FASTER AND FASTER CONVERGENT SERIES FOR $\zeta(3)$

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Abstract. Using WZ pairs we present accelerated series for computing $\zeta(3)$

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Alf van der Poorten [P] gave a delightful account of Apéry's proof [A] of the irrationality of $\zeta(3)$. Using WZ forms, that came from [WZ1], Doron Zeilberger [Z] embedded it in a conceptual framework.

We recall [Z] that a discrete function A(n,k) is called Hypergeometric (or Closed Form (CF)) in two variables when the ratios $A(n+1,k)/A(n,k)$ and $A(n,k+1)/A(n,k)$ are both rational functions. A pair (F,G) of CF functions is a WZ pair if $F(n+1,k)-F(n,k)=G(n,k+1)-G(n,k)$. In this paper, after choosing a particular F (where its companion G is then produced by the amazing Maple package EKHAD accompanying [PWZ]), we will give a list of accelerated series calculating $\zeta(3)$. Our choice of F is

$$F(n,k)=\frac{(-1)^k k!^2(sn-k-1)!}{(sn+k+1)!(k+1)}$$

where s may take the values s=1,2,3, ... [AZ] (the section pertaining to this can be found in http://www.math.temple.edu/~tewodros). In order to arrive at the desired series we apply the following result:

Theorem: ([Z], Theorem 7, p.596) For any WZ pair (F,G)

$$\sum_{n=0}^{\infty} G(n,0)=\sum_{n=1}^{\infty}\left(F(n,n-1)+G(n-1,n-1)\right),$$

whenever either side converges.

The case s=1 is Apéry's celeberated sum [P] (see also [Z]):

$$\zeta(3)=\frac{5}{2}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{\binom{2n}{n}n^3}$$

where the corresponding G is

$$G(n,k)=\frac{2(-1)^k k!^2(n-k)!}{(n+k+1)!(n+1)^2}.$$

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For s=2 we obtain

$$\zeta(3)=\frac{1}{4}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{56n^2-32n+5}{(2n-1)^2}\frac{1}{\binom{3n}{n}\binom{2n}{n}n^3}$$

where G is

$$G(n,k)=\frac{(-1)^k k!^2(2n-k)!(3+4n)(4n^2+6n+k+3)}{2(2n+k+2)!(n+1)^2(2n+1)^2}.$$

For s=3 we have

$$\zeta(3)=\sum_{n=0}^{\infty}\frac{(-1)^n}{72\binom{4n}{n}\binom{3n}{n}}\{\frac{6120n+5265n^4+13761n^2+13878n^3+1040}{(4n+1)(4n+3)(n+1)(3n+1)^2(3n+2)^2}\},$$

and so on.

References

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[PWZ] M. Petkovšek, H.S. Wilf, D.Zeilberger, *"A=B"*, A.K. Peters Ltd., 1996.
The package EKHAD is available by the www at http://www.math.temple.edu/~zeilberg/programs.html

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