maj}\,(I)}}{(1-q^{k_1})(1-q^{k_1+k_2}) \cdots (1-q^{k_1+k_2+\cdots+k_s})}, \tag{201}$$

where $I = \mathrm{Ev}(\mathrm{spack}(u,\epsilon))$ and $K = \mathrm{Ev}(u)$, which implies the result. ■

**Lemma 7.14** *Let $I$ and $K$ be two compositions of $n$ such that $K \geqslant I$. Then*

$$\sum_{J \in [I,K]} M_J(1/(1-q)) = \frac{1}{1-q^n} \frac{q^{\mathrm{maj}\,(I)}}{\prod_{d \in \mathrm{Des}(K)} 1-q^d}. \tag{202}$$

*Proof* – We have

$$M_J(1/(1-q)) = \frac{1}{1-q^n} \prod_{d \in Des(J)} \frac{q^d}{1-q^d}. \tag{203}$$

Factorizing by the common denominator of all these elements and by $q^{\mathrm{maj}\,(K)}$, we have to evaluate

$$\sum_{D \subseteq \mathrm{Des}(K) \setminus \mathrm{Des}(I)} \prod_{d \in D} (1-q^d) q^{-d} \tag{204}$$

which is equal to

$$\prod_{d \in \mathrm{Des}(K) / \mathrm{Des}(I)} (1 - 1 + q^{-d}) = q^{-(\mathrm{maj}\,(K) - \mathrm{maj}\,(I))}. \tag{205}$$

■

Putting together Proposition 7.11 and Lemma 7.14, one obtains:

**Corollary 7.15** *Let $u$ be a word of size $n$. Then*

$$(q)_{\mathrm{Ev}(u)}\mathbf{M}_u(\mathbb{X}) = \sum_{(u,\epsilon) regular} (-1)^{m'(u,\epsilon)} t^{m(\epsilon)} q^{\mathrm{maj}\,(\mathrm{spack}(u,\epsilon))}, \tag{206}$$

*where $(q)_I$ is defined as $(1-q^n)\prod_{d\in\mathrm{Des}(I)}(1-q^d)$.*

**Corollary 7.16** *Let $u$ be a word of size $n$. Then the generating function of signed packed words of unsigned part $u$ by major index of their super-packed word and number of signs is:*

$$\sum_{(u,\epsilon)\ \ regular} t^{m(\epsilon)} q^{\mathrm{maj}\,(\mathrm{spack}(u,\epsilon))} = (1+t^{i_p}) \prod_{k=1}^{p-1} (q^{i_1+\cdots+i_k} + t^{i_k}). \tag{207}$$

**Example 7.17** For example, with $u = 112333344$, one has:

$$\sum_{(u,\epsilon)\ \ \mathrm{regular}} t^{m(\epsilon)} q^{\mathrm{maj}\,(\mathrm{spack}(u,\epsilon))} = (1+t^2)(q^2+t^2)(q^3+t)(q^7+t^4), \tag{208}$$

since we have the following 16 words with their $t$ and $q$ statistics:

| $(u,\epsilon)$ | $m(\epsilon)$ | $\mathrm{maj}\,(\mathrm{spack}(u,\epsilon))$ | $(u,\epsilon)$ | $m(\epsilon)$ | $\mathrm{maj}\,(\mathrm{spack}(u,\epsilon))$ |
|---|---|---|---|---|---|
| $112333344$ | 0 | 12 | $1123333\overline{44}$ | 2 | 12 |
| $112\overline{3333}44$ | 4 | 5 | $112\overline{333344}$ | 6 | 5 |
| $11\overline{2}333344$ | 1 | 9 | $11\overline{2}3333\overline{44}$ | 3 | 9 |
| $11\overline{23333}44$ | 5 | 2 | $11\overline{2333344}$ | 7 | 2 |
| $\overline{11}2333344$ | 2 | 10 | $\overline{11}23333\overline{44}$ | 4 | 10 |
| $\overline{11}2\overline{3333}44$ | 6 | 3 | $\overline{11}2\overline{333344}$ | 8 | 3 |
| $\overline{112}333344$ | 3 | 7 | $\overline{112}3333\overline{44}$ | 5 | 7 |
| $\overline{1123333}44$ | 7 | 0 | $\overline{112333344}$ | 9 | 0 |

(209)

# 8 Tridendriform operations and $(q,t)$-specialization

## 8.1 Tridendriform structure of WQSym

A *dendriform trialgebra* [29] is an associative algebra whose multiplication $\cdot$ splits into three pieces

$$x \cdot y = x \prec y + x \circ y + x \succ y\,, \tag{210}$$

where $\circ$ is associative, and

$$(x \prec y) \prec z = x \prec (y \cdot z)\,, \quad (x \succ y) \prec z = x \succ (y \prec z)\,, \quad (x \cdot y) \succ z = x \succ (y \succ z)\,, \tag{211}$$

$$(x \succ y) \circ z = x \succ (y \circ z)\,, \quad (x \prec y) \circ z = x \circ (y \succ z)\,, \quad (x \circ y) \prec z = x \circ (y \prec z)\,. \tag{212}$$

It has been shown in [36] that the augmentation ideal $\mathbb{K}\langle A\rangle^+$ has a natural structure of dendriform trialgebra: for two non empty words $u, v \in A^*$, we set

$$u \prec v = \begin{cases} uv & \text{if } \max(u) > \max(v) \\ 0 & \text{otherwise,} \end{cases} \tag{213}$$

$$u \circ v = \begin{cases} uv & \text{if } \max(u) = \max(v) \\ 0 & \text{otherwise,} \end{cases} \tag{214}$$

$$u \succ v = \begin{cases} uv & \text{if } \max(u) < \max(v) \\ 0 & \text{otherwise.} \end{cases} \tag{215}$$

$\mathbf{WQSym}^+$ is a sub-dendriform trialgebra of $\mathbb{K}\langle A\rangle^+$, the partial products being given by

$$\mathbf{M}_{w'} \prec \mathbf{M}_{w''} = \sum_{w=u\cdot v \in w' *_W w'', |u|=|w'|; \max(v)<\max(u)} \mathbf{M}_w, \tag{216}$$

$$\mathbf{M}_{w'} \circ \mathbf{M}_{w''} = \sum_{w=u\cdot v \in w' *_W w'', |u|=|w'|; \max(v)=\max(u)} \mathbf{M}_w, \tag{217}$$

$$\mathbf{M}_{w'} \succ \mathbf{M}_{w''} = \sum_{w=u\cdot v \in w' *_W w'', |u|=|w'|; \max(v)>\max(u)} \mathbf{M}_w, \tag{218}$$

where the *convolution* $u' *_W u''$ of two packed words is defined as

$$u' *_W u'' = \sum_{v,w; u=v\cdot w \,\in\, \mathrm{PW}, \mathrm{pack(v)=u'}, \mathrm{pack(w)=u''}} u\,. \tag{219}$$

## 8.2 Specialization of the partial products

If $w$ is a packed word, let $\mathrm{nmax}(w)$ be the number of maximal letters of $w$ in $w$.

**Theorem 8.1** *Let $u_1 \in \mathrm{PW(n)}$ and $u_2 \in \mathrm{PW(m)}$. Then*

$$(\mathbf{M}_{u_1} \prec \mathbf{M}_{u_2})(\mathbb{X}) = \frac{1-q^n}{1-q^{n+m}}\ \frac{q^m - t^{\mathrm{nmax}(u_2)}}{1-t^{\mathrm{nmax}(u_2)}}\ \mathbf{M}_{u_1}(\mathbb{X})\mathbf{M}_{u_2}(\mathbb{X}), \tag{220}$$

$$(\mathbf{M}_{u_1} \circ \mathbf{M}_{u_2})(\mathbb{X}) = \frac{(1-q^n)(1-q^m)}{1-q^{n+m}}\ \frac{1-t^{\mathrm{nmax}(u_1)+\mathrm{nmax}(u_2)}}{(1-t^{\mathrm{nmax}(u_1)})(1-t^{\mathrm{nmax}(u_2)})}\mathbf{M}_{u_1}(\mathbb{X})\mathbf{M}_{u_2}(\mathbb{X}), \tag{221}$$

*and*

$$(\mathbf{M}_{u_1} \succ \mathbf{M}_{u_2})(\mathbb{X}) = \frac{1-q^m}{1-q^{n+m}}\ \frac{q^n - t^{\mathrm{nmax}(u_1)}}{1-t^{\mathrm{nmax}(u_1)}}\ \mathbf{M}_{u_1}(\mathbb{X})\mathbf{M}_{u_2}(\mathbb{X}). \tag{222}$$

*Proof* – Thanks to the combinatorial interpretation of $\mathbf{M}_u(\mathbb{X})$ in terms of signed words (Proposition 7.11 and Lemma 7.14), one only has to check what happens to the major index of the evaluation of signed words in the cases of the left, middle, or right tridendriform products. The analysis is similar to (but simpler than) the analysis in the **FQSym** case done previously. ∎

**Example 8.2** Note that the left tridendriform product does not depend on the actual value of $w_1$ but only on its length. For example,

$$(\mathbf{M}_{111} \prec \mathbf{M}_{2122})(\mathbb{X}) = \frac{1-q^3}{1-q^7}\,\frac{q^4-t^3}{1-t^3}\,\mathbf{M}_{111}(\mathbb{X})\mathbf{M}_{2122}(\mathbb{X}) \tag{223}$$

$$(\mathbf{M}_{132} \prec \mathbf{M}_{2122})(\mathbb{X}) = \frac{1-q^3}{1-q^7}\,\frac{q^4-t^3}{1-t^3}\,\mathbf{M}_{132}(\mathbb{X})\mathbf{M}_{2122}(\mathbb{X}) \tag{224}$$

But the result depends on the number of maximum values of $w_2$:

$$(\mathbf{M}_{121} \prec \mathbf{M}_{3122})(\mathbb{X}) = \frac{1-q^3}{1-q^7}\,\frac{q^4-t}{1-t}\,\mathbf{M}_{121}(\mathbb{X})\mathbf{M}_{3122}(\mathbb{X}) \tag{225}$$

One can check on these examples the relation of dendriform trialgebras: $\mathbf{M}_u\mathbf{M}_v = \mathbf{M}_u \prec \mathbf{M}_v + \mathbf{M}_u \circ \mathbf{M}_v + \mathbf{M}_u \succ \mathbf{M}_v$:

$$(\mathbf{M}_{1212} \prec \mathbf{M}_{33231})(\mathbb{X}) = \frac{1-q^4}{1-q^9}\,\frac{q^5-t^3}{1-t^3}\,\mathbf{M}_{1212}(\mathbb{X})\mathbf{M}_{33231}(\mathbb{X}) \tag{226}$$

$$(\mathbf{M}_{1212} \circ \mathbf{M}_{33231})(\mathbb{X}) = \frac{(1-q^4)(1-q^5)}{1-q^9}\,\frac{1-t^5}{(1-t^2)(1-t^3)}\,\mathbf{M}_{1212}(\mathbb{X})\mathbf{M}_{33231}(\mathbb{X}) \tag{227}$$

$$(\mathbf{M}_{1212} \succ \mathbf{M}_{33231})(\mathbb{X}) = \frac{1-q^5}{1-q^9}\,\frac{q^4-t^2}{1-t^2}\,\mathbf{M}_{1212}(\mathbb{X})\mathbf{M}_{33231}(\mathbb{X}). \tag{228}$$

# 9 The free dendriform trialgebra

## 9.1 A subalgebra of WQSym

Recall that $\mathfrak{TD}$, the Loday-Ronco algebra of plane trees [29], or the free tridendriform algebra on one generator, is naturally a subalgebra of **WQSym** [37], the embedding being

$$\mathcal{M}_T(A) = \sum_{\mathcal{T}(u)=T} \mathbf{M}_u(A)\,, \tag{229}$$

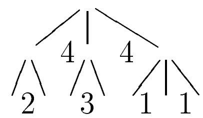
where $\mathcal{T}(u)$ is the plane tree associated with $u$ [37]. Hence, $\mathcal{M}_T(\mathbb{X})$ is well-defined.

$\mathfrak{TD}$ was originally defined [29] as the free tridendriform algebra on one generator as follows: if $T$ is a planar tree and $T_1, \ldots, T_k$ are its subtrees, then

$$\mathcal{M}_T = (\mathcal{M}_{T_1} \succ \mathcal{M}_1 \prec \mathcal{M}_{T_2}) \circ (\mathcal{M}_1 \prec \mathcal{M}_{T_3}) \circ \ldots \circ (\mathcal{M}_1 \prec \mathcal{M}_{T_k}). \tag{230}$$

## 9.2 A hook-content formula for plane trees

Let $T$ be a plane tree. Let $\mathrm{Int}(T)$ denote all internal nodes of $T$ (*i.e.*, nodes which are not leaves) except the root. Let us define a *region* of $T$ as any part of the plane between two edges coming from the same vertex. The regions are the places where one writes the values of a packed word when inserting it (see [37]). For example, with $w = 243411$, one gets




(231)

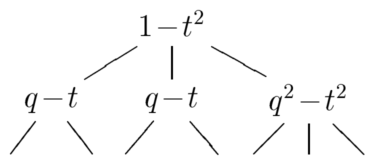
**Theorem 9.1** *Let $T$ be a plane tree with $n$ regions. Then*

$$\mathcal{M}_T(\mathbb{X}) = \frac{1-t^{a(\mathrm{root})-1}}{1-q^n} \prod_{i\in\mathrm{Int}(T)} \frac{q^{r(i)}-t^{a(i)-1}}{1-q^{r(i)}}, \tag{232}$$

*where $a(i)$ is the number of children of $i$, and $r(i)$ the total number of regions of $T$ below $i$.*

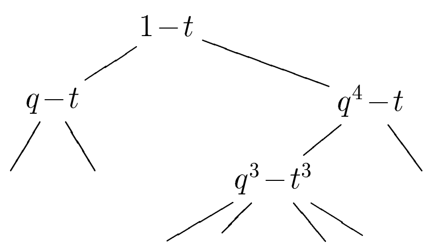
*Proof* – This is obtained by applying the tridendriform operations in **WQSym**, according to the decomposition (230), exactly as Formulas (143) and (152) are obtained from (130) in the cases of **PBT** and **FQSym**. ■

Writing for each node the numerator of its $(q,t)$ contribution, one has for the tree (231)




(233)

A more complicated example would be




(234)

Putting together Corollaries 7.15 and 7.16, Definition (229) and Theorem 9.1, one obtains finally

**Corollary 9.2** *Let $T$ be a plane tree with $n$ regions. Then*

$$\sum_{\substack{(u,\epsilon)\ \text{regular} \\ \mathcal{T}(u)=T}} \frac{t^{m(\epsilon)} q^{\mathrm{maj}\,(\mathrm{spack}(u,\epsilon))}}{(q)_{\mathrm{Ev}(u)}} = \frac{1+t^{a(\mathrm{root})-1}}{1-q^n} \prod_{i\in\mathrm{Int}(T)} \frac{q^{r(i)}+t^{a(i)-1}}{1-q^{r(i)}}. \tag{235}$$

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