# THE DISTRIBUTION OF DESCENTS AND LENGTH IN A COXETER GROUP


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Abstract. We give a method for computing the *q-Eulerian distribution*

$$W(t,q) = \sum_{w \in W} t^{\mathrm{des}(w)} q^{l(w)}$$

as a rational function in $t$ and $q$, where $(W,S)$ is an arbitrary Coxeter system, $l(w)$ is the *length function* in $W$, and $\mathrm{des}(w)$ is the number of simple reflections $s \in S$ for which $l(ws) < l(w)$. Using this we compute generating functions encompassing the $q$-Eulerian distributions of the classical infinite families of finite and affine Weyl groups.


## I. Introduction.

Let $(W,S)$ be a Coxeter system (see [Hu] for definitions and terminology). There are two statistics on elements of the Coxeter group $W$

$$\begin{aligned} l(w) &= min\{l : w = s_{i_1} s_{i_2} \cdots s_{i_l} \text{ for some } s_{i_k} \in S\} \\ \mathrm{des}(w) &= |\{s \in S : l(ws) < l(w)\}| \end{aligned}$$

which generalize the well-known permutation statistics *inversion number* and *descent number* in the case $W$ is the symmetric group $S_n$. The polynomial

$$\sum_{w \in S_n} t^{\mathrm{des}(w)}$$

is known in the combinatorial literature as the *Eulerian polynomial*, which has generating function

$$\sum_{n \geq 0} \frac{x^n}{n!} \sum_{w \in S_n} t^{\mathrm{des}(w)} = \frac{(1-t)\, e^{x(1-t)}}{1 - t\, e^{x(1-t)}}$$

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and a $q$-analogue first computed by Stanley [St, §3]:

$$\sum_{n\geq 0}\frac{x^n}{[n]!_q}\sum_{w\in S_n} t^{\mathrm{des}(w)}q^{l(w)} = \frac{(1-t)\exp(x(1-t);q)}{1-t\,\exp(x(1-t);q)} \tag{1}$$

where $\exp(x;q)$ is the $q$-exponential given by

$$\exp(x;q)=\sum_{n\geq 0}\frac{x^n}{[n]!_q}$$

using the notation

$$\begin{aligned}[n]!_q &= [n]_q[n-1]_q\cdots[2]_q[1]_q=\frac{(q;q)_n}{(1-q)^n}\\ [n]_q &= \frac{1-q^n}{1-q}\\ (x;q)_n &= (1-x)(1-qx)(1-q^2x)\cdots(1-q^{n-1}x)\end{aligned}$$

For this reason, we call

$$W(t,q)=\sum_{w\in W} t^{\mathrm{des}(w)}q^{l(w)}$$

the *$q$-Eulerian distribution* of the Coxeter system $(W,S)$, or the $q$-Eulerian distribution of $W$ by abuse of notation. (We caution the reader that this is not the same notion as the *$q$-Eulerian polynomial* considered in [Br] for $W=B_n,D_n$). Analogous generating functions to equation (1) for the infinite families of finite Coxeter groups $W=B_n(=C_n),D_n$ were computed in [Re1,Re2].

Note that in the case of an infinite Coxeter group $W$, the Eulerian distribution $\sum_{w\in W} t^{\mathrm{des}(w)}$ does not make sense as a formal power series in $t$, since there are only finitely many values $\{0,1,2,\ldots,|S|-1\}$ of $\mathrm{des}(w)$ and hence infinitely many group elements $w$ with the same value of $\mathrm{des}(w)$. On the other hand, the length distribution

$$W(q)=\sum_{w\in W} q^{l(w)}$$

*does* make sense in $\mathbb{Z}[[q]]$, and is known to be a computable rational function in $q$ (see equation (6)). The formula for $W(t,q)$ (equation (2)), which essentially comes from inclusion-exclusion, shows that $W(t,q)$ is a computable polynomial in $t$ having coefficients given by rational functions in $q$. Both this expression for $W(t,q)$ and this corollary are known as folklore within the subject of Coxeter groups, but are hard to find written down.

For some of the classical infinite families of finite and affine Coxeter groups, an encoding trick can be used to produce a generating function encompassing the $q$-Eulerian distributions of the entire family of groups as in equation (1). We derive a general result (Theorem 4) along these lines, and use it to recover known generating functions for the classical Weyl groups of types $A_n(=S_{n+1}),B_n(=C_n),D_n$ (see [St,Re1,Re2]) and derive new results for the infinite families $\tilde{A}_n,\tilde{B}_n,\tilde{C}_n,\tilde{D}_n$ of affine Weyl groups. For example, we show for the affine Weyl groups $\tilde{S}_n (= \tilde{A}_{n-1})$ associated to the symmetric groups $S_n$ that

$$\sum_{n \geq 1} \frac{x^n}{1-q^n} \tilde{S}_n(t,q) = \left[ \frac{x \frac{\partial}{\partial x} \log(\exp(x;q))}{1 - t \exp(x;q)} \right]_{x \mapsto x \frac{1-t}{1-q}}.$$

Theorem 4 explains why the factor

$$1 - t \exp(x;q)$$

naturally appears in the denominator in all of these generating functions.

The paper is structured as follows. Section II collects folklore, known results, and straightforward extensions concerning the computation of the $q$-Eulerian polynomial $W(t,q)$ of a general Coxeter system $(W,S)$. In Section III, we apply this to compute a generating function analogous to equation (1) for a general class of infinite families of Coxeter groups (Theorem 4). Section IV then specializes this to produce explicit generating functions for all of the infinite families of finite and affine Weyl groups (Theorems 5,6,7,8).

## II. How to calculate $W(t,q)$.

We recall here some facts about Coxeter systems $(W,S)$ and refer the reader to [Hu] for proofs and definitions which have been omitted. Given $w \in W$, let its *descent set* $Des(w)$ be defined by

$$Des(w) = \{s \in S : l(ws) < l(w)\}$$

For any subset $J \subseteq S$, the *parabolic subgroup* $W_J$ is the subgroup generated by $J$. The set

$$W^J = \{w \in W : \mathrm{Des}(w) \subseteq S - J\}$$

form a set of coset representatives for $W/W_J$, and furthermore when $w \in W$ is written uniquely in the form $w = u \cdot v$ where $u \in W^J, v \in W_J$, then we have $l(u) + l(v) = l(w)$. As a consequence,

$$W_J(q) \left( \sum_{w \in W : \mathrm{Des}(w) \subseteq S - J} q^{l(w)} \right) = W(q)$$

$$\sum_{w \in W : \mathrm{Des}(w) \subseteq S - J} q^{l(w)} = \frac{W(q)}{W_J(q)}$$

where recall that we are using the notation

$$W(q) = \sum_{w \in W} q^{l(w)}.$$

We will consider not only subsets $S \subseteq T$, but also *multisets* $T$ on the ground set $S$, which we think of as *functions* $T : S \to \mathbb{N}$ specifying a multiplicity $T(s)$ for each element of $s$ in $S$. For any such function $T$ in $S^{\mathbb{N}}$, let $\hat{T}$ denote its *support*, i.e. the subset $\hat{T} \subseteq S$ defined by

$$\hat{T} = \{s \in S : T(s) > 0\}.$$

Also denote by $|T|$ the *cardinality* $\sum_{s \in S} T(s)$ of the multiset or function.

**Theorem 1.** *For any Coxeter system $(W, S)$ we have*

$$W(t,q) = \sum_{T \subseteq S} t^{|T|}(1-t)^{|S-T|} \; \frac{W(q)}{W_{S-T}(q)} \tag{2}$$

$$\frac{W(t,q)}{(1-t)^{|S|}} = \sum_{T \in S^{\mathbb{N}}} t^{|T|} \frac{W(q)}{W_{S-\hat{T}}(q)} \tag{3}$$

*Proof.* We prove equation (2), from which (3) follows easily. Starting with the right-hand side of (2), one has

$$\begin{aligned}
&\sum_{T \subseteq S} t^{|T|}(1-t)^{|S-T|} \; \frac{W(q)}{W_{S-T}(q)} \\
&= \sum_{T \subseteq S} t^{|T|}(1-t)^{|S-T|} \sum_{w \in W : \mathrm{Des}(w) \subseteq T} q^{l(w)} \\
&= \sum_{w \in W} q^{l(w)} \sum_{\mathrm{Des}(w) \subseteq T \subseteq S} t^{|T|}(1-t)^{|S-T|} \\
&= \sum_{w \in W} q^{l(w)} t^{\mathrm{des}(w)} \sum_{\varnothing \subseteq T' \subseteq S - \mathrm{Des}(w)} t^{|T'|}(1-t)^{|S-\mathrm{Des}(w)-T'|} \\
&= \sum_{w \in W} q^{l(w)} t^{\mathrm{des}(w)} (t + (1-t))^{|S-\mathrm{Des}(w)|} \\
&= \sum_{w \in W} q^{l(w)} t^{\mathrm{des}(w)} \\
&= W(t,q) \blacksquare
\end{aligned}$$

*Remarks.* The specialization of equation (2) to $q = 1$ appears as [Ste, Proposition 2.2(b)], and the special case of (2) in which $W$ is of type $A_n$ appears in slightly different form as [DF, equation (2.5)].

It is just as easy to refine equations (2), (3) to keep track of the entire descent set $\mathrm{Des}(w)$ by giving each $s \in S$ its own indeterminate $t_s$. One can also refine this computation to incorporate other statistics than the length function $l(w)$, as long as the statistic $n(w)$ in question is *additive* under every parabolic coset decomposition in the following sense: for all $J \subseteq S$, when $w \in W$ is written uniquely as $w = u \cdot v$ with $u \in W^J, v \in W_J$, we have $n(w) = n(u) + n(v)$. The following theorem is then proven in exactly the same fashion as Theorem 1:

**Theorem 1′.** *Let $(W, S)$ be a Coxeter system, and $n_1(w), n_2(w), \ldots$ a series of* *additive statistics. Then using the notations*

$$\begin{aligned} \mathbf{q}^{\mathbf{n}(w)} &= \prod_i q_i^{n_i(w)} \\ \mathbf{t}^T &= \prod_{s\in T} t_s \\ (1-\mathbf{t})^T &= \prod_{s\in T}(1-t_s) \\ W(\mathbf{q}) &= \sum_{w\in W} \mathbf{q}^{\mathbf{n}(w)} \\ W(\mathbf{t},\mathbf{q}) &= \sum_{w\in W} \mathbf{t}^{\mathrm{Des}(w)}\mathbf{q}^{\mathbf{n}(w)} \end{aligned}$$

*we have*

$$W(\mathbf{t},\mathbf{q}) = \sum_{subset\ T\subseteq S} \mathbf{t}^T(1-\mathbf{t})^{S-T}\ \frac{W(\mathbf{q})}{W_{S-T}(\mathbf{q})} \tag{4}$$

$$\frac{W(\mathbf{t},\mathbf{q})}{(1-\mathbf{t})^S} = \sum_{T\in S^{\mathbb{N}}} \mathbf{t}^T \frac{W(\mathbf{q})}{W_{S-\hat{T}}(\mathbf{q})} \blacksquare \tag{5}$$

In light of this theorem, it is useful to know a classification of the additive statistics on $W$:

**Proposition 2.** *Let $(W,S)$ be a Coxeter system, and let $n : W \to \mathbb{Z}$ be an additive statistic in the above sense. Then*

1. *The statistic $n$ is completely determined by its values on $S$ via the formula*

$$n(w) = \sum_{j=1}^{l(w)} n(s_{i_j})$$

*for any reduced decomposition $w = s_{i_1}s_{i_2}\cdots s_{i_{l(w)}}$.*

2. *The statistic $n$ is well-defined if and only if it is constant on the $W$-conjugacy classes restricted to $S$, which are well-known (see e.g.* [Hu, Exercise §5.3]*) to coincide with the connected components of nodes in the subgraph induced by the odd-labelled edges of the Coxeter diagram.*

*As a consequence, there is a universal tuple of additive statistics $n_1, n_2, \ldots$ whose multivariate distribution specializes to that of any other additive statistics, defined by setting $n_i|_S$ to be the characteristic function of the $i^{th}$ $W$-conjugacy class restricted to $S$.*

*Proof.* If $n$ is additive, then the decomposition $1 = 1\cdot 1$ implies $n(1) = n(1)+n(1)$ so $n(1) = 0$. If the values of $n$ on $S$ are specified, then $n(w)$ is determined by the formula in the proposition for any $w$, using induction on $l(w)$: choose any $s \in \mathrm{Des}(w)$, and then $w = ws\cdot s$ is the unique decomposition in $W^{\{s\}}\cdot W_{\{s\}}$, so $n(w) = n(ws)+n(s)$.

To prove the second assertion, note that if $s, s'$ are connected by an odd-labelled edge in the Coxeter diagram, then the longest element of $W_{\{s,s'\}}$ has two reduced decompositions

$$s\,s's\,\cdots = s'\,s\,s'\,\cdots$$

and the formula for $n$ forces $n(s) = n(s')$. So $n$ must be constant on the $W$-conjugacy classes restricted to $S$, and Tits' solution to the word problem for $(W, S)$ [Hu, §8.1] shows that any such function on $S$ will extend (by the above formula) to a well-defined additive function on $W$. ■

Recall [Hu, §1.11, §5.12] the fact that $W(q)$ is a rational function in $q$, which may be computed using the recursion

$$W(q) = f(q)\left(\sum_{J\subsetneq S}\frac{(-1)^{|J|}}{W_J(q)}\right)^{-1} \tag{6}$$

where

$$f(q) = \begin{cases} (-1)^{|S|+1} & \text{if } W \text{ is infinite} \\ q^{l(w_0)} + (-1)^{|S|+1} & \text{if } W \text{ is finite}\end{cases}$$

and $w_0$ is the element of maximal length in $W$ when $W$ is finite. From equation (2), we conclude that $W(t, q)$ is also a rational function in $t$ and $q$ (in fact a polynomial in $t$ with coefficients given by rational functions of $q$, i.e. $W(t, q) \in \mathbb{Z}(q)[t]$). More generally, the $\mathbf{q}$-analogue of recursion (6) in which $q$ is replaced by $\mathbf{q}$ and $l(w)$ by $\mathbf{a}(w)$ follows from the same proof as (6). Therefore $W(\mathbf{q}) \in \mathbb{Z}(\mathbf{q})$ for any additive statistics $a_1(w), a_2(w), \ldots$, and from equation (4) we conclude that $W(\mathbf{t}, \mathbf{q}) \in \mathbb{Z}(\mathbf{q})[\mathbf{t}]$.

Before leaving this folklore section, we note a happy occurrence when the Coxeter diagram for $W$ is *linear*, i.e. when it has no nodes of degree greater than or equal to 3. In this situation and with $q = 1$, Stembridge [Ste, Proposition 2.3, Remark 2.4] observed that the right-hand side of (2) has a concise determinantal expression, and the proof given there generalizes in a straightforward fashion to prove the following:

**Theorem 3.** *Let $(W, S)$ be a Coxeter system with linear Coxeter diagram, and label the nodes $1, 2, \ldots, n$ in linear order. Then*

$$W(\mathbf{t}, \mathbf{q}) = W(\mathbf{q})\ det[a_{ij}]_{0\le i,j\le n}$$

*where*

$$a_{ij} = \begin{cases} 0 & i - j > 1 \\ t_i - 1 & i - j = 1 \\ \frac{t_i}{W_{[i+1,j]}(\mathbf{q})} & i \le j \end{cases}$$

*and by convention $t_0 = 1$, and $W_{[i+1,i]}$ is the trivial group with 1 element.*■

For example, if $W$ is the Weyl group of type $B_n(= C_n)$, then the Coxeter diagram is a path with $n$ nodes having all edges labelled 3 except for one on the end labelled 4. An interesting additive statistic $n(w)$ is the number of times the Coxeter generator on the end with the edge labelled 4 occurs in a reduced word for $w$ (this is the same as the number of negative signs occurring in $w$ when considered as a *signed* *permutation*). It is not hard to check (see e.g. [Re1, Lemma 3.1]) that if we let $\mathbf{q}^{\mathbf{n}(w)} = a^{n(w)} q^{l(w)}$, then

$$B_n(\mathbf{q}) = (-aq;q)_n [n]!_q$$

and hence the above determinant is very explicit. For example when $n = 2$,

$$B_2(\mathbf{t},\mathbf{q}) = (-aq;q)_2\,[2]!_q\; det \begin{bmatrix} 1 & \frac{1}{[2]!_q} & \frac{1}{(-aq;q)_2[2]!_q} \\ t_1 - 1 & t_1 & \frac{t_1}{(-aq;q)_1[1]!_q} \\ 0 & t_2 - 1 & t_2 \end{bmatrix}$$
$$= 1 + qt_1 + aq^2t_1 + aq^3t_1 + aqt_2 + aq^2t_2 + a^2q^3t_2 + a^2q^4t_1t_2.$$

**III.** $W(t,q)$ **for infinite families.**

In this section we use equation (2) to compute the generating function encompassing $W^{(n)}(t,q)$ for all $n$, where $W^{(n)}$ is an infinite family of Coxeter groups which grows in a certain prescribed fashion. It turns out that all of the infinite families of finite and affine Coxeter groups fit this description, and we deduce generating functions for their $q$-Eulerian polynomials (and some more general infinite families) as corollaries.

We begin by describing the infinite family $W^{(n)}$. Let $(W,S)$ be a Coxeter system, and choose a particular generator $v \in S$ to distinguish. Partition the neighbors of $v$ in the Coxeter diagram for $(W,S)$ into two blocks $B_1, B_2$, and define $(W^{(n)}, S^{(n)})$ for $n \in \mathbb{N}$ to be the Coxeter system whose diagram is obtained from that of $(W,S)$ as follows: replace the node $v$ with a path having $n+1$ vertices $s_0, \ldots, s_n$ and $n$ edges all labelled 3, then connect $s_0$ to the elements of $B_1$ using the same edge labels as $v$ used, and similarly connect $s_n$ to the elements of $B_2$. For example, $(W^{(0)}, S^{(0)}) = (W,S)$, while $(W^{(1)}, S^{(1)})$ will have one more node and one more edge (labelled 3) in its diagram than $(W,S)$ had. The goal of this section is to compute an expression for the generating function

$$\sum_{n \geq 0} \frac{x^n}{W^{(n)}(q)} W^{(n)}(t,q)$$

For a subset $J \subseteq S - v$, let $(W_J^{(n)}, S_J^{(n)})$ be the Coxeter system corresponding to the parabolic subgroup generated by $J \cup \{s_0, \ldots, s_n\}$. Also define for $J \subseteq S - v$ and $a, b \in \mathbb{N}$ the Coxeter system $(W_J^{(a,b)}, S_J^{(a,b)})$ to be the one corresponding to the parabolic subgroup of $(W^{(a+b)}, S^{(a+b)})$ generated by $J \cup (\{s_0, \ldots, s_n\} - s_a)$. Let

$$\exp_{W_J}(x;q) = \sum_{n \geq 0} \frac{x^n}{W_J^{(n)}(q)}$$
$$\operatorname{dex}_{W_J}(x;q) = \sum_{a,b \geq 0} \frac{x^{a+b}}{W_J^{(a,b)}(q)}$$

The terminologies "exp" and "dex" are intended to be suggestive of the fact that in the special cases of interest, $\exp_{W_J}(x;q)$ will be related to a $q$-analogue of the exponential function $\exp(x)$, and $\operatorname{dex}_{W_J}(x;q)$ will either be a product of two such $q$-analogues of exponentials (so a *double* exponential) or the *derivative* of such a $q$-analogue.

**Theorem 4.**

$$\sum_{n\geq 0} \frac{x^n}{W^{(n)}(q)} W^{(n)}(t,q) =$$
$$\left[ \sum_{J\subseteq S-v} t^{|J|}(1-t)^{|S-J|} \left( \exp_{W_{S-v-J}}(x;q) + \frac{t\,\mathrm{dex}_{W_{S-v-J}}(x;q)}{1-t\,\exp(x;q)} \right) \right]_{x\mapsto x(1-t)}$$

*Proof.* From equation (2) we have

$$W^{(n)}(t,q) = \sum_{T\subseteq S^{(n)}} t^{|T|}(1-t)^{|S^{(n)}-T|}\ \frac{W^{(n)}(q)}{W^{(n)}_{S^{(n)}-T}(q)}$$

so that

$$\begin{aligned}
&\frac{W^{(n)}(t,q)}{W^{(n)}(q)\ (1-t)^n} \\
&= \sum_{J\subseteq S-v} t^{|J|}(1-t)^{|S-J|} \sum_{K\subseteq\{s_0,\dots,s_n\}} \frac{t^{|K|}}{(1-t)^{|K|}}\ \frac{1}{W^{(n)}_{S^{(n)}-J-K}(q)} \\
&= \sum_{J\subseteq S-v} t^{|J|}(1-t)^{|S-J|} \sum_{K\in\mathbb{N}^{\{s_0,\dots,s_n\}}} t^{|K|}\ \frac{1}{W^{(n)}_{S^{(n)}-J-\hat{K}}(q)} \\
&= \sum_{J\subseteq S-v} t^{|J|}(1-t)^{|S-J|} \left( \frac{1}{W^{(n)}_{S-v-J}(q)} + \sum_{k\geq 1} t^k \sum_{\substack{K\in\mathbb{N}^{\{s_0,\dots,s_n\}}\\ |K|=k}} \frac{1}{W^{(n)}_{S^{(n)}-J-\hat{K}}(q)} \right)
\end{aligned}$$

At this stage, we use an encoding for the functions $K:\{s_0,\dots,s_n\}\to\mathbb{N}$ having $|K|=k$. Let $\omega_i\in\mathbb{R}^n$ be the vector $e_1+e_2+\ldots+e_i$, where $e_i$ is the $i^{th}$ standard basis vector, so that $\omega_0=(0,0,\dots,0)$ and $\omega_n=(1,1,\dots,1)$. Given $K:\{s_0,\dots,s_n\}\to\mathbb{N}$, encode it as the vector $c(K)=\sum_{i=0}^n K(s_i)\omega_i\in\mathbb{R}^n$. Note that once we have fixed the cardinality $|K|=k\geq 1$, then $K$ is completely determined by $c(K)$, which is a decreasing sequence with entries in the range $[0,k]$. Hence $K$ is also completely determined by the sequence $a(K)=(a_0,\dots,a_k)$ where $a_i$ is the number of occurrences of $i$ in $c(K)$. Furthermore, it is easy to check that the parabolic subgroup $W_{S^{(n)}-J-\hat{K}}$ is then isomorphic to

$$W^{(a,b)}_{S-v-J}\times S_{a_1}\times\cdots\times S_{a_{k-1}}.$$

Therefore we may continue the calculation

$$\frac{W^{(n)}(t,q)}{W^{(n)}(q)\ (1-t)^n} = \sum_{J \subseteq S-v} t^{|J|}(1-t)^{|S-J|} \times$$

$$\left( \frac{1}{W^{(n)}_{S-v-J}(q)} + \sum_{k \geq 1} t^k \sum_{\substack{(a_0,\dots,a_k)\in\mathbb{N}^{k+1} \\ \sum a_i = n}} \frac{1}{W^{(a,b)}_{S-v-J}(q)\,[a_1]!_q \cdots [a_{k-1}]!_q} \right)$$

$$\sum_{n\geq 0} \frac{W^{(n)}(t,q)}{W^{(n)}(q)} \ \frac{x^n}{(1-t)^n} = \sum_{J \subseteq S-v} t^{|J|}(1-t)^{|S-J|} \times$$

$$\left( \sum_{n\geq 0} \frac{x^n}{W^{(n)}_{S-v-J}(q)} + \sum_{k\geq 1} t^k \sum_{n\geq 0} \sum_{\substack{(a_0,\dots,a_k)\in\mathbb{N}^{k+1} \\ \sum a_i = n}} \frac{x^{a_0+a_k}}{W^{(a_0,a_k)}_{S-v-J}(q)} \frac{x^{a_1}}{[a_1]!_q} \cdots \frac{x^{a_{k-1}}}{[a_{k-1}]!_q} \right)$$

$$= \sum_{J \subseteq S-v} t^{|J|}(1-t)^{|S-J|} \times$$

$$\left( \exp_{W_{S-v-J}}(x;q) + \sum_{a_0,a_k \geq 0} \frac{x^{a_0+a_k}}{W^{(a,b)}_{S-v-J}(q)} \sum_{k\geq 1} t^k (\exp(x;q))^k \right)$$

$$= \sum_{J \subseteq S-v} t^{|J|}(1-t)^{|S-J|} \left( \exp_{W_{S-v-J}}(x;q) + \mathrm{dex}_{W_{S-v-J}}(x;q) \frac{t}{1-t\ \exp(x;q)} \right)$$

The theorem now follows upon replacing $x$ by $x(1-t)$.■

*Remarks.*

1. The crucial encoding of functions $K : \{s_0, \dots, s_n\} \to \mathbb{N}$ used in the middle of the preceding proof is a translation and generalization of the "direct encoding" used in [GG, §1] for type $A_n$.
2. There is an obvious **q**-analogue of Theorem 3 involving additive statistics on $(W, S)$, with the same proof.

## IV. Explicit generating functions for classical Weyl groups and affine Weyl groups.

This section (and the remainder of the paper) is devoted to specializing Theorem 4 to compute generating functions for descents and length in all of the classical finite and affine Weyl groups, and certain families which generalize them. In all cases where $W$ is a finite or affine Weyl group, the denominators $W(q)$ occurring in the left-hand side of Theorem 4 can be made explicit for the following reason: if $W$ is a finite *Weyl* group of rank $n$, then there is an associated multiset of numbers $e_1, e_2, \dots, e_n$ called the *exponents* of $W$, satisfying

$$W(q) = \prod_{i=1}^{n} [e_i + 1]_q \tag{7}$$

$$\tilde{W}(q) = \prod_{i=1}^{n} \frac{[e_i+1]_q}{1-q^{e_i}} \tag{8}$$

where $\tilde{W}$ is the affine Weyl group associated to $W$. The first formula is a theorem of Chevalley [Hu, §3.15], the second a theorem of Bott [Hu, §8.9]. We should mention that Bott's proof, although extremely elegant and unified, is not completely elementary, and more elementary proofs of some cases of his theorem have recently appeared in [BB, BE , EE, ER].

We first consider an infinite family of Coxeter systems with *linear* diagrams. Let $W_n^{r,s}$ be the family of Coxeter groups whose Coxeter diagram is a path with $n$ nodes, in which the labels on almost all of the edges are 3 *except* for the leftmost edge labelled $r$ and the rightmost edge labelled $s$. Let $W_n^r$ be the family defined by $W_n^r = W_n^{r,3}$ The next result uses Theorem 4 to compute a generating function for $W_n^{r,s}(t,q)$. Note that $W_n^{r,s}$ contains as special cases the finite Coxeter groups of type $A_n, B_n(=C_n), H_3, H_4$, and the affine Weyl groups $\tilde{C}_n$, as well as some hyperbolic Coxeter groups (see [Hu, §2.4, 2.5, 6.9]).

Before stating the theorem, we establish some more notation. Let

$$\exp_{W^r}(x;q) = \sum_{n\geq 0} \frac{x^n}{W_n^r(q)}$$
$$\exp_{W^{r,s}}(x;q) = \sum_{n\geq 0} \frac{x^n}{W_n^{r,s}(q)}$$

where by convention we define $W_0^{r,s} = W_0^r$ to be the trivial group with 1 element, $W_1^{r,s} = W_1^r$ is the unique Coxeter system of rank 1, and $W_2^{r,s} = W_2^r = I_2(r)$ is the rank 2 (dihedral) Coxeter system of order $2r$.

**Theorem 5.**

(9)
$$\sum_{n\geq 0} \frac{x^n}{W_n^{r,s}(q)} W_n^{r,s}(t,q) = \exp_{W^{r,s}}(x(1-t);q) + \frac{t\,x\,(1-t)\,\exp_{W^r}(x(1-t);q)\,\exp_{W^s}(x(1-t);q)}{1-t\,\exp(x(1-t);q)}$$

(10)
$$\sum_{n\geq 0} \frac{x^n}{W_n^r(q)} W_n^r(t,q) = \frac{(1-t)\,\exp_{W^r}(x(1-t);q)}{1-t\,\exp(x(1-t);q)}$$

*Proof.* Equation (10) follows from equation (9) by setting $s=3$ and noting that

$$\exp_{W^{r,3}}(x;q) = \exp_{W^r}(x;q)$$
$$\exp_{W^3}(x;q) = \frac{\exp(x;q)-1}{x}.$$

We wish to derive equation (9) from Theorem 4. In the notation preceding Theorem 4, choose $(W,S)$ to have Coxeter diagram with 3 nodes $s_1, s_2, s_3$ forming a path with two edges $\{s_1,s_2\},\{s_2,s_3\}$ labelled $r$ and $s$ respectively, and let $v =$

$s_2, B_1 = \{s_1\}, B_2 = \{s_2\}$. One can then check that

$$\begin{aligned}
W^{(n)} &= W^{r,s}_{n+3} \\
\exp_{W_{s_1,s_3}}(x;q) &= x^{-3}\left(\exp_{W^{r,s}}(x;q) - 1 - \frac{x}{[2]_q} - \frac{x^2}{[2]_q[r]_q}\right) \\
\exp_{W_{s_1}}(x;q) &= x^{-2}\left(\exp_{W^r}(x;q) - 1 - \frac{x}{[2]_q}\right) \\
\exp_{W_{s_3}}(x;q) &= x^{-2}\left(\exp_{W^s}(x;q) - 1 - \frac{x}{[2]_q}\right) \\
\exp_{W_\varnothing}(x;q) &= x^{-2}(\exp(x;q) - 1 - x) \\
\mathrm{dex}_{W_{s_1,s_3}}(x;q) &= x^{-2}(\exp_{W^r}(x;q) - 1)(\exp_{W^s}(x;q) - 1) \\
\mathrm{dex}_{W_{s_1}}(x;q) &= x^{-2}(\exp_{W^r}(x;q) - 1)(\exp(x;q) - 1)) \\
\mathrm{dex}_{W_{s_3}}(x;q) &= x^{-2}(\exp_{W^s}(x;q) - 1)(\exp(x;q) - 1)) \\
\mathrm{dex}_{W_\varnothing}(x;q) &= x^{-2}(\exp(x;q) - 1)^2
\end{aligned}$$

and using these facts, equation (9) follows from Theorem 4 with a little algebra. ■

We now specialize Theorem 5 to obtain generating functions for the types $A_{n-1}(= S_n), B_n(= C_n)$, and $\tilde{C}_n$.

If $r = 3$ then $W^r_n$ coincides with the finite Weyl group $A_n(= S_{n+1})$ which has exponents $1, \ldots, n$, and one can check that equation (10) is equivalent to Stanley's formula (1). It is interesting to note that $\exp_{W^r}(x;q)$ has an alternate expression in this case in terms of an infinite product, since $\exp_{W^r}(x;q) = x^{-1}(\exp(x;q) - 1)$ as noted earlier, and

$$\exp(x;q) = \sum_{n\geq 0} \frac{(x(1-q))^n}{(q;q)_n} = (x(1-q);q)^{-1}_\infty$$

where the last equality is by the $q$-binomial theorem [GR, Appendix II.3]:

$$\sum_{n\geq 0} \frac{(z;q)_n}{(q;q)_n} x^n = \frac{(zx;q)_\infty}{(x;q)_\infty}.$$

If $r = 4$ then $W^r_n$ coincides with the finite Weyl group $B_n$ or $C_n$ which has exponents $1, 3, \ldots, 2n-1$. In this case equation (10) is equivalent to [Re1, §3] specialized to $a = q = 1$. Again we note that $\exp_{W^r}(x;q)$ has an alternate expression in this case as an infinite product, since

$$\exp_{W^4}(x;q) = \sum_{n\geq 0} \frac{(x(1-q))^n}{(q^2;q^2)_n} = (x(1-q);q^2)^{-1}_\infty$$

again by the $q$-binomial theorem. Furthermore, since the Coxeter diagram in the case has an edge labelled 4, there exists another additive statistic $n(w)$, equal to the number of negative signs in $w$ considered as a signed permutation (see example after Theorem 3). Using the known distribution

$$B_n(\mathbf{q}) = \sum_{w\in B_n} a^{n(w)}q^{l(w)} = (-aq;q)_n[n]!_q$$

the proof of Theorem 3 for $r=4$ can be refined to a result equivalent to [Re1, §3, specialized to $q=1$]. On the other hand, it does not seem to be true that the generalization of $\exp_{W^4}(x;q)$ defined by

$$\exp_{W^4}(x;a,q) = \sum_{n\geq 0}\frac{x^n}{(-aq;q)_n[n]!_q}$$

has a nice infinite product expression.

If $r=s=4$ then $W^{r,s}_{n+1}$ coincides with the affine Weyl group $\tilde{C}_n$ for $n\geq 2$, so equation (9) says

$$\begin{aligned}
&1+\frac{x(1+tq)}{[2]_q}+\frac{x^2C_2(t,q)}{[2]_q[4]_q}+x\sum_{n\geq 2}\frac{x^n}{\tilde{C}_n(q)}\tilde{C}_n(t,q)\\
&=\exp_{W^{4,4}}(x(1-t);q)+\frac{t\,x\,(1-t)\,\left[\exp_{W^4}(x(1-t);q)\right]^2}{1-t\,\exp(x(1-t);q)}.
\end{aligned}$$

Again we can replace $\exp(x;q), \exp_{W^4}(x;q)$ by their infinite product formulas as before, and $\exp_{W^{4,4}}(x;q)$ also has an expression involving an infinite product: since the associated Weyl group $C_n$ has exponents $1,3,\ldots,2n-1$, by (8) we have

$$W^{4,4}_{n+1} = \tilde{C}_n(q) = \frac{(q^2;q^2)_n}{(1-q)^n(q;q^2)_n} \tag{11}$$

for $n\geq 2$ and hence

$$\begin{aligned}
\exp_{W^{4,4}}(x;q) &= 1+\frac{x}{[2]_q}+\frac{x^2}{[2]_q[4]_q}+x\sum_{n\geq 2}\frac{(q;q^2)_n}{(q^2;q^2)_n}(x(1-q))^n\\
&= 1+\frac{x}{[2]_q}+\frac{x^2}{[2]_q[4]_q}+x\left(\frac{(xq(1-q);q^2)_\infty}{(x(1-q);q^2)_\infty}-\frac{x(1-q)}{1+q}-1\right)
\end{aligned}$$

where the last equality comes from the $q$-binomial theorem. Furthermore, since the Coxeter diagram in this case has its two extreme edges labelled 4, there exist two other additive statistic $n(w), m(w)$ equal to the number of occurrences of the two endpoint Coxeter generators occurring in a reduced word for $w$. One can prove the following refinement of equation (11) (a special case of Bott's Theorem) for $W^{4,4}_{n+1}=\tilde{C}_n$:

$$\begin{aligned}
\tilde{C}_n(\mathbf{q}) &= \sum_{w\in\tilde{C}_n} a^{n(w)}b^{m(w)}q^{l(w)}\\
&= \frac{(-aq;q)_n(-bq;q)_n[n]!_q}{(abq^{n+1};q)_n}
\end{aligned} \tag{12}$$

by using the $\mathbf{q}$-generalization of recursion (6) to show

$$\begin{aligned}\frac{1}{\tilde{C}_n(\mathbf{q})} &= \sum_{i=0}^{n} \frac{q^{i^2} a^i}{B_i(a,q)B_{n-i}(b,q)} \\ &= \sum_{i=0}^{n} \frac{q^{i^2} a^i}{(-aq;q)_i [i]!_q (-bq;q)_{n-i} [n-i]!_q}\end{aligned}$$

and then applying the $q$-Vandermonde summation formula [GR, Appendix II.6]. The $\mathbf{q}$-refinement of Theorem 5 with $r = s = 4$ then gives a very explicit generating function generalization enumerating $\tilde{C}_n$ by the quadruple of statistics

$$(n(w), m(w), l(w), \mathrm{des}(w)).$$

On the other hand, it no longer seems to be true that there is a nice infinite product expression for the relevant generalization of $\exp_{W^{4,4}}(x;q)$ defined by

$$\begin{aligned}\exp_{W^{4,4}}(x;a,b,q) &= \sum_{n\geq 0} \frac{(abq^{n+1};q)_n}{(-bq;q)_n(-aq;q)_n[n]!_q} x^n \\ &= {}_4\phi_3\left( \begin{matrix} \sqrt{abq} & -\sqrt{abq} & \sqrt{abq} & -\sqrt{abq} \\ -aq & -bq & abq & \end{matrix} \middle|\, x(1-q);q \right)\end{aligned}$$

where the last equation is basic hypergeometric series notation (see e.g. [GR]).

We next deal with the affine symmetric groups. Let $\tilde{A}_{n-1} = \tilde{S}_n$ be the affine Weyl group corresponding to the Weyl group $A_{n-1} = S_n$, so $\tilde{S}_n$ has as its Coxeter diagram a cycle with $n$ vertices and label 3 on every edge. Let

$$\exp_{\tilde{S}}(x:q) = \sum_{n\geq 1} \frac{x^n}{\tilde{S}_n(q)}.$$

We now prove a formula claimed in the Introduction:

**Theorem 6.**

$$\sum_{n\geq 1} \frac{x^n}{1-q^n} \tilde{S}_n(t,q) = \left[ \frac{x \frac{\partial}{\partial x} \log(\exp(x;q))}{1 - t\,\exp(x;q)} \right]_{x \mapsto x\,\frac{1-t}{1-q}}.$$

*Proof.* In the notation preceding Theorem 4, choose $(W,S)$ to have Coxeter diagram with 3 nodes $s_1, s_2, s_3$ arranged in a triangle with the three edges labelled 3. Let

$v = s_3$ and $B_1 = \{s_1\}, B_2 = \{s_2\}$. One can then check that

$$\begin{aligned}
W^{(n)} &= \tilde{S}_{n+3} \\
\exp_{W_{s_1,s_2}}(x;q) &= x^{-3}\left(\exp_{\tilde{S}}(x;q) - x - \frac{x^2(1-q)}{[2]_q}\right) \\
\exp_{W_{s_1}}(x;q) = \exp_{W_{s_2}}(x;q) &= x^{-3}\left(\exp(x;q)1 - x - \frac{x^2}{[2]_q}\right) \\
\exp_{W_\varnothing}(x;q) &= x^{-2}(\exp(x;q) - 1 - x) \\
\operatorname{dex}_{W_{s_1,s_2}}(x;q) &= -2x^{-3}\left(\exp(x;q) - 1 - x - \frac{x^2}{[2]_q}\right) \\
&\quad + x^{-2}\left(\frac{\partial}{\partial x}\exp(x;q) - 1 - \frac{2x}{[2]_q}\right) \\
\operatorname{dex}_{W_{s_1}}(x;q) = \operatorname{dex}_{W_{s_2}}(x;q) &= x^{-3}(\exp(x;q) - 1)(\exp(x;q) - 1 - x) \\
\operatorname{dex}_{W_\varnothing}(x;q) &= x^{-2}(\exp(x;q) - 1)^2
\end{aligned}$$

and using these facts one can simplify Theorem 4 in this case to

$$\sum_{n\geq 1} \frac{x^n}{\tilde{S}_n(q)}\tilde{S}_n(t,q) = \left[\exp_{\tilde{S}}(x;q) + \frac{t\,x\,\frac{\partial}{\partial x}\exp(x;q)}{1 - t\,\exp(x;q)}\right]_{x\mapsto x\,(1-t)}.$$

To rewrite this more explicitly, we note that the exponents of $S_n = A_{n-1}$ are $1, 2, \ldots, n-1$, so that equation (8) gives

$$\tilde{S}_n(q) = \frac{[n]!_q}{(q;q)_{n-1}} = \frac{1-q^n}{(1-q)^n}.$$

Therefore

$$\begin{aligned}
\exp_{\tilde{S}}(x;q) &= \sum_{n\geq 1} \frac{x^n(1-q)^n}{1-q^n} \\
&= \sum_{n\geq 1}\sum_{m\geq 0} (x(1-q))^n q^{nm} \\
&= \sum_{m\geq 0}\sum_{n\geq 1} (x(1-q)q^m)^n \\
&= \sum_{m\geq 0} \frac{x(1-q)\,q^m}{1 - x(1-q)\,q^m} \\
&= \sum_{m\geq 0} x\frac{\partial}{\partial x}\log[(1 - x(1-q)\,q^m)^{-1}] \\
&= x\frac{\partial}{\partial x}\log[(x(1-q);q)_\infty^{-1}] \\
&= x\frac{\partial}{\partial x}\log(\exp(x;q))
\end{aligned}$$

Substituting this into the last equation and replacing $x$ by $\frac{x}{1-q}$ gives

$$\sum_{n\geq 1} \frac{x^n}{1-q^n}\tilde{S}_n(t,q) = \left[ x\frac{\partial}{\partial x}\log(\exp(x;q)) + \frac{t\,x\,\frac{\partial}{\partial x}\exp(x;q)}{1-t\,\exp(x;q)}\right]_{x\mapsto x\frac{1-t}{1-q}}$$

which is equivalent to the theorem by a little algebra.■

Next we move on to a common generalization of the Weyl groups $D_n$ and the affine Weyl groups $\tilde{B}_n$. Let $D_n^r$ be the Coxeter system whose graph is obtained from the graph for $D_n$ by replacing the label of 3 on the edge farthest from the "fork" with a label of $r$. Note that $D_n^3 = D_n$ and $D_n^4 = \tilde{B}_n$ (see [Hu, §2.4, 2.5]). We adopt the notation

$$\begin{aligned}\exp_{D^r}(x;q) &= \sum_{n\geq 2} \frac{x^n}{D_n^r(q)}\\ \exp_D(x;q) &= \sum_{n\geq 2} \frac{x^n}{D_n(q)}\end{aligned}$$

where by convention we define $D_2^r = A_1 \oplus A_1$ and $D_3^r = A_3$.

**Remark:** The notation $\exp_D(x;q)$ is slightly different from the notation $\exp_D(u)$ used in [Re2, Corollary 4.5], and in fact, there is an error in this previous reference, which we correct here: the definition of $\exp_D(u)$ given there as

$$\exp_D(u) = \sum_{n\geq 0} \frac{u^n}{(-q;q)_{n-1}[n]!_q}$$

should actually read

$$\exp_D(u) = 2 + \sum_{n\geq 1} \frac{u^n}{(-q;q)_{n-1}[n]!_q}$$

Therefore this previous definition of $\exp_D(u)$ differs from our present notation $\exp_D(u;q)$ in the coefficients of $u^0, u^1$.

**Theorem 7.**

(13)

$$\sum_{n\geq 4} \frac{x^n}{D_n^r(q)} D_n^r(t,q) =$$

$$\sum_{n\geq 4} \langle x^n\rangle \left[\exp_{D^r}(x;q) + \frac{t\,x\,\exp_{W^r}(x;q)}{1-t\,\exp(x;q)}\left(2 - \frac{tx}{1-t} + \exp_D(x;q)\right)\right]_{x\mapsto x(1-t)} \cdot x^n$$

*Proof.* In the notation preceding Theorem 4, choose $(W,S)$ to have Coxeter diagram with 4 nodes $s_1, s_2, s_3, s_4$ in which $s_4$ is connected by an edge labelled 3 to $s_1, s_2,$

and connected to $s_3$ by an edge labelled $r$, with no other edges in the diagram. Let $v = s_4$ and $B_1 = \{s_1, s_2\}, B_2 = \{s_3\}$. One can then check that

$$\begin{aligned}
W^{(n)} &= D^r_{n+4} \\
\exp_{W_{s_1,s_2,s_3}}(x;q) &= x^{-4}\left(\exp_{D^r}(x;q) - \frac{x^2}{([2]_q)^2} - \frac{x^3}{[3]!_q}\right) \\
\exp_{W_{s_1,s_2}}(x;q) &= x^{-3}\left(\exp_{D}(x;q) - \frac{x^2}{([2]_q)^2}\right) \\
\exp_{W_{s_1,s_3}}(x;q) = \exp_{W_{s_2,s_3}}(x;q) &= x^{-3}\left(\exp_{W^r}(x;q) - 1 - \frac{x}{[2]!_q} - \frac{x^2}{[2]_q[r]_q}\right) \\
\exp_{W_{s_1}}(x;q) = \exp_{W_{s_2}}(x;q) &= x^{-3}\left(\exp(x;q) - 1 - x - \frac{x^2}{[2]!_q}\right) \\
\exp_{W_{s_3}}(x;q) &= x^{-2}\left(\exp_{W^r}(x;q) - 1 - \frac{x}{[2]!_q}\right) \\
\exp_{W_\varnothing}(x;q) &= x^{-2}(\exp(x;q) - 1 - x) \\
\operatorname{dex}_{W_{s_1,s_2,s_3}}(x;q) &= x^{-3}\exp_D(x;q)(\exp_{W^r}(x;q) - 1) \\
\operatorname{dex}_{W_{s_1,s_2}}(x;q) &= x^{-3}\exp_D(x;q)(\exp(x;q) - 1) \\
\operatorname{dex}_{W_{s_1,s_3}}(x;q) = \operatorname{dex}_{W_{s_2,s_3}}(x;q) &= x^{-3}(\exp_{W^r}(x;q) - 1)(\exp(x;q) - 1 - x) \\
\operatorname{dex}_{W_{s_1}}(x;q) = \operatorname{dex}_{W_{s_2}}(x;q) &= x^{-3}(\exp(x;q) - 1)(\exp(x;q) - 1 - x) \\
\operatorname{dex}_{W_{s_3}}(x;q) &= x^{-2}(\exp_{W^r}(x;q) - 1)(\exp(x;q) - 1) \\
\operatorname{dex}_{W_\varnothing}(x;q) &= x^{-2}(\exp(x;q) - 1)^2
\end{aligned}$$

and using these the result follows from Theorem 4. ■

We now specialize Theorem 7 to $r = 3, 4$. If $r = 3$, then $D^r_n = D_n$ and then one can check that our conventions for $D_3$ and $D_2$ have been chosen correctly so that the generating function on the right-hand side of equation (13) agrees with the left-hand side in its coefficient of $x^2, x^3$ (as well as $x^n$ for $n \geq 4$). Therefore we obtain

$$2tx + \sum_{n\geq 2}\frac{x^n}{D_n(q)}D_n(t,q) = \\ \frac{(1-t)\exp_D(x(1-t);q) + t\,(2-tx)\,(\exp(x(1-t);q) - 1)}{1 - t\,\exp(x(1-t);q)}.$$

which one can easily check agrees with [Re2, Corollary 4.5]. Note that since $D_n$

has exponents $1, 3, \ldots, 2n-3, n-1$ by equation (7) we have

$$\begin{aligned}
\exp_D(x;q) &= \sum_{n\geq 2} \frac{x^n}{(-q;q)_{n-1}[n]!_q} \\
&= \sum_{n\geq 2} \frac{x^n(1-q)^n(1+q^n)}{(q^2;q^2)_n} \\
&= \sum_{n\geq 2} \left( \frac{(x(1-q))^n}{(q^2;q^2)_n} + \frac{(xq(1-q))^n}{(q^2;q^2)_n} \right) \\
&= (x(1-q);q^2)_\infty^{-1} + (xq(1-q);q^2)_\infty^{-1} - 2 - x
\end{aligned}$$

so one can again replace the exponential functions $\exp(x;q), \exp_D(x;q)$ appearing above by expressions involving infinite products if desired.

If $r = 4$, then $D_n^r = \tilde{B}_{n-1}$, so equation (13) gives a closed form for the generating function

$$\sum_{n\geq 3} \frac{x^n}{\tilde{B}_n(q)} \tilde{B}_n(t,q).$$

Since $\tilde{B}_n$ is the affine Weyl group associated to $B_n$, which has the same exponents as $C_n$, we must have $\tilde{B}_n(q) = \tilde{C}_n(q)$ by equation (7). Therefore we have already seen how all of the functions $\exp_{D^4}(x;q)$, $\exp_{W^4}(x;q)$, $\exp(x;q)$ appearing in the generating function can be made more explicit, and replaced by expressions involving infinite products if desired. Furthermore, since the Coxeter diagram for $\tilde{B}_n$ has an edge labelled 4, there is another additive statistic $n(w)$ which counts how many times the Coxeter generator at that end of the diagram is used in a reduced word for $w$, and one can derive (similarly to (12)) the following refinement of equation (10) for $W = \tilde{B}_n$:

$$\begin{aligned}
\tilde{B}_n(\mathbf{q}) &= \sum_{w\in\tilde{B}_n} a^{n(w)} q^{l(w)} \\
&= \frac{(-aq;q)_n(-q;q)_{n-1}[n]!_q}{(aq^n;q)_n}
\end{aligned} \tag{14}$$

This allows one to refine equation (13) when $r = 4$ so as to incorporate the statistic $n(w)$. However, as before the generalization $\exp_{D^4}(x;a,q)$ of $\exp_{D^4}(x;q)$ does not seem to have a nice infinite product expression.

The only affine Weyl group remaining to be discussed is $\tilde{D}_n$, whose Coxeter diagram looks like a path with forks at both ends having $n$ nodes total, and all edges labelled 3 (see [Hu, §2.5]). We use the notation

$$\exp_{\tilde{D}}(x;q) = \sum_{n\geq 4} \frac{x^n}{\tilde{D}_n(q)}.$$

**Theorem 8.**

$$\sum_{n\geq 4} \frac{x^n}{\tilde{D}_n(q)} \tilde{D}_n(t,q) = \sum_{n\geq 4} \langle x^n \rangle$$
$$\Bigg\{ (1-t)\, \exp_{\tilde{D}}(x(1-t);q) + \frac{t}{1-t\, \exp(x(1-t);q)} \times$$
$$\Bigg[ \frac{t^2(2+2x-tx)^2 + t(2+tx)(2-4t-3tx+2t^2x)\, \exp(x(1-t);q)}{1-t}$$
$$+2(1-t)(2-tx)\, \exp_D(x(1-t);q) + (1-t)\, \exp_D(x(1-t);q)^2 \Bigg] \Bigg\}\; x^n$$

*Proof.* In the notation preceding Theorem 4, choose $(W,S)$ to have Coxeter diagram with 5 nodes $s_1, s_2, s_3, s_4, s_5$ in which $s_5$ is connected by an edge labelled 3 to $s_1, s_2, s_3, s_4$, and there are no other edges in the diagram. Let $v = s_5$ and $B_1 = \{s_1, s_2\}, B_2 = \{s_3, s_4\}$. One can then check that

$$\begin{aligned}
W^{(n)} &= \tilde{D}^r_{n+4} \\
\exp_{W_{s_1,s_2,s_3,s_4}}(x;q) &= x^{-4} exp_{\tilde{D}}(x;q) \\
\exp_{W_{s_1,s_2,s_3}}(x;q) &= \exp_{W_{s_1,s_2,s_4}}(x;q) = \exp_{W_{s_1,s_3,s_4}}(x;q) = \exp_{W_{s_2,s_3,s_4}}(x;q) \\
&= x^{-4}\left( \exp_D(x;q) - \frac{x^2}{([2]_q)^2} - \frac{x^3}{[3]!_q} \right) \\
\exp_{W_{s_1,s_2}}(x;q) &= \exp_{W_{s_3,s_4}}(x;q) = x^{-3}\left( \exp_D(x;q) - \frac{x^2}{([2]_q)^2} \right) \\
\exp_{W_{s_1,s_3}}(x;q) &= \exp_{W_{s_1,s_4}}(x;q) = \exp_{W_{s_2,s_3}}(x;q) = \exp_{W_{s_2,s_4}}(x;q) \\
&= x^{-4}\left( \exp(x;q) - 1 - x - \frac{x^2}{[2]!_q} - \frac{x^3}{[3]!_q} \right) \\
\exp_{W_{s_1}}(x;q) &= \exp_{W_{s_2}}(x;q) = \exp_{W_{s_3}}(x;q) = \exp_{W_{s_4}}(x;q) \\
&= x^{-3}\left( \exp(x;q) - 1 - x - \frac{x^2}{[2]!_q} \right) \\
\exp_{W_\varnothing}(x;q) &= x^{-2}(\exp(x;q) - 1 - x) \\
\mathrm{dex}_{W_{s_1,s_2,s_3,s_4}}(x;q) &= x^{-4} \exp_{\tilde{D}(x;q)} \\
\mathrm{dex}_{W_{s_1,s_2,s_3}}(x;q) &= \mathrm{dex}_{W_{s_1,s_2,s_4}}(x;q) = \mathrm{dex}_{W_{s_1,s_3,s_4}}(x;q) = \mathrm{dex}_{W_{s_2,s_3,s_4}}(x;q) \\
&= x^{-4} \exp_D(x;q)(\exp(x;q) - 1 - x) \\
\mathrm{dex}_{W_{s_1,s_2}}(x;q) &= \mathrm{dex}_{W_{s_3,s_4}}(x;q) = x^{-3} \exp_D(x;q)(\exp(x;q) - 1) \\
\mathrm{dex}_{W_{s_1,s_3}}(x;q) &= \mathrm{dex}_{W_{s_1,s_4}}(x;q) = \mathrm{dex}_{W_{s_2,s_3}}(x;q) = \mathrm{dex}_{W_{s_2,s_4}}(x;q) \\
&= x^{-4}(\exp(x;q) - 1 - x)^2 \\
\mathrm{dex}_{W_{s_1}}(x;q) &= \mathrm{dex}_{W_{s_2}}(x;q) = \mathrm{dex}_{W_{s_3}}(x;q) = \mathrm{dex}_{W_{s_4}}(x;q) \\
&= x^{-3}(\exp(x;q) - 1)(\exp(x;q) - 1 - x) \\
\mathrm{dex}_{W_\varnothing}(x;q) &= x^{-2}(\exp(x;q) - 1)^2
\end{aligned}$$

and using these facts, the result follows from Theorem 4 with a little algebra. ■

As in the previous cases of affine Weyl groups, it is possible to replace $\exp_{\tilde{D}}(x)$ by an expression involving infinite products, if desired. Since $D_n$ has exponents $1, 3, 5, \ldots, 2n-5, 2n-3, n-1$, by equation (8) we have

$$\begin{aligned}\tilde{D}_n(q) &= \frac{(-q;q)_{n-1}[n]!_q}{(1-q)^n(q;q^2)_{n-1}(1-q^{n-1})}\\ &= \frac{(q^2;q^2)_n}{(1-q)^n(q^{-1};q^2)_n}\frac{(1-q^{-1})}{(1+q^n)(1-q^{n-1})}\end{aligned}$$

Therefore

$$\begin{aligned}\exp_{\tilde{D}}(x;q) &= \sum_{n\geq 4}(x(1-q))^n\frac{(q^{-1};q^2)_n}{(q^2;q^2)_n}\frac{(1+q^n)(1-q^{n-1})}{(1-q^{-1})}\\ &= \sum_{n\geq 4}(x(1-q))^n\frac{(q^{-1};q^2)_n}{(q^2;q^2)_n}\left[(1+q^n)+\frac{1}{q-1}(1-q^{2n})\right]\\ &= \frac{(xq^{-1}(1-q);q^2)_\infty}{(x(1-q);q^2)_\infty}+\frac{(x(1-q);q^2)_\infty}{(xq(1-q);q^2)_\infty}\\ &+ xq^{-1}(1-q)\frac{(xq(1-q);q^2)_\infty}{(x(1-q);q^2)_\infty}-\sum_{i=0}^{3}c_i(q)\,x^i\\ &= \frac{(xq(1-q);q^2)_\infty}{(x(1-q);q^2)_\infty}+\frac{(x(1-q);q^2)_\infty}{(xq(1-q);q^2)_\infty}-\sum_{i=0}^{3}c_i(q)\,x^i\end{aligned}$$

for some $c_i(q)$ which are rational functions of $q$.

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