Some New Optimum Golomb Rectangles

James B. Shearer
IBM Research Division
T.J. Watson Research Center
P.O. Box 218
Yorktown Heights, N.Y. 10598
jbs@watson.ibm.com

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Abstract: We give some new optimum Golomb rectangles found by computer search.

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In [2] Robinson defined a Golomb rectangle as an $N \times M$ array of ones and zeros such that the two-dimensional autocorrelation has three values: $0, 1$ and K, where K is the number of ones in the array. This means that the positions of the ones in any nonzero integral translation of the rectangle will overlap with the positions of the ones in the original position of the rectangle in at most one place. Equivalently, the differences between the positions of every pair of ones in the rectangle, considered as vectors, are distinct. See also [1]. Let $G(N, M)$ be the maximum number of ones that can be present in an $N \times M$ Golomb rectangle. For example $G(2, 2) = 3$. Robinson defined an optimum Golomb rectangle to be one containing $G(N, M)$ ones. We prefer to add the conditions $G(N, M) > G(N-1, M)$ and $G(N, M) > G(N, M-1)$.

In table 1 we give a number of new optimum Golomb rectangles found by computer search. In most cases they are far from unique. Note there exists a 2×18 rectangle with 9 ones. The rectangle in Robinson's table V is 2×20 an apparent misprint.

A brief description of the computer program used to find these rectangles follows. Recall that a Golomb ruler is a set of integers $a_1 < a_2 < \cdots < a_k$ for which the $\binom{k}{2}$ differences $\{a_j - a_i | 1 \leq i < j \leq k\}$ are distinct. We will use the following easy lemma.

Lemma 1: *$N \times M$ Golomb rectangles with K ones correspond $1-1$ with K element Golomb rulers with elements chosen from the set $\{i + (2N-1)(j-1) | 1 \leq i \leq N, 1 \leq j \leq M\}$.*

Proof: Let $\{(b_i, c_i) | 1 \leq b_i \leq N, 1 \leq c_i \leq M, 1 \leq i \leq K\}$ be a set of K positions in a $N \times M$ rectangle. We claim this set consists of the positions of the ones in a $N \times M$ Golomb rectangle with K ones iff the set $\{a_i = b_i + (2N-1)(c_i - 1) | 1 \leq i \leq K\}$, is a

Golomb ruler with elements chosen from the set $\{i + (2N-1)(j-1) | 1 \leq i \leq N, 1 \leq j \leq M\}$. For suppose $(b_{i_1}, c_{i_1}) - (b_{i_2}, c_{i_2}) = (b_{i_3}, c_{i_3}) - (b_{i_4}, c_{i_4})$. Then

$$\begin{aligned} a_{i_1} - a_{i_2} &= (b_{i_1} + (2N-1)(c_{i_1} - 1)) - (b_{i_2} + (2N-1)(c_{i_2} - 1)) \\ &= (b_{i_1} - b_{i_2}) + (2N-1)(c_{i_1} - c_{i_2}) \\ &= (b_{i_3} - b_{i_4}) + (2N-1)(c_{i_3} - c_{i_4}) \\ &= (b_{i_3} + (2N-1)(c_{i_3} - 1)) - (b_{i_4} + (2N-1)(c_{i_4} - 1)) \\ &= a_{i_3} - a_{i_4}. \end{aligned}$$

Conversely suppose $(a_{i_1} - a_{i_2}) = (a_{i_3} - a_{i_4})$. Then

$$\begin{aligned} (b_{i_1} &+ (2N-1)(c_{i_1} - 1)) - (b_{i_2} + (2N-1)(c_{i_2} - 1)) \\ &= (b_{i_3} + (2N-1)(c_{i_3} - 1)) - (b_{i_4} + (2N-1)(c_{i_4} - 1)). \end{aligned}$$

It follows that

$$(b_{i_1} - b_{i_2}) - (b_{i_3} - b_{i_4}) + (2N-1)((c_{i_1} - c_{i_2}) - (c_{i_3} - c_{i_4})) = 0.$$

Now $1 \leq b_{i_1}, b_{i_2}, b_{i_3}, b_{i_4} \leq N$. It follows that

$$-(2N-1) < (b_{i_1} - b_{i_2}) + (b_{i_3} - b_{i_4}) < (2N-1).$$

Therefore we must have $(b_{i_1} - b_{i_2}) - (b_{i_3} - b_{i_4}) = 0$ and $(c_{i_1} - c_{i_2}) - (c_{i_3} - c_{i_4}) = 0$. Hence $(b_{i_1}, c_{i_1}) - (b_{i_2}, c_{i_2}) = (b_{i_3}, c_{i_3}) - (b_{i_4}, c_{i_4})$. This suffices to prove the claim and the lemma. □

Lemma 1 means that searches for Golomb rectangles can be performed with a modified version of the author's Golomb ruler search program (see [3]). This program performs a straightforward depth first backtrack search. It builds up a Golomb ruler by picking $a_1, a_2, \ldots, a_k$ in order (a_j being picked at level j of the search tree). At each node of the search tree the program keeps track of which differences have been used (i.e. are formed by pairs of the elements which have already been picked) and of which integers can be adjoined to the current ruler without violating the distinct difference condition. The sons of a node at level j are formed by setting a_{j+1} to each of the elements in the level j eligibility list in turn. The search tree is pruned (i.e. the program backtracks) when too few integers remain eligible to allow completion of the ruler, when not enough small differences remain unused to allow completion of the ruler or when allowed by symmetry conditions. (Symmetry conditions allow search trees to be pruned because we need only generate one member of each symmetry class of solutions. Clever use of symmetry can produce dramatic improvements in running times.) The following symmetry conditions were used in the Golomb rectangle program. Assume at least half the ones are in the left half of the rectangle (flip left and right if necessary). Assume the top half of the first column contains a one (clearly in an optimum rectangle the first column must contain a one, flip top and bottom if necessary). These conditions are probably not the best (in particular the second condition could assume the top half of the first column contains at least half the ones in the first column) but they were simple to implement. One can choose to search for $N \times M$ rectangles or for $M \times N$ rectangles. The main runs were done with $N \le M$ although that is not always the best choice.

The entire program amounts to about 100 lines of VS Fortran code. As is often the case for this sort of program running times (on a 3090 IBM mainframe) increased rapidly with the problem size. For example showing $G(7,11) < 14$ required 640 seconds of cpu time but showing $G(9,12) < 16$ required 44000 seconds of cpu time. On average each node visited in the search tree required about 6 microseconds of cpu time.

A skeptical reader might ask why he should believe the program is correct. Checking that the rectangles found are in fact Golomb rectangles is fairly simple. An independent program checked this for the rectangles in table 1. It is possible (although tedious) to do these checks by hand. The validity of the assertion that the rectangles in table 1 are optimum (i.e. that Golomb rectangles with better parameters do not exist) is more problematic. However, the following factors give the author confidence that his search program has not missed any superior rectangles. The program successfully reproduced (with the exception noted above) the results Robinson obtained with a completely different program. The program is reasonably small (100 lines of code). Additionally, the basic algorithm and much of the code is the same as for the author's Golomb ruler program. The results obtained by the Golomb ruler program are in agreement with those obtained by other researchers using independent programs. Of course, additional checking is always possible. For example, it would be nice to do searches on $N \times M$ rectangles and on $M \times N$ rectangles as the search trees will be completely different (when $N \neq M$). However, as noted above this was not done for the large cases.

Table 1

G(2,18)=9	110000010000000010
	100101000000010001
G(2,29)=11	11000000100000000000010100010
	10010000000010000100000000001
G(2,35)=12	10100000000000110000000010000000100
	10010001000000000000000100001000001
G(2,43)=13	1100000000000010100000000000000000100010010

1000001000010000000000010000000010000000001

G(2,52)=14 1100000001000000000000001000000100000000000001000101
1001000000000100000000000000010000000000010000100000

G(2,59)=15 11000000001000000100000000000000000000001000001000000010001
10010000000000000000001010000000000100000000000000100001000

G(3,18)=11 100100000000010100
100000000100000001
010000011000100000

G(3,21)=12 100001000001000000011
100000000000000010100
001000100100000000010

G(3,26)=13 10001000000001000000001101
01000000100000000000000001
10000010000000010000100000

G(3,31)=14 1000010001000000010000001000001
1000000000000000000010010000000
0110000000000100000000000000101

G(3,37)=15 1000010000000001000000000000000000101
1001000000000000000000000010001000000
0010000000100000010000010000000000110

G(4,13)=11 1001000000011
1000000000000
0100001010000
1000100000100

G(4,16)=12 1001000000001010
1000000000000001
0000000100010000
1000011000000100

G(4,19)=13 1010000100000000100
1000000000100000001
0000000001000001000

	0100110000000000100
G(4,23)=14	11000001000100001000000
	10000000000000000010010
	00000010000000000001000
	10100000000000100000001
G(4,27)=15	100000000100010100000001000
	100100000000000000000000100
	000000011000000000010000001
	100001000000000000000000010
G(5,11)=11	11000000001
	10000000000
	00000010010
	10100001000
	00001000001
G(5,12)=12	110001000001
	100000000000
	000000010010
	100000001000
	001010000001
G(5,15)=13	100001000001001
	100000000000010
	000000000010001
	001010000000000
	100000011000000
G(5,18)=14	100000001000100001
	100000010010000000
	000010000000000100
	100000000000000000
	010100000000000011
G(5,20)=15	10000000010000000100
	00001000100000000011
	10100000000000000000
	00000000000001001000
	01000010000001000001

G(6,8)=11
```
11000100
00000001
00000001
01010000
10000010
01001000
```

G(6,10)=12
```
1100000001
1000000100
0001000001
0000100000
0000000010
1010010000
```

G(6,12)=13
```
100010000110
000010000001
100000000000
000101000000
001000000000
100000001001
```

G(6,14)=14
```
10010000000101
00000110000010
00100000000000
10000000000010
10000000000000
00001000100001
```

G(6,17)=15
```
11000000000000101
10000001000000000
00000000010000010
00010000000001000
10000000000000000
01000100001001000
```

G(7,7)=11
```
1000100
0100001
1000001
0000000
0001000
```

	0000001
	0110100
G(7,9)=12	110000101
	100000000
	000100000
	100000000
	000000001
	010010000
	001000100
G(7,10)=13	1100000010
	0000010000
	0000100100
	1000000000
	0000000001
	1000000001
	0101000100
G(7,12)=14	001010000001
	000000001000
	100000000001
	110000000000
	000000000010
	000100100000
	010001000001
G(7,15)=15	100000001000011
	100000010001000
	000000000000000
	101000000000000
	000010000000001
	000001000000000
	010000000010010
G(8,8)=12	11001010
	10000000
	00000001
	10000000
	00010000
	00000100

	00100000
	10000001
G(8,11)=14	10000001001
	10001000010
	00000000000
	00001000000
	00000000010
	11000000000
	00000001000
	00100000101
G(8,13)=15	1000010001001
	1000000000000
	0000000001000
	0100000000000
	1000000000010
	0000000000001
	0000011000000
	0010000010100
G(9,9)=13	110000001
	100000000
	000010000
	100000000
	000000100
	010000010
	000000000
	100000000
	000101001
G(9,10)=14	1000001000
	1010000001
	0000000000
	1001000000
	0000000100
	0000000000
	0000000001
	1000000010
	0100011000

G(9,12)=15

```
110000010001
100000000000
000000001010
100000000000
000010000000
000001000000
000000000000
100000000100
000100100001
```

G(10,10)=15

```
1000010010
0100000000
0000000001
1000100000
0010000000
0000000000
0000000001
0000000001
0101000000
1000001100
```

References

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