The Fundamental Group of Balanced Simplicial Complexes and Posets

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Dedicated to Anders Björner on the occasion of his 60th birthday

Abstract

We establish an upper bound on the cardinality of a minimal generating set for the fundamental group of a large family of connected, balanced simplicial complexes and, more generally, simplicial posets.

1 Introduction

One commonly studied combinatorial invariant of a finite $(d-1)$-dimensional simplicial complex Δ is its f-vector $f = (f_0, \ldots, f_{d-1})$ where f_i denotes the number of i-dimensional faces of Δ. This leads to the study of the h-numbers of Δ defined by the relation $\sum_{i=0}^{d} h_i \lambda^{d-i} = \sum_{i=0}^{d} f_{i-1}(\lambda-1)^{d-i}$. A great deal of work has been done to relate the f-numbers and h-numbers of Δ to the dimensions of the singular homology groups of Δ with coefficients in a certain field; see, for example, the work of Björner and Kalai in [2] and [3], and Chapters 2 and 3 of Stanley [13]. In comparison, very little seems to be known about the relationship between the f-numbers of a simplicial complex and various invariants of its homotopy groups. In this paper, we bound the minimal number of generators of the fundamental group of a balanced simplicial complex in terms of h_2. More generally, we bound the minimal number of generators of the fundamental group of a balanced simplicial poset in terms of h_2.

It was conjectured by Kalai [7] and proved by Novik and Swartz in [8] that if Δ is a $(d-1)$-dimensional manifold that is orientable over the field $\mathbf{k}$, then

$$h_2 - h_1 \geq \binom{d+1}{2}\beta_1,$$

where β_1 is the dimension of the singular homology group $H_1(\Delta; \mathbf{k})$. The Hurewicz Theorem (see Spanier [10]) says that $H_1(X;\mathbb{Z})$ is isomorphic to the abelianization of $\pi_1(X,*)$

for a connected space X. We will see below that $\pi_1(\Delta, *)$ is finitely generated. Thus the Hurewicz Theorem says that the minimal number of generators of the fundamental group of a simplicial complex Δ is greater than or equal to the number of generators of $H_1(\Delta; \mathbb{Z})$. By the universal coefficient theorem, $H_1(\Delta; \mathbf{k}) \approx H_1(\Delta; \mathbb{Z}) \otimes \mathbf{k}$ for any field $\mathbf{k}$; and, consequently, the minimal number of generators of $\pi_1(\Delta, *)$ is greater than or equal to $\beta_1(\Delta)$ for any field $\mathbf{k}$.

In this paper, we study simplicial complexes and simplicial posets Δ that are pure and balanced with the property that every face $F \in \Delta$ of codimension at least 2 (including the empty face) has connected link. This includes the class of balanced triangulations of compact manifolds and, using the language of Goresky and MacPherson in [5], the more general class of balanced normal pseudomanifolds. Under these weaker assumptions, we show that

$$h_2 \geq \binom{d}{2} m(\Delta),$$

where $m(\Delta)$ denotes the minimal number of generators of $\pi_1(\Delta, *)$.

The paper is structured as follows. Section 2 contains all necessary definitions and background material. In Section 3, we outline a sequence of theorems in algebraic topology that are used to give a description of the fundamental group in terms of a finite set of generators and relations. In Section 4, we use the theorems in Section 3 to prove Theorem 4.5. This theorem gives the desired bound on $m(\Delta)$. In Section 5, after giving some definitions related to simplicial posets, we extend the topological results in Section 3 and the result of Theorem 4.5 to the class of simplicial posets.

2 Notation and Conventions

Throughout this paper, we assume that Δ is a $(d-1)$-dimensional simplicial complex on vertex set $V = \{v_1, \ldots, v_n\}$. We recall that the dimension of a face $F \in \Delta$ is $\dim F = |F| - 1$, and the dimension of Δ is $\dim \Delta = \max\{\dim F : F \in \Delta\}$. A simplicial complex is *pure* if all of its facets (maximal faces) have the same dimension. The *link* of a face $F \in \Delta$ is the subcomplex

$$\mathrm{lk}_\Delta F = \{G \in \Delta : F \cap G = \emptyset, F \cup G \in \Delta\}.$$

Similarly, the *closed star* of a face $F \in \Delta$ is the subcomplex

$$\overline{\mathrm{st}}_\Delta F = \{G \in \Delta : F \cup G \in \Delta\}.$$

The *geometric realization* of Δ, denoted by $|\Delta|$, is the union over all faces $F \in \Delta$ of the convex hull in $\mathbb{R}^n$ of $\{e_i : v_i \in F\}$ where $\{e_1, \ldots, e_n\}$ denotes the standard basis in $\mathbb{R}^n$. Given this geometric realization, we will make little distinction between the combinatorial object Δ and the topological space $|\Delta|$. For example, we will often abuse notation and write $H_i(\Delta; \mathbf{k})$ instead of the more cluttered $H_i(|\Delta|; \mathbf{k})$.

The *f-vector* of Δ is the vector $f = (f_{-1}, f_0, f_1, \ldots, f_{d-1})$ where f_i denotes the number of i-dimensional faces of Δ. By convention, we set $f_{-1} = 1$, corresponding to the empty face. If it is important to distinguish the simplicial complex Δ, we write $f(\Delta)$ for the f-vector of Δ, and $f_i(\Delta)$ for its f-numbers (i.e. the entries of its f-vector). Another important combinatorial invariant of Δ is the *h-vector* $h = (h_0, \ldots, h_d)$ where

$$h_i = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{d-i} f_{j-1}.$$

For us, it will be particularly important to study a certain class of complexes known as balanced simplicial complexes, which were introduced by Stanley in [11].

Definition 2.1 *A $(d-1)$-dimensional simplicial complex Δ is **balanced** if its 1-skeleton, considered as a graph, is d-colorable. That is to say there is a coloring $\kappa : V \to [d]$ such that for all $F \in \Delta$ and distinct $v, w \in F$, we have $\kappa(v) \neq \kappa(w)$. We assume that a balanced complex Δ comes equipped with such a coloring κ.*

The order complex of a rank-d graded poset is one example of a balanced simplicial complex. If Δ is a balanced complex and $S \subseteq [d]$, it is often important to study the *S-rank selected subcomplex* of Δ, which is defined as

$$\Delta_S = \{F \in \Delta : \kappa(F) \subseteq S\};$$

that is, for a fixed coloring κ, we define Δ_S to be the subcomplex of faces whose vertices are colored with colors from S. In [11] Stanley showed that

$$h_i(\Delta) = \sum_{|S|=i} h_i(\Delta_S). \tag{1}$$

3 The Edge-Path Group

In order to obtain a concrete description of $\pi_1(\Delta, *)$ that relies only on the structure of Δ as a simplicial complex, we introduce the *edge-path group* of Δ (see, for example, Seifert and Threlfall [9] or Spanier [10]). This will ultimately allow us to relate the combinatorial data of $f(\Delta)$ to the fundamental group of Δ.

An *edge* in Δ is an ordered pair of vertices (v, v') with $\{v, v'\} \in \Delta$. An *edge path* γ in Δ is a finite nonempty sequence $(v_0, v_1)(v_1, v_2) \cdots (v_{r-1}, v_r)$ of edges in Δ. We say that γ is an edge path from v_0 to v_r, or that γ starts at v_0 and ends at v_r. A *closed* edge path at v is an edge path γ such that $v_0 = v = v_r$.

We say that two edge paths γ and γ' are *simply equivalent* if there exist vertices v, v', v'' in Δ with $\{v, v', v''\} \in \Delta$ such that the unordered pair $\{\gamma, \gamma'\}$ is equal to one of the following unordered pairs:

- $\{(v, v''), (v, v')(v', v'')\}$,
- $\{\gamma_1(v, v''), \gamma_1(v, v')(v', v'')\}$ for some edge path γ_1 ending at v,

- $\{(v,v'')\gamma_2, (v,v')(v',v'')\gamma_2\}$ for some edge path γ_2 starting at v'',
- $\{\gamma_1(v,v'')\gamma_2, \gamma_1(v,v')(v',v'')\gamma_2\}$ for edge paths γ_1, γ_2 as above.

We note that the given vertices $v, v', v'' \in \Delta$ need not be distinct. For example, (v,v) is a valid edge (the edge that does not leave the vertex v), and we have the simple equivalence $(v,v')(v',v) \sim (v,v)$. We say that two edge paths γ and γ' are *equivalent*, and write $\gamma \sim \gamma'$, if there is a finite sequence of edge paths $\gamma_0, \gamma_1, \ldots, \gamma_s$ such that $\gamma = \gamma_0$, $\gamma' = \gamma_s$ and γ_i is simply equivalent to γ_{i+1} for $0 \leq i \leq s-1$. It is easy to check that this defines an equivalence relation on the collection of edge paths γ in Δ starting at v and ending at v'. Moreover, for two edge paths γ and γ' with the terminal vertex of γ equal to the initial vertex of γ', we can form their product edge path $\gamma\gamma'$ by concatenation.

Now we pick a base vertex $v_0 \in \Delta$. Let $E(\Delta, v_0)$ denote the set of equivalence classes of closed edge paths in Δ based at v_0. We multiply equivalence classes by $[\gamma] * [\gamma'] = [\gamma\gamma']$ to give $E(\Delta, v_0)$ a group structure called the *edge path group* of Δ based at v_0.

The Cellular Approximation Theorem ([10] VII.6.17) tells us that any path in Δ is homotopic to a path in the 1-skeleton of Δ. We use this fact to motivate the proof of the following theorem from Spanier.

Theorem 3.1 ([10] III.6.17) *If Δ is a simplicial complex and $v_0 \in \Delta$, then there is a natural isomorphism*

$$E(\Delta, v_0) \approx \pi_1(\Delta, v_0).$$

For a connected simplicial complex Δ we will also consider the group G, defined as follows. Let T be a spanning tree in the 1-skeleton of Δ. Since Δ is connected, such a spanning tree exists. We define G to be the free group generated by edges $(v, v') \in \Delta$ modulo the relations

[R1]. $(v,v') = 1$ if $(v,v') \in T$, and

[R2]. $(v,v')(v',v'') = (v,v'')$ if $\{v,v',v''\} \in \Delta$.

The following theorem will be crucial in our study of the fundamental group.

Theorem 3.2 ([10] III.7.3) *With the above notation,*

$$E(\Delta, v_0) \approx G.$$

We note for later use that this isomorphism is given by the map

$$\Phi : E(\Delta, v_0) \to G$$

that sends $[(v_0,v_1)(v_1,v_2)\cdots(v_{r-1},v_r)]_E \mapsto [(v_0,v_1)(v_1,v_2)\cdots(v_{r-1},v_r)]_G$. Here, $[-]_E$ and $[-]_G$ denote the equivalence classes of an edge path in $E(\Delta, v_0)$ and G, respectively. The inverse to this map is defined on the generators of G as follows. For $(v,v') \in \Delta$, there is an edge path γ from v_0 to v along T and an edge path γ' from v' to v_0 along T. Using these paths, we map $\Phi^{-1}[(v,v')]_G = [\gamma(v,v')\gamma']_E$.

4 The Fundamental Group and h-numbers

Our goal now is to use Theorem 3.2 to bound the minimal number of generators of $\pi_1(\Delta, *)$. For ease of notation, let $m(\Delta, *)$ denote the minimal number of generators of $\pi_1(\Delta, *)$. When the basepoint is understood or irrelevant (e.g. when Δ is connected) we will write $m(\Delta)$ in place of $m(\Delta, *)$. For the remainder of this section, we will be concerned with simplicial complexes Δ of dimension $(d-1)$ with the following properties:

(I). Δ is pure,

(II). Δ is balanced,

(III). $\mathrm{lk}_\Delta F$ is connected for all faces $F \in \Delta$ with $0 \leq |F| < d-1$.

In particular, property (III) implies that Δ is connected by taking F to be the empty face.

Since results on balanced simplicial complexes are well-suited to proofs by induction, we begin with the following observation.

Proposition 4.1 *Let Δ be a simplicial complex with $d \geq 2$ that satisfies properties (I)–(III). If $F \in \Delta$ is a face with $|F| < d-1$, then $\mathrm{lk}_\Delta F$ satisfies properties (I)–(III) as well.*

Proof: When $d = 2$, the result holds trivially since the only such face F is the empty face. When $d > 3$ and F is nonempty, it is sufficient to show that the result holds for a single vertex $v \in F$. Indeed, if we set $G = F \setminus \{v\}$, then $\mathrm{lk}_\Delta F = \mathrm{lk}_{\mathrm{lk}_\Delta v} G$ at which point we may appeal to induction on $|F|$.

We immediately see that $\mathrm{lk}_\Delta v$ inherits properties (I) and (II) from Δ. Finally, if $\sigma \in \mathrm{lk}_\Delta v$ is a face with $|\sigma| < d-2$, then $\mathrm{lk}_{\mathrm{lk}_\Delta v} \sigma = \mathrm{lk}_\Delta(\sigma \cup v)$ is connected by property (III). $\square$

Lemma 4.2 *Let Δ be a $(d-1)$-dimensional simplicial complex with $d \geq 2$ that satisfies properties (I) and (III). If F and F' are facets in Δ, then there is a chain of facets*

$$F = F_0, F_1, \ldots, F_m = F' \qquad (*)$$

such that $|F_i \cap F_{i+1}| = d-1$ for all i.

Remark 4.3 *We say that a pure simplicial complex satisfying property (*) is **strongly connected**.*

Proof: We proceed by induction on d. When $d = 2$, Δ is a connected graph, and such a chain of facets is a path from some vertex $v \in F$ to a vertex $v' \in F'$. We now assume that $d \geq 3$.

First, we note that the closed star of each face in Δ is strongly connected. Indeed, by induction the link (and hence the closed star $\overline{\mathrm{st}}_\Delta \sigma$) of each face $\sigma \in \Delta$ with $|\sigma| < d-1$

is strongly connected. On the other hand, if $\sigma \in \Delta$ is a face with $|\sigma| = d-1$, then every facet in $\overline{\mathrm{st}}_\Delta \sigma$ contains σ and so $\overline{\mathrm{st}}_\Delta \sigma$ is strongly connected as well. Finally, if σ is a facet, then $\overline{\mathrm{st}}_\Delta \sigma$ is strongly connected as it only contains a single facet.

It is also clear that if Δ' and Δ'' are strongly connected subcomplexes of Δ such that $\Delta' \cap \Delta''$ contains a facet, then $\Delta' \cup \Delta''$ is strongly connected as well. Finally, suppose $\Delta_0 \subseteq \Delta$ is a maximal strongly connected subcomplex of Δ. If $F \in \Delta_0$ is any face, then $\overline{\mathrm{st}}_\Delta F$ intersects Δ_0 in a facet. Since $\overline{\mathrm{st}}_\Delta F \cup \Delta_0$ is strongly connected and Δ_0 is maximal, we must have $\overline{\mathrm{st}}_\Delta F \subseteq \Delta_0$. Thus Δ_0 is a connected component of Δ. Since Δ is connected, $\Delta = \Delta_0$. $\square$

Lemma 4.4 *Let Δ be a $(d-1)$-dimensional simplicial complex with $d \geq 2$ that satisfies properties (I)–(III). For any $S \subseteq [d]$ with $|S| = 2$, the rank selected subcomplex Δ_S is connected.*

Proof: Say $S = \{c_1, c_2\}$. Pick vertices $v, v' \in \Delta_S$ and facets $F \ni v$, $F' \ni v'$. By Lemma 4.2, there is a chain of facets $F = F_1, \ldots, F_m = F'$ for which F_i intersects F_{i+1} in a codimension 1 face. We claim that a path from v to v' in Δ_S can be found in $\cup_{i=1}^m F_i$.

When $m = 1$, $\{v, v'\}$ is an edge in $F = F_1 = F'$. For $m > 1$, we examine the facet F_1. Without loss of generality, say $\kappa(v) = c_1$, and let $w \in F_1$ be the vertex with $\kappa(w) = c_2$. If $\{v, w\} \in F_1 \cap F_2$, then the facet F_1 in our chain is extraneous, and we could have taken $F = F_2$ instead. Inductively, we can find a path from v to v' in Δ_S that is contained in $\cup_{i=2}^m F_i$. On the other hand, if $v \notin F_2$, then we can find a path from w to v' in Δ_S that is contained in $\cup_{i=2}^m F_i$ by induction. Since $(v, w) \in \Delta_S$, this path extends to a path from v to v' in Δ_S. $\square$

Theorem 4.5 *Let Δ be a $(d-1)$-dimensional simplicial complex with $d \geq 2$ that satisfies properties (I)–(III), and $S \subseteq [d]$ with $|S| = 2$. If v, v' are vertices in Δ_S, then any edge path γ from v to v' in Δ is equivalent to an edge path from v to v' in Δ_S.*

Proof: When $d = 2$, $\Delta_S = \Delta$, and the result holds trivially, so we can assume $d \geq 3$.

We may write our edge path γ as a sequence

$$\gamma = (v_0, v_1)(v_1, v_2) \cdots (v_{r-1}, v_r)$$

where $v_0 = v$, $v_r = v'$, and $\{v_i, v_{i+1}\} \in \Delta$ for all i. We establish the claim by induction on r. When $r = 1$, the edge (v, v') is already an edge in Δ_S. Now we assume $r > 1$. If $v_1 \in \Delta_S$, the sequence $(v_1, v_2) \cdots (v_{r-1}, v_r)$ is equivalent to an edge path $\widetilde{\gamma}$ from v_1 to v' in Δ_S by our induction hypothesis on r. Hence γ is equivalent to $(v_0, v_1)\widetilde{\gamma}$.

On the other hand, suppose that $v_1 \notin \Delta_S$. Since $\kappa(v_1) \notin S$ and Δ is pure and balanced, there is a vertex $\widetilde{v} \in \Delta_S$ such that $\{v_1, v_2, \widetilde{v}\} \in \Delta$. By Proposition 4.1, $\mathrm{lk}_\Delta v_1$ is a simplicial complex of dimension at least 1 satisfying properties (I)–(III). Thus by Lemma 4.4, there is an edge path $\gamma' = (u_0, u_1) \cdots (u_{k-1}, u_k)$ such that $u_0 = v_0$, $u_k = \widetilde{v}$, and each edge $\{u_i, u_{i+1}\} \in (\mathrm{lk}_\Delta v_1)_S$. Since each edge $\{u_i, u_{i+1}\} \in \mathrm{lk}_\Delta v_1$, it follows that $\{u_i, u_{i+1}, v_1\} \in \Delta$ for all i.

We now use the fact that $(u,u')(u',u'') \sim (u,u'')$ for all $\{u,u',u''\} \in \Delta$ to see the following simple equivalences of edge paths.

$$\begin{aligned}
(v_0,v_1)(v_1,\widetilde{v}) &= (u_0,v_1)(v_1,\widetilde{v}) \\
&\sim (u_0,u_1)(u_1,v_1)(v_1,\widetilde{v}) \\
&\sim (u_0,u_1)(u_1,u_2)(u_2,v_1)(v_1,\widetilde{v}) \\
&\quad \ldots \\
&\sim (u_0,u_1)(u_1,u_2)\cdots(u_{k-2},u_{k-1})(u_{k-1},v_1)(v_1,\widetilde{v}) \\
&\sim (u_0,u_1)(u_1,u_2)\cdots(u_{k-2},u_{k-1})(u_{k-1},\widetilde{v}).
\end{aligned}$$

For convenience, we write $\gamma_1 = (u_0,u_1)(u_1,u_2)\cdots(u_{k-2},u_{k-1})(u_{k-1},\widetilde{v})$. Now we observe that $(v_0,v_1)(v_1,v_2) \sim (v_0,v_1)(v_1,\widetilde{v})(\widetilde{v},v_2)$ so that

$$\begin{aligned}
\gamma &= (v_0,v_1)(v_1,v_2)(v_2,v_3)\cdots(v_{r-1},v_r) \\
&\sim (v_0,v_1)(v_1,\widetilde{v})(\widetilde{v},v_2)(v_2,v_3)\cdots(v_{r-1},v_r) \\
&\sim \gamma_1(\widetilde{v},v_2)(v_2,v_3)\cdots(v_{r-1},v_r).
\end{aligned}$$

By induction on r, there is an edge path γ_2 in Δ_S from $\widetilde{v}$ to v_r that is equivalent to $(\widetilde{v},v_2)(v_2,v_3)\cdots(v_{r-1},v_r)$ so that $\gamma \sim \gamma_1\gamma_2$. Thus, indeed, γ is equivalent to an edge path in Δ_S. □

Setting $v = v' = v_0$, we have the following corollary.

Corollary 4.6 *If $v_0 \in \Delta_S$, every class in $E(\Delta, v_0)$ can be represented by a closed edge path in Δ_S.*

Now we have an explicit description of a smaller generating set of $\pi_1(\Delta, v_0)$.

Lemma 4.7 *Let Δ be a $(d-1)$-dimensional simplicial complex with $d \geq 2$ that satisfies properties (I)–(III). For a fixed $S \subseteq [d]$ with $|S| = 2$, the group G of Theorem 3.2 is generated by the edges (v,v') with $\{v,v'\} \in \Delta_S$.*

Proof: In order to use Theorem 3.2, we must choose some spanning tree T in the 1-skeleton of Δ. We will do this in a specific way. Since Δ_S is a connected graph, we can find a spanning tree $\widetilde{T}$ in Δ_S. Since Δ is connected, we can extend $\widetilde{T}$ to a spanning tree T in Δ so that $\widetilde{T} \subseteq T$.

By Corollary 4.6, each class in $E(\Delta, v_0)$ is represented by a closed edge path in Δ_S, and hence the isomorphism Φ of Theorem 3.2 maps $E(\Delta, v_0)$ into the subgroup of $H \subseteq G$ generated by edges $(v,v') \in \Delta_S$. Since Φ is surjective, we must have $H = G$. □

Corollary 4.8 *With Δ and S as in Lemma 4.7, we have*

$$m(\Delta) \leq h_2(\Delta_S).$$

Proof: Lemma 4.7 tells us that the $f_1(\Delta_S)$ edges in Δ_S generate the group G. Since our spanning tree T contains a spanning tree in Δ_S, $f_0(\Delta_S) - 1$ of these generators will be identified with the identity. Thus

$$m(\Delta) \leq f_1(\Delta_S) - f_0(\Delta_S) + 1 = h_2(\Delta_S).$$

□

While the proof of the above corollary requires specific information about the set S and a specific spanning tree $T \subset \Delta$, its result is purely combinatorial. Since Δ is connected, $\pi_1(\Delta, *)$ is independent of the basepoint, and so we can sum over all such sets $S \subset [d]$ with $|S| = 2$ to get

$$\begin{aligned} \binom{d}{2} m(\Delta) &\leq \sum_{|S|=2} h_2(\Delta_S) \\ &= h_2(\Delta) \qquad \text{by Equation (1).} \end{aligned}$$

This gives the following theorem.

Theorem 4.9 *Let Δ be a pure, balanced simplicial complex of dimension $(d-1)$ with the property that $\mathrm{lk}_\Delta F$ is connected for all faces $F \in \Delta$ with $|F| < d-1$. Then*

$$\binom{d}{2} m(\Delta) \leq h_2(\Delta).$$

5 Extensions and Further Questions

5.1 Simplicial Posets

We now generalize the results in Section 4 to the class of simplicial posets. A *simplicial poset* is a poset P with a least element $\hat{0}$ such that for any $x \in P \setminus \{\hat{0}\}$, the interval $[\hat{0}, x]$ is a Boolean algebra (see Björner [1] or Stanley [12]). That is to say that the interval $[\hat{0}, x]$ is isomorphic to the face poset of a simplex. Thus P is graded by $\mathrm{rk}(\sigma) = k + 1$ if $[\hat{0}, \sigma]$ is isomorphic to the face poset of a k-simplex. The face poset of a simplicial complex is a simplicial poset. Following [1], we see that every simplicial poset P has a geometric interpretation as the face poset of a regular CW-complex $|P|$ in which each cell is a simplex and each pair of simplices is joined along a possibly empty subcomplex of their boundaries. We call $|P|$ the *realization* of P. With this geometric picture in mind, we refer to elements of P as *faces* and work interchangeably between P and $|P|$. In particular, we refer to rank-1 elements of P as vertices and maximal rank elements of P as facets. As in the case of simplicial complexes, we say that the *dimension* of a face $\sigma \in P$ is $\mathrm{rk}(\sigma) - 1$, and the dimension of P is $d - 1$ where $d = \mathrm{rk}(P) = \max\{\mathrm{rk}(\sigma) : \sigma \in P\}$. We say that P is *pure* if each of its facets has the same rank. In addition, we can form the order complex $\Delta(\overline{P})$ of the poset $\overline{P} = P \setminus \{\hat{0}\}$, which gives a barycentric subdivision of $|P|$.

As with simplicial complexes, we define the link of a face $\tau \in P$ as

$$\mathrm{lk}_P\tau = \{\sigma \in P : \sigma \geq \tau\}.$$

It is worth noting that $\mathrm{lk}_P\tau$ is a simplicial poset whose minimal element is τ, but $\mathrm{lk}_P\tau$ is not necessarily a subcomplex of $|P|$. All hope is not lost, however, since for any saturated chain $F = \{\tau_0 < \tau_1 < \ldots < \tau_r = \tau\}$ in $(\hat{0}, \tau]$ we have $\mathrm{lk}_{\Delta(\overline{P})}(F) \cong \Delta(\overline{\mathrm{lk}_P(\tau)})$. Here we say F is *saturated* if each relation $\tau_i < \tau_{i+1}$ is a covering relation in P.

We are also concerned with balanced simplicial posets and strongly connected simplicial posets. Suppose P is a pure simplicial poset of dimension $(d-1)$, and let V denote the vertex set of P. We say that P is *balanced* if there is a coloring $\kappa : V \to [d]$ such that for each facet $\sigma \in P$ and distinct vertices $v, w < \sigma$, we have $\kappa(v) \neq \kappa(w)$. If $S \subseteq [d]$, we can form the S-rank selected poset of P, defined as

$$P_S = \{\sigma \in P : \kappa(\sigma) \subseteq S\} \text{ where } \kappa(\sigma) = \{\kappa(v) : v < \sigma, \mathrm{rk}(v) = 1\}.$$

We say that P is *strongly connected* if for all facets $\sigma, \sigma' \in P$ there is a chain of facets

$$\sigma = \sigma_0, \sigma_1, \ldots, \sigma_m = \sigma',$$

and faces τ_i of rank $d-1$ such that τ_i is covered by σ_i and σ_{i+1} for all $0 \leq i \leq m-1$. For simplicial complexes, the face τ_i is naturally $\sigma_i \cap \sigma_{i+1}$; however, for simplicial posets, the face τ_i is not necessarily unique.

As in Section 4, we are concerned with simplicial posets P of rank d satisfying the following three properties:

(i). P is pure,

(ii). P is balanced,

(iii). $\mathrm{lk}_P\sigma$ is connected for all faces $\sigma \in P$ with $0 \leq \mathrm{rk}(\sigma) < d-1$.

Our first task is to understand the fundamental group of a simplicial poset by constructing an analogue of the edge-path group of a simplicial complex. We have to be careful because there can be several edges connecting a given pair of vertices. An *edge* in P is an oriented rank-2 element $e \in P$ with an initial vertex, denoted $\mathrm{init}(e)$, and a terminal vertex, denoted $\mathrm{term}(e)$. If e is an edge, we let e^{-1} denote its *inverse edge*, that is, we interchange the initial and terminal vertices of e, reversing the orientation of e. We note that the initial and terminal vertices of e are distinct since $[\hat{0}, e]$ is a Boolean algebra. We also allow for the degenerate edge $e = (v, v)$ for any vertex $v \in P$. An *edge path* γ in P is a finite nonempty sequence $e_0 e_1 \cdots e_r$ of edges in P such that $\mathrm{term}(e_i) = \mathrm{init}(e_{i+1})$ for all $0 \leq i \leq r-1$. A *closed* edge path at v is an edge path γ such that $\mathrm{init}(e_0) = v = \mathrm{term}(e_r)$. Given edge paths γ from v to v' and γ' from v' to v'', we can form their product edge path $\gamma\gamma'$ from v to v'' by concatenation.

Suppose $\sigma \in P$ is a rank-3 face with (distinct) vertices v, v' and v'' and edges e, e' and e'' with $\mathrm{init}(e) = v = \mathrm{init}(e'')$, $\mathrm{init}(e') = v' = \mathrm{term}(v)$ and $\mathrm{term}(e'') = v'' = \mathrm{term}(e')$. Analogously to Section 3, we say that two edge paths γ and γ' are *simply equivalent* if the unordered pair $\{\gamma, \gamma'\}$ is equal to one of the following unordered pairs:

- $\{e'', ee'\}$ or $\{(v,v), ee^{-1}\}$;
- $\{\gamma_1 e'', \gamma_1 ee'\}$ or $\{\gamma_1, \gamma_1 ee^{-1}\}$ for some edge path γ_1 ending at v;
- $\{e''\gamma_2, ee'\gamma_2\}$ or $\{\gamma_2, (e')^{-1}e'\gamma_2\}$ for some edge path γ_2 starting at v'';
- $\{\gamma_1 e''\gamma_2, \gamma_1 ee'\gamma_2\}$ for edge paths γ_1, γ_2 as above.

We say that two edge paths γ and γ' are equivalent and write $\gamma \sim \gamma'$ if there is a finite sequence of edge paths $\gamma = \gamma_0, \ldots, \gamma_s = \gamma'$ such that γ_i is simply equivalent to γ_{i+1} for all i. As in the case of simplicial complexes, this forms an equivalence relation on the collection of edge paths in P with initial vertex v and terminal vertex v'. We pick a base vertex v_0 and let $\widetilde{E}(P, v_0)$ denote the collection of equivalence classes of closed edge paths in P at v_0. We give $\widetilde{E}(P, v_0)$ a group structure by loop multiplication, and the resulting group is called the *edge path group* of P based at v_0.

Now we ask if the groups $\pi_1(P, v_0)$ and $\widetilde{E}(P, v_0)$ are isomorphic. As topological spaces, $|P|$ and $\Delta(\overline{P})$ are homeomorphic and so their fundamental groups are isomorphic. The latter space is a simplicial complex, and so we know that $E(\Delta(\overline{P}), v_0) \approx \pi_1(P, v_0)$. The following theorem will show that indeed $\pi_1(P, v_0) \approx \widetilde{E}(P, v_0)$.

Theorem 5.1 *Let P be a simplicial poset of rank d satisfying properties (i) and (iii). If v_0 is a vertex in P, then*

$$\widetilde{E}(P, v_0) \approx E(\Delta(\overline{P}), v_0).$$

Proof: Given an edge $e \in P$ with initial vertex v and terminal vertex v', we define an edge path in $\Delta(\overline{P})$ from v to v' by barycentric subdivision as $\mathrm{Sd}(e) = (v, e)(e, v')$. We define $\Phi : \widetilde{E}(P, v_0) \to E(\Delta(\overline{P}), v_0)$ by

$$\Phi([e_0 e_1 \cdots e_r]_{\widetilde{E}}) = [\mathrm{Sd}(e_0)\mathrm{Sd}(e_1) \cdots \mathrm{Sd}(e_r)]_E.$$

It is easy to check that Φ is well-defined, as it respects simple equivalences.

We now claim that $\Delta(\overline{P})$ in fact satisfies properties (I)–(III) of Section 4. Since $\Delta(\overline{P})$ is the order complex of a pure poset, it is pure and balanced. Indeed, the vertices in $\Delta(\overline{P})$ are elements $\sigma \in \overline{P}$, colored by their rank in P. Finally, for a saturated chain $F = \{\tau_1 < \tau_2 < \ldots < \tau_r = \tau\}$ in $\overline{P}$ for which $r < d-1$, we see that $\mathrm{lk}_{\Delta(\overline{P})} F \cong \Delta(\overline{\mathrm{lk}_P(\tau)})$ is connected since $\mathrm{lk}_P \tau$ is connected. By Proposition 3.3 in [4], we need only consider saturated chains here. By Theorem 4.7, it follows that any class in $E(\Delta(\overline{P}), v_0)$ can be represented by a closed edge path in $(\Delta(\overline{P}))_{\{1,2\}}$. In particular, we can represent any class in $E(\Delta(\overline{P}), v_0)$ by an edge path $\gamma = \mathrm{Sd}(e_0)\mathrm{Sd}(e_1) \cdots \mathrm{Sd}(e_r)$ for some edge path $e_0 e_1 \cdots e_r$ in P. This gives a well-defined inverse to Φ. $\square$

With Theorem 5.1 and the above definitions, the proofs of Proposition 4.1, Lemmas 4.2 and 4.4, and Theorem 4.5 carry over almost verbatim to the context of simplicial posets and can be used to prove the following Lemma.

Lemma 5.2 *Let P be a simplicial poset of rank $d \geq 2$ that satisfies properties (i)–(iii).*

a. *If $\sigma \in P$ is a face and $\mathrm{rk}(\sigma) < d-1$, then $\mathrm{lk}_P\sigma$ satisfies properties (i)–(iii) as well.*

b. *P is strongly connected.*

c. *For any $S \subseteq [d]$ with $|S| = 2$, the rank selected subcomplex P_S is connected.*

d. *If v and v' are vertices in P_S, then any edge path γ from v to v' in P is equivalent to an edge path from v to v' in P_S.*

As in Section 4, part (d) of this Lemma implies the following generalization of Theorem 4.9.

Theorem 5.3 *Let P be a pure, balanced simplicial poset of rank d with the property that $\mathrm{lk}_P\sigma$ is connected for each face $\sigma \in P$ with $\mathrm{rk}(\sigma) < d-1$. Then*

$$\binom{d}{2} m(P) \leq h_2(P).$$

5.2 How Tight are the Bounds?

We now turn our attention to a number of examples to determine if the bounds given by Theorems 4.9 and 5.3 are tight. We begin by studying a family of simplicial posets constructed by Novik and Swartz in [8]. Lemma 7.6 in [8] constructs a simplicial poset $X(1,d)$ of dimension $(d-1)$ satisfying properties (i)–(iii) whose geometric realization is a $(d-2)$-disk bundle over $\mathbb{S}^1$ and $h_2(X(1,d)) = \binom{d}{2}$. As X is a bundle over $\mathbb{S}^1$ with contractible fiber, we have $\pi_1(X(1,d),*) \approx \mathbb{Z}$ so that $m(X(1,d)) = 1$. This construction shows that the bound in Theorem 5.3 is tight. Moreover, taking connected sums of r copies of $X(1,d)$ (when $d \geq 4$) gives a simplicial poset P whose fundamental group is isomorphic to $\mathbb{Z}^r$ and $h_2(P) = r\binom{d}{2}$. We do not know, however, if the bound in Theorem 5.3 is tight when $\pi_1(P,*)$ is either non-free or non-Abelian. We would also like to know if Theorem 5.3 holds if we drop the condition that P is balanced.

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