On the shadow of squashed families of k-sets

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Abstract: The *shadow* of a collection $\mathcal{A}$ of k-sets is defined as the collection of the $(k-1)$-sets which are contained in at least one k-set of $\mathcal{A}$. Given $|\mathcal{A}|$, the size of the shadow is minimum when $\mathcal{A}$ is the family of the first k-sets in *squashed order* (by definition, a k-set A is smaller than a k-set B in the squashed order if the largest element of the symmetric difference of A and B is in B). We give a tight upper bound and an asymptotic formula for the size of the shadow of squashed families of k-sets.

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1 Introduction

A hypergraph is a collection of subsets (called *edges*) of a finite set S. If a hypergraph $\mathcal{A}$ is such that $A_i, A_j \in \mathcal{A}$ implies $A_i \not\subseteq A_j$, then $\mathcal{A}$ is called an *antichain*. In other words $\mathcal{A}$ is a collection of incomparable sets. Antichains are also known under the names *simple hypergraph* or *clutter*.

The *shadow* of a collection $\mathcal{A}$ of k-sets (set of size k) is defined as the collection of the $(k-1)$-sets which are contained in at least one k-set of $\mathcal{A}$. The shadow of $\mathcal{A}$ is denoted by $\Delta(\mathcal{A})$.

In the following we assume that S is a set of integers. The *squashed order* is defined on the the set of k-sets. Given two k-sets A and B, we say that A is smaller than B in the squashed order if the largest element of the symmetric difference of A and B is in B. The first 3-sets in the squashed order are

$$\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,5\},\{1,3,5\},\cdots$$

Let $F_k(x)$ denote the family of the first x k-sets in the squashed order. We will prove the following.

Theorem 1 *If* $x \leq \binom{n}{k}$ *then*

$$|\Delta(F_k(x))| \leq kx - x(x-1) \times q_{n,k} \text{ where } q_{n,k} = \frac{k}{\binom{n}{k}-1} \times \frac{n-k}{n-k+1}$$

Equality holds when $x = 0$ *or* $x = \binom{n}{k}$.

Theorem 2 *When* $x \to \infty$, $|\Delta(F_k(x))| \sim \frac{k}{\sqrt[k]{k!}}\, x^{1-\frac{1}{k}}$

The squashed order is very useful when dealing with the size of the shadow of a collection of k-sets. The main result is that if you want to minimize the shadow then you have to take the first sets in the squashed order. This is a consequence of the Kruskal-Katona theorem [4, 3]. Before stating their theorem, recall the definition of the l-binomial representation of a number.

Theorem 3 *Given positive integers* x *and* l, *there exists a unique representation of* x *(called the* l*-binomial representation) in the form*

$$x = \binom{x_l}{l} + \binom{x_{l-1}}{l-1} + \cdots + \binom{x_t}{t}$$

where $x_l > x_{l-1} > \cdots > x_t \geq t$.

See [1] or [2] for more details.

Theorem 4 (Kruskal-Katona) *Let* $\mathcal{A}$ *be a collection of* l*-sets, and suppose that the* l*-binomial representation of* $|\mathcal{A}|$ *is*

$$|\mathcal{A}| = \binom{x_l}{l} + \binom{x_{l-1}}{l-1} + \cdots + \binom{x_t}{t}$$

where $x_l > x_{l-1} > \cdots > x_t \geq t$. *Then*

$$|\Delta(\mathcal{A})| \geq \binom{x_l}{l-1} + \binom{x_{l-1}}{l-2} + \cdots + \binom{x_t}{t-1}$$

There is equality when $\mathcal{A}$ *is the collection of the first* $|\mathcal{A}|$ l*-sets in the squashed order.*

Though the above theorem gives the exact values of the shadow when the antichain is squashed, it is awkward to manipulate. Because of this, theorem 1 may be more useful for some problems such as those of construction of completely separating systems (see [5], for example).

2 Proofs

2.1 Proof of theorem 1

We need a few lemmas before proving theorem 1.

Lemma 1 *The inequality of theorem 1 holds when $n \leq 6$.*

Proof of lemma 1: Done by computer check. Can be done by hand too. □

Lemma 2 *The inequality of theorem 1 holds when $k = 1$.*

Proof of lemma 2: We have $q_{n,1} = 1/n$. So the inequality to prove is;

$$|\Delta(F_1(x))| \leq x - x(x-1) \times \frac{1}{n}$$

The right hand side of the inequality can be rewritten as

$$\frac{x}{n}(n - x + 1)$$

As $|\Delta(F_1(x))|$ is equal to 1 (because $\Delta(F_1(x)) = \{\emptyset\}$), all we have to prove is that

$$\frac{n}{x} \leq n - x + 1$$

i.e.

$$x^2 - (n+1)x + n \leq 0$$

The zeroes of this polynomial are 1 and n. This implies that for x in the interval $[1, \binom{n}{1}]$, the inequality holds.□

Lemma 3 *The inequality of theorem 1 holds when $k = n - 1$.*

Proof of lemma 3: We have $q_{n,n-1} = \frac{1}{2}$. So the inequality to prove is;

$$|\Delta(F_{n-1}(x))| \leq x[n - 1 - \frac{x-1}{2}]$$

The value of x is in the range $[1, n]$. If $x = n$ then both sides of the inequality are equal to $\binom{n}{2}$. Now, assume that x is in the range $[1, n-1]$. The $(n-1)$-binomial representation of x is:

$$x = \binom{x_{n-1}}{n-1} + \binom{x_{n-2}}{n-2} + \cdots + \binom{x_t}{t}$$

where $x_{n-1} > x_{n-2} > \cdots > x_t \geq t$. As $x \leq n-1$, we have $x_{n-1} = n-1$. And, therefore $x_{n-i} = n-i$ for all $i \in [1, n-t]$. Hence $x = n-t$. Because of the $(n-1)$-binomial representation of x, the size of the shadow of $F_{n-1}(x)$ is given by the formula:

$$|\Delta(F_{n-1}(x))| = \binom{n-1}{n-2} + \binom{n-2}{n-3} + \cdots + \binom{t}{t-1}$$

i.e.

$$|\Delta(F_{n-1}(x))| = \binom{n-1}{1} + \binom{n-2}{1} + \cdots + \binom{t}{1}$$

Finally, we have

$$|\Delta(F_{n-1}(x))| = \frac{n(n-1)}{2} - \frac{t(t-1)}{2} = \frac{1}{2}(n-t)(n+t-1)$$

As $x = n-t$. By substituting $n-x$ to t in the right hand side, we find that

$$|\Delta(F_{n-1}(x))| = x[n-1-\frac{x-1}{2}]$$

Which is what we wanted to prove. □

Lemma 4 *The inequality of theorem 1 holds when $k = n$.*

Proof of lemma 4: obvious. □

Lemma 5 *The function $n \longmapsto q_{n,k}$ is decreasing on $[k+1, \infty]$.*

Proof of lemma 5:

$$q_{n+1,k} - q_{n,k} = \frac{k}{\binom{n+1}{k} - 1} \times \frac{n+1-k}{n+2-k} - \frac{k}{\binom{n}{k} - 1} \times \frac{n-k}{n+1-k}$$

which has the same sign as

$$k(n+1-k)^2 \times (\binom{n}{k} - 1) - k(n-k)(n+2-k) \times (\binom{n+1}{k} - 1)$$

which has the same sign as

$$(n+1-k)^2 \times (\binom{n}{k} - 1) - (n-k)(n+2-k) \times (\binom{n}{k} + \binom{n}{k-1} - 1)$$

$$= \binom{n}{k} - 1 - (n-k)(n-k+2) \times \binom{n}{k-1}$$

$$= \binom{n}{k} - 1 - \binom{n}{k} \frac{k(n-k)(n-k+2)}{n-k+1} < 0$$

□

To prove theorem 1, we use a double induction on k then n. The case $k = 1$ has been considered in lemma 2. If $x \le \binom{n-1}{k}$ then as the function $n \longmapsto q_{n,k}$ is decreasing, using the induction hypothesis we are done. Thus, we can assume that $x = \binom{n-1}{k} + j$ with $j \le \binom{n-1}{k-1}$. It is a classical result (see [2] or [1]) that

$$|\Delta(F_k(x))| = \binom{n-1}{k-1} + |\Delta(F_{k-1}(j))|$$

By induction hypothesis

$$|\Delta(F_{k-1}(j))| \le j(k-1) - j(j-1) \times q_{n-1,k-1}$$

Combining these inequalities we get:

Claim 1

$$|\Delta(F_k(x))| \le \binom{n-1}{k-1} + j(k-1) - j(j-1)q_{n-1,k-1}$$

If theorem 1 is true then $|\Delta(F_k(x))| \le kx - x(x-1) \times q_{n,k}$ with equality when $j = \binom{n-1}{k-1}$. Hence, to prove theorem 1 it is sufficient to prove that we have:

$$\binom{n-1}{k-1} + j(k-1) - j(j-1)q_{n-1,k-1} \le kx - x(x-1) \times q_{n,k} \qquad (\star)$$

As $k\binom{n-1}{k} = (n-k)\binom{n-1}{k-1}$ and $x = \binom{n-1}{k} + j$, $(\star)$ is equivalent to

$$x(x-1)q_{n,k} \le (n-k-1)\binom{n-1}{k-1} + j + j(j-1)q_{n-1,k-1}$$

To simplify the expressions we introduce some new variables. Let $q_0 = q_{n,k}$ and $q_1 = q_{n-1,k-1}$. Let $y = \binom{n-1}{k-1}$. We will use later the facts that $\binom{n}{k} = \frac{n}{k}y$, and that $\binom{n-1}{k} = \frac{n-k}{k}y$. With this notation $(\star)$ is equivalent to

$$x(x-1)q_0 \le (n-k-1)y + j(j-1)q_1 + j$$

As $x = \frac{n-k}{k}y + j$, we have

$$x(x-1)q_0 = q_0 j^2 + q_0(2\frac{n-k}{k}y - 1)j + q_0(\frac{n-k}{k}y)^2 - \frac{n-k}{k}yq_0$$

Therefore, $(\star)$ is equivalent to

$$0 \leq j^2(q_1 - q_0) - j(-1 + q_1 - q_0 + 2\frac{n-k}{k}yq_0) + (n-k-1)y - q_0(\frac{n-k}{k}y)^2 + \frac{n-k}{k}yq_0$$

Finally we have,

Claim 2 $(\star)$ *is equivalent to*

$$0 \leq j^2(q_1 - q_0) - j(-1 + q_1 - q_0 + 2\frac{n-k}{k}yq_0) + (n-k-1)y + q_0\frac{n-k}{k}y(1 - \frac{n-k}{k}y)$$

Let $\Phi(j) = j^2(q_1 - q_0) - j(-1 + q_1 - q_0 + 2\frac{n-k}{k}yq_0) + (n-k-1)y + q_0\frac{n-k}{k}y(1 - \frac{n-k}{k}y)$. We will prove that this polynomial in j is positive on the interval $[0, \binom{n-1}{k-1}]$, by proving that $\Phi'' \geq 0$, $\Phi'(y) \leq 0$ and $\Phi(y) = 0$. Let's prove that $\Phi'' = q_1 - q_0$ is positive.

$$q_0 - q_1 = [\frac{k}{\binom{n}{k} - 1} - \frac{k-1}{\binom{n-1}{k-1} - 1}]\frac{n-k}{n-k+1}$$

i.e.

$$q_0 - q_1 = [\frac{k}{\frac{n}{k}y - 1} - \frac{k-1}{y-1}]\frac{n-k}{n-k+1}$$

The sign of $q_0 - q_1$ is the same as the sign of

$$k(y-1) - (k-1)(\frac{n}{k}y - 1) = ky - k - ny + k + \frac{n}{k}y - 1 = y(k - n + \frac{n}{k}) - 1$$

Notice that $k - n + \frac{n}{k}$ is negative because $k \in [2, n-2]$. Indeed, the sign of $k - n + \frac{n}{k}$ is the same as the sign of $k^2 - nk + n$. It's easy to check that this polynomial in k is negative on $[2, n-1]$ as soon as $n \geq 5$. Hence, $q_0 - q_1$ is negative.

Let's check that $(\star)$ becomes an equality when j takes the value of $y = \binom{n-1}{k-1}$. By substituting $\binom{n}{k}$ to x in the right hand side of the inequality of theorem 1, we get $\binom{n}{k-1}$ as expected. By substituting $y = \binom{n-1}{k-1}$ to j in the inequality of claim 1, we obtain also $\binom{n}{k-1}$ (use the induction hypothesis that $|\Delta(F_{k-1}(y))| = \binom{n-1}{k-2}$). This implies that $\binom{n-1}{k-1}$ is a root of the polynomial $\Phi(j)$.

To finish the proof of theorem 1 we will prove that $y = \binom{n-1}{k-1}$ is the smaller root of $\Phi(j)$, by showing that at that point the derivative of $\Phi(j)$ is negative. This will sufficient as we already know that the second derivative is positive. We have

$$\Phi'(y) = 2y(q_1 - q_0) - (-1 + q_1 - q_0 + 2\frac{n-k}{k}yq_0)$$

$\Phi'(y) \leq 0$ is equivalent to

$$2y(q_1 - q_0) \leq -1 + q_1 - q_0 + 2\frac{n-k}{k} y q_0$$

which is equivalent to

$$2y(\frac{k-1}{y-1} - \frac{k}{\frac{n}{k}y - 1})\frac{n-k}{n-k+1} \leq -1 + q_1 - q_0 + 2\frac{n-k}{k} y \frac{k}{\frac{n}{k}y-1}\frac{n-k}{n-k+1}$$

which is equivalent to

$$2y(\frac{k-1}{y-1} - \frac{k^2}{ny-k}) + \frac{n-k+1}{n-k} \leq (q_1 - q_0)\frac{n-k+1}{n-k} + \frac{2(n-k)ky}{ny-k}$$

i.e.

$$\frac{2y(k-1)}{y-1} + \frac{n-k+1}{n-k} \leq (q_1 - q_0)\frac{n-k+1}{n-k} + \frac{2nky}{ny-k}$$

It is sufficient to prove that

$$\frac{2y(k-1)}{y-1} + \frac{3}{2} \leq \frac{2nky}{ny-k}$$

The left hand side is equal to $2k - \frac{1}{2} + \frac{2(k-1)}{y-1}$. The right hand side is equal to $2k + \frac{2k^2}{ny-k}$. The function $t \mapsto \frac{-1}{2} + \frac{2(k-1)}{t-1}$ is negative as soon as $t \geq 4(k-1)+1$. As $n \geq 7$ and $k \in [2, n-2]$, we have $y = \binom{n-1}{k-1} \geq 4(k-1)+1$. Therefore,

$$\frac{2y(k-1)}{y-1} + 3/2 \leq \frac{2nky}{ny-k}$$

This finishes the proof of theorem 1. $\square$

2.2 Proof of theorem 2

Consider the k-binomial representation of x :

$$x = \binom{x_k}{k} + \binom{x_{k-1}}{k-1} + \cdots + \binom{x_t}{t} \text{ where } x_k > x_{k-1} > \cdots > x_t \geq t$$

It is easy to prove that

$$\text{when } x \to \infty, \quad x \sim \binom{x_k}{k} \text{ and similarly , } \quad |\Delta(F_k(x))| \sim \binom{x_k}{k-1}$$

As $x \sim \binom{x_k}{k}$, we have $x \sim \frac{{x_k}^k}{k!}$. This implies that $x_k \sim (x(k!))^{\frac{1}{k}}$. Therefore

$$\frac{|\Delta(F_k(x))|}{x} \sim \frac{\binom{x_k}{k-1}}{\binom{x_k}{k}} \sim \frac{k}{x_k - k + 1}$$

Hence $\frac{|\Delta(F_k(x))|}{x} \sim \frac{k}{(x(k!))^{\frac{1}{k}}}$ $\square$

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