# For which graphs does every edge belong to exactly two chordless cycles?

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**Abstract**

A graph is *2-cycled* if each edge is contained in exactly two of its chordless cycles. The 2-cycled graphs arise in connection with the study of balanced signing of graphs and matrices. The concept of balance of a $\{0, +1, -1\}$-matrix or a signed bipartite graph has been studied by Truemper and by Conforti *et al.* The concept of $\alpha$-balance is a generalization introduced by Truemper. Truemper exhibits a family $\mathcal{F}$ of planar graphs such that a graph $G$ can be signed to be $\alpha$-balanced if and only if each induced subgraph of $G$ in $\mathcal{F}$ can. We show here that the graphs in $\mathcal{F}$ are exactly the 2-connected 2-cycled graphs.

# 1 Introduction

A graph is said to be *2-cycled* if each of its edges is contained in exactly two chordless cycles. The 2-cycled graphs arise in connection with the study of balanced signing of graphs and matrices by Truemper [3] and by Conforti *et al.* [2], as indicated in the next three paragraphs.

A *signed graph* is a graph $G = (V, E)$ together with a mapping $f : E \longrightarrow \{+1, -1\}$. Consider a mapping $\alpha : \mathcal{C} \longrightarrow \{0, 1, 2, 3\}$, where $\mathcal{C}$ is the set of chordless cycles of $G$. If $\Sigma_{e \in C} f(e) \equiv \alpha(C) \pmod 4$ for all $C \in \mathcal{C}$, we say that the signed graph is *$\alpha$-balanced.* A trivial necessary condition, which we assume throughout, is that $|C| \equiv \alpha(C) \pmod 2$ for all $C \in \mathcal{C}$. When $\alpha = 0$, this condition means that $G$ is bipartite, in which case it can be specified by its adjacency matrix $A$, and $A$ is balanced in the usual sense if and only if the signed graph consisting of $G$ and the constant mapping $f = 1$ is 0-balanced. Similarly, a $\{0, +1, -1\}$-matrix $A$ specifies a signed bipartite graph, and $A$ is said to be balanced when the signed bipartite graph is 0-balanced.

It is easy to check that each graph of the following types is 2-cycled (See Figure 1):

**Star-subdivision of $K_4$:** The result of subdividing zero or more of the three edges incident to a single vertex of $K_4$;

**Rim-subdivision of a wheel:** The result of subdividing zero or more rim edges of the wheel $W_k$, $k \geq 3$;

**Subdivision of $K_{2,3}$:** The result of subdividing zero or more edges of $K_{2,3}$.;

**Triangles-joining:** Two vertex-disjoint triangles with three vertex-disjoint paths joining them.

Note that if two nonadjacent edges of $K_4$ and possibly other edges are subdivided, the resulting graph is not 2-cycled. It is called a *bad subdivision of* $K_4$.

Truemper [3] showed that a graph $G$ possesses a mapping $f$ that makes it $\alpha$-balanced if and only if each induced subgraph of $G$ that is a star-subdivision of $K_4$, a rim-subdivision of a wheel, a subdivision of $K_{2,3}$ or a triangles-joining enjoys the same property. Our main result is that these are all the 2-connected 2-cycled graphs (Clearly, a graph s 2-cycled if and only if all its 2-connected components are, so without loss of generality we may consider only 2-connected graphs):

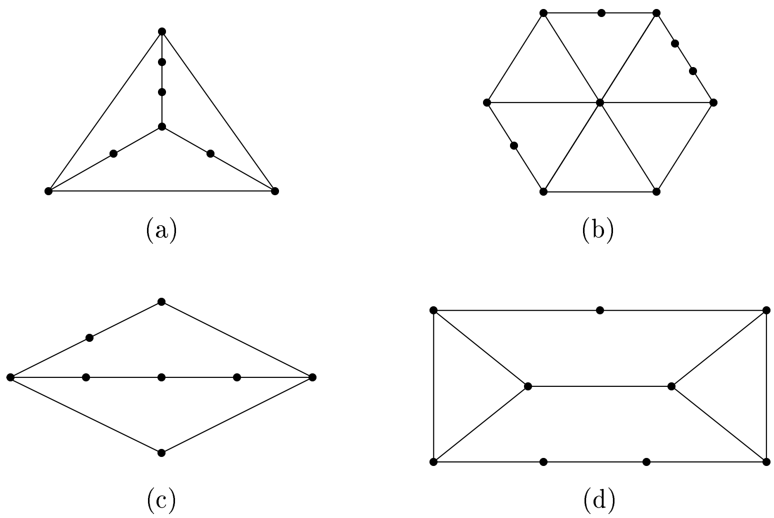


Figure 1: 2-cycled graphs. (a): Star-subdivision of $K_4$; (b): Rim-subdivision of a wheel; (c): Subdivision of $K_{2,3}$; (d): Triangles-joining.

**Theorem 1 (Main Theorem)** *A 2-connected graph is 2-cycled if and only if it is a star-subdivision of $K_4$, a rim-subdivision of a wheel, a subdivision of $K_{2,3}$ or a triangles-joining.*

This paper is organized as follows. In Section 2 we give definitions of some new concepts. In Section 3 we define and characterize the upper and lower 2-cycled graphs; these graphs are defined so that a graph is 2-cycled if and only if it is both upper 2-cycled and lower 2-cycled. In Section 4 we study the structure of 2-cycled graphs and prove the Main Theorem. Early on (in Corollary 2) we show that the upper 2-cycled graphs are planar, and this planarity plays an important part in the proofs.

# 2 Preliminaries

We discuss only finite simple graphs and use standard terminology and notation from [1], except as indicated. We denote by $N_G(u)$ or simply $N(u)$ the set of vertices adjacent to a vertex $u$ in a graph $G$, and by $N_G(S)$ or $N(S)$ the set $\bigcup_{u\in S} N_G(u)$ for a vertex subset $S$. A *chord* of a path or a cycle is an edge joining two non-consecutive vertices of the path or cycle. A *chordless* path or cycle is one having no chord. For a path $P = (x_1, x_2, \ldots, x_k)$, we use the notation $P[x_i, x_j]$ for the subpath $(x_i, \ldots, x_j)$, where $1 \leq i < j \leq n$. If $e = ab$ is an edge of $G$, the *contraction* $G/e$ *of* $G$ *with respect to* $e$ is the graph obtained from $G$ by replacing $a$ and $b$ with a new vertex $c$ and joining $c$ to those vertices that are adjacent to $a$ or $b$. The edge set of $G/e$ may be regarded as a subset of the edge set of $G$. A *minor* of $G$ is a graph that can be obtained from $G$ by a sequence of vertex-deletions, edge-deletions and contractions. By *subdividing an edge* $e$ we mean replacing $e$ by a path $P$ joining the ends of $e$, where $P$ has length at least 2 and all of its internal vertices have degree 2. A *subdivision* of $G$ is a graph obtained by subdividing zero or more of the edges of $G$. The *intersection (union)* $G_1 \cap G_2$ $(G_1 \cup G_2)$ of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph with vertex set $V_1 \cap V_2$ $(V_1 \cup V_2)$ and edge set $E_1 \cap E_2$ $(E_1 \cup E_2)$. If $C_1$ and $C_2$ are cycles of a plane graph $G$, we say that $C_1$ *is within (surrounds)* $C_2$ if the area enclosed by $C_1$ is contained in (contains) that enclosed by $C_2$.

Two cycles $C$ and $C'$ are said to be *harmonic* if $C \cap C'$ is a path, as illustrated in Figure 2. If $C$ and $C'$ are harmonic cycles of a plane graph, we can find an appropriate plane drawing of the graph such that $C'$ is within $C$, if it is not already the case, by selecting a face within $C$ and making it the outer face.

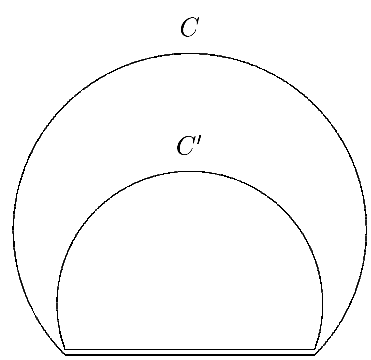


Figure 2: Harmonic cycles.

Let $C$ and $C'$ be two cycles with a common edge $e$, and $u$ a vertex of $C' - C$. Let $P'$ be the maximal subpath of $C'$ that contains $u$ and does not have internal vertices on $C$, and let $P$ be the subpath of $C$ joining the two ends of $P'$ and containing $e$. Then $P' \cup P$ is a cycle $C''$, as illustrated in Figure 3. The operation transforming $C'$ into $C''$ is called *grafting* $C'$ *with* *respect to $C$, $e$ and $u$.* An important property of this operation is that the new cycle $C''$ is harmonic with $C$. Furthermore, if the graph is a plane graph and $u$ is within $C$ (or $C'$ surrounds $C$), then $C''$ is within (surrounds) $C$.

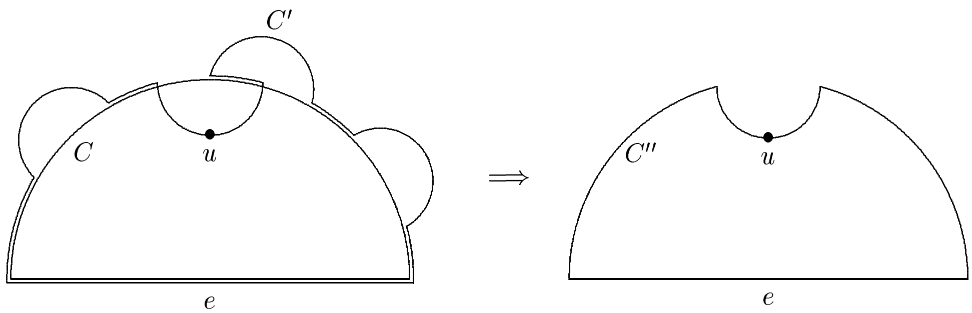


Figure 3: Grafting.

Let $P = (x_1, x_2, \ldots, x_k)$ be a path in $G$. If $P$ has a chord $x_i x_j$ for some $i < j-1$, we can obtain another path $P' = (x_1, \ldots, x_i, x_j, \ldots, x_k)$ by deleting the vertices between $x_i$ and $x_j$ and adding the edge $x_i x_j$ to $P$. If $P'$ still has chords, we can apply the same operation to $P'$, and so on until we obtain a chordless path $P^*$ connecting $x_1$ to $x_k$. For a cycle $C$ of $G$ and an edge $e$ of $C$, we can apply the above operation to $C - e$ to obtain a chordless cycle $C^*$ containing $e$. We call the operation transforming $C$ into $C^*$ *chord-cutting $C$ with respect to $e$.* We note that if the graph is a plane graph and $C$ surrounds (is within, is harmonic with) a chordless cycle $\widehat{C}$ and $e$ is a common edge of $C$ and $\widehat{C}$, then the cycle obtained by chord-cutting $C$ with respect to $e$ again surrounds (is within, is harmonic with) $\widehat{C}$.

Let $C$ and $C'$ be cycles of $G$, where $C$ is chordless, $e$ a common edge of $C$ and $C'$, and $u$ a vertex of $C' - C$. By grafting $C'$ with respect to $C$, $e$ and $u$, and then chord-cutting the resulting cycle with respect to $C$ and $e$, we obtain a chordless cycle $C^*$. We call the operation transforming $C'$ into $C^*$ *harmonizing $C'$ to $C$ with respect to $e$ and $u$.* Note that the new cycle $C^*$ still contains $e$ and is harmonic with $C$ and chordless. Furthermore, if $G$ is a plane graph and $u$ is within $C$ ($C'$ surrounds $C$), then $C^*$ is within (surrounds) $C$. After the harmonization operation we forget $C'$ and rename $C^*$ as $C'$.

# 3 Upper and lower 2-cycled graphs

We say that a graph is *upper (lower) 2-cycled* if each of its edges is contained in at most (at least) two of its chordless cycles. Clearly, a graph possesses this property if and only if each 2-connected component does, but in the rest of this section we do not assume 2-connectivity. The following lemma is crucial in characterizing upper 2-cycled graphs.

**Lemma 1** *If $G = (V, E)$ is upper 2-cycled, so are its minors.*

**Proof.** It suffices to show that if $G'$ results from $G$ by deleting or contracting an edge $uv$ and $G'$ is not upper 2-cycled, neither is $G$. Let $e = ab$ be an edge of $G'$ that is contained in distinct chordless cycles $C'_1$, $C'_2$ and $C'_3$ of $G'$.
*Case 1:* $G' = G - uv$. Note that if $uv$ is not a chord of $C'_i$, then $C'_i$ is also a chordless cycle of $G$; in this case, we put $C_i = C'_i$. If $uv$ is a chord of $C'_i$, then $C'_i \cup uv$ is split into two chordless cycles of $G$, each consisting of $uv$ and a subpath of $C'$ connecting $u$ to $v$; we call the one containing $e$ $C_i$ and the other one $\widetilde{C}_i$. If $C_1$, $C_2$ and $C_3$ are distinct, then they are distinct chordless cycles of $G$ containing $e$. If they are not, we may assume $C_1 = C_2$. Then $C'_1$ and $C'_2$ must have $uv$ as a chord, and $C_1$, $\widetilde{C}_1$ and $\widetilde{C}_2$ are distinct chordless cycles of $G$ containing $uv$.
*Case 2:* $G' = G/uv$. The edge $uv$ of $G$ is contracted to a vertex $w$ of $G'$. Because $uv \neq ab$, $\{a, b\} \cap \{u, v\}$ is empty or has one vertex. If it is nonempty, we assume $u = a$ without loss of generality.

If $E(C'_i)$ forms a cycle of $G$, it must be a chordless cycle, and we let $C_i$ be that cycle. If not, $w$ must be a vertex of $C'_i$, and $E(C'_i)$ forms a path $P_i$ in $G$ connecting $u$ to $v$. Let $u'_i, v'_i$ be the neighbors of $u, v$ on $P_i$, respectively. Then $P_i \cup uv$ forms a cycle $C^*_i$ of $G$, and its only possible chords are $uv'_i$ and $u'_i v$. By chord-cutting $C^*_i$ with respect to $e$, we find a chordless cycle $C_i$ containing $e$.

Note that if $e$ and $uv$ are not adjacent, or if the chord $u'_i v$ does not exist, then $E(C_i)$ is contracted to $E(C'_i)$ when we contract the edge $uv$.

Now we have three chordless cycles $C_1$, $C_2$ and $C_3$ containing $e$. If they are not all distinct, say $C_1 = C_2$, then $C_1$ is the triangle $\{u = a, v, b = u'_1 = u'_2\}$, $C^*_1$ and $C^*_2$ both have $bv$ as a chord, and $bv$ is contained in three distinct chordless cycles of $G$, namely $\{a, v, b\}$, $bv \cup P'_1 - e$, $bv \cup P'_2 - e$. ∎

We note that $K_{3,3} - e$ and $K_2 \oplus 3K_1$ (the graph obtained by joining every vertex of $K_2$ to every vertex of $3K_1$) are not upper 2-cycled. These graphs are illustrated in Figure 4. Therefore we have the following corollary of Lemma 1.

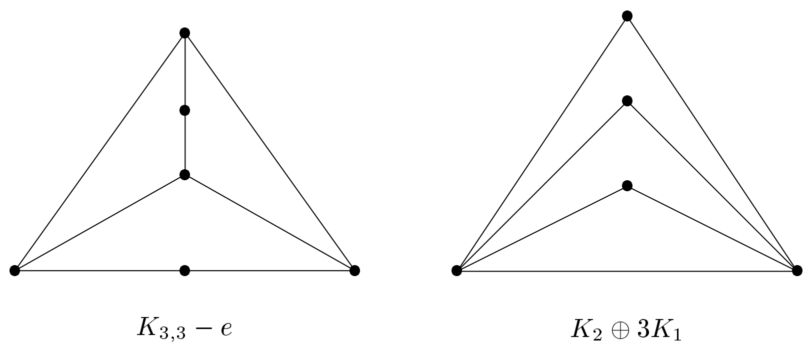


Figure 4: Forbidden minors of upper 2-cycled graphs.

**Corollary 1** *An upper 2-cycled graph contains no $K_2 \oplus 3K_1$ or $K_{3,3} - e$ as a minor.*

Note that $K_2 \oplus 3K_1$ is a minor of $K_5$ and $K_{3,3} - e$ is a minor of $K_{3,3}$. By Kuratowski's Theorem, we have the following consequence of Corollary 1.

**Corollary 2** *An upper 2-cycled graph must be planar.*

The next theorem characterizes the upper 2-cycled graphs. Although we only use its necessity part to prove the Main Theorem, it has an independent interest.

**Theorem 2** *A graph is upper 2-cycled if and only if it contains no $K_2 \oplus 3K_1$ or $K_{3,3} - e$ as a minor.*

**Proof.** The necessity is Corollary 1 above. Now we prove the sufficiency. By the argument leading to Corollary 2, $G$ must be planar. Assume that, if possible, $G$ is not upper 2-cycled. We assert that $G$ has three cycles $C_1$, $C_2$ and $C_3$ and an edge $e$ such that the following properties hold for an appropriate plane drawing of $G$:

1. $C_1$, $C_2$ and $C_3$ are distinct chordless cycles containing $e$;

2. $C_2$ is within $C_1$ and $C_3$ is within $C_2$;

3. $C_1$, $C_2$ and $C_3$ are harmonic with each other.

In proving the assertion, we make use of a weaker version of Property 3, namely,

4. $C_2$ is harmonic with $C_1$ and $C_3$.

By the assumption that $G$ is not upper 2-cycled, it has three cycles $C_1$, $C_2$ and $C_3$ and an edge $e$ satisfying Property 1. If two of the cycles are harmonic, we rename them as $C_1$ and $C_3$. If not, we harmonize $C_3$ to $C_1$ with respect to $e$, and the new $C_3$ is still different from $C_1$ and $C_2$. In any case, we may assume $C_3$ is within $C_1$. For the new $C_1$, $C_2$ and $C_3$, Property 1 still holds, but now $C_3$ is within and harmonic with $C_1$.

Next, let us consider three cases about $C_2$.

*Case 1:* $C_2$ has a vertex $u$ inside $C_3$. We harmonize $C_2$ to $C_3$ with respect to $u$ and $e$, and switch the names of $C_3$ and $C_2$. The cycles $C_1$, $C_2$ and $C_3$ now satisfy Properties 1, 2 and 4.

*Case 2:* $C_2$ has a vertex $u$ outside $C_1$. We select a face within $C_3$, make it the outer face, and switch the names of $C_1$ and $C_3$, and we are back to Case 1.

*Case 3:* $C_2$ is between $C_1$ and $C_3$. We harmonize $C_1$ to $C_2$ and $C_3$ to $C_2$ with respect to $e$. The cycles $C_1$, $C_2$ and $C_3$ now satisfy Properties 1, 2 and 4.

Thus in all cases, Properties 1, 2 and 4 hold for $C_1$, $C_2$, $C_3$ and $e$. By planarity and Property 2 we have $C_1 \cap C_3 \subset C_2$, hence $C_1 \cap C_3 = (C_1 \cap C_2) \cap (C_2 \cap C_3)$. Since each of $C_1 \cap C_2$ and $C_2 \cap C_3$ is a subpath of $C_2$, $C_1 \cap C_3$ must be a path or the union of two disjoint paths. In the former case, $C_1$ is harmonic with $C_3$, as required. In the latter case, illustrated in Figure 5, the symmetric difference of $E(C_2)$ and $E(C_3)$ forms a cycle $C'$, and we can find an edge $e'$ in $C_1 \cap C_2$ such that $e'$ is also on $C'$. Renaming $C'$ as $C_3$ and $e'$ as $e$ and chord-cutting $C_3$ with respect to the new edge $e$, we achieve Property 3 for the new $C_1$, $C_2$, $C_3$ and $e$ while Properties 1 and 2 remain valid. This completes the proof of the assertion.

It follows from the assertion that $P_{13} = C_1 \cap C_3$ is a path contained in $C_2$ and containing $e$. Let $P'_{13}$ ($P'_{31}$) be the subpath of $C_1 - e$ ($C_3 - e$) between the ends $a$ and $b$ of $P_{13}$.

Suppose no internal vertex of $P'_{31}$ is on $C_2$. Let $P'_{13} = (a = x_0, x_1, \ldots, x_k = b)$, and let $i$ ($j$) be the largest (smallest) index such that $x_0, \ldots, x_i$ ($x_j, \ldots, x_k$) are on $C_2$, as illustrated in Figure 6. Since $C_1$ and $C_2$ are chordless, $P'_{31}$ and $P'_{13}[x_i, x_j]$ are not single edges, i.e., each has an internal vertex. For the same reason, the subpath of $C_2$ from $x_i$ to $x_j$ that does not contain $e$ has

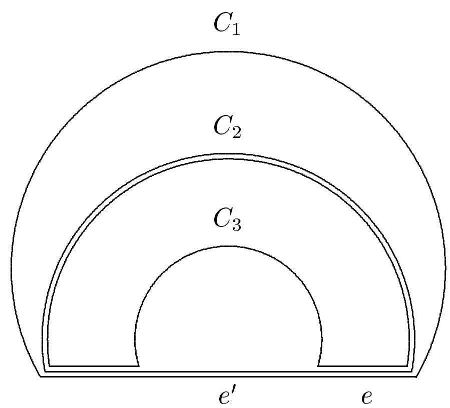


Figure 5: An illustration for the proof of the assertion.

an internal vertex. We contract $x_0, \ldots, x_i$ into one vertex and $x_j, \ldots, x_k$ into another vertex, and now $C_1 \cup C_2 \cup C_3$ is a subdivision of $K_2 \oplus 3K_1$, which has $K_2 \oplus 3K_1$ as a minor, contrary to the hypothesis. A similar argument holds if no internal vertex of $P'_{13}$ is on $C_2$.

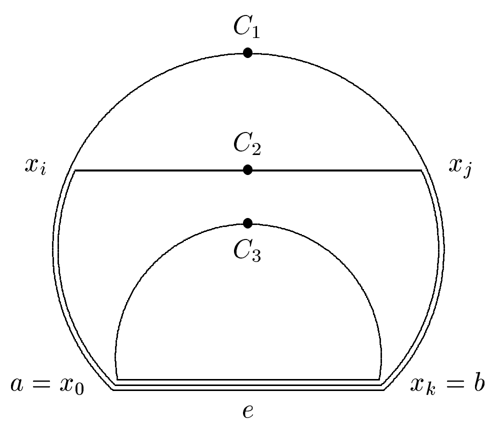


Figure 6: An illustration for the proof of Theorem 2.

If both $P'_{13}$ and $P'_{31}$ have an internal vertex on $C_2$, there is a subpath $P$ of $C_2$ connecting an internal vertex $d$ of $P'_{13}$ to an internal vertex $c$ of $P'_{31}$ such that $P$ has no internal vertex on $C_1$ or $C_3$. Without loss of generality, we assume that the cycle $C_2$ passes through the vertices $a, b, c, d$ in this order. Then, since $C_2$ is harmonic with both $C_1$ and $C_3$, $C_2$ must be $P_{13}[a, b] \cup$

$P'_{31}[b,c] \cup P[c,d] \cup P'_{13}[d,a]$, as illustrated in Figure 7. Because $C_2$ is a chordless cycle, each of $P'_{13}[b,d]$ and $P'_{31}[a,c]$ has an internal vertex. Hence $C_1 \cup C_2 \cup C_3$ is a bad subdivision of $K_4$, which can always be contracted to $K_{3,3} - e$, contrary to the hypothesis. ■

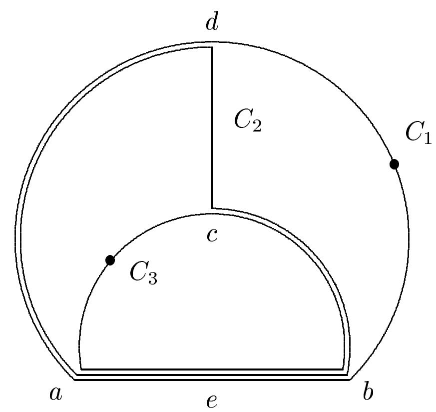


Figure 7: An illustration for the proof of Theorem 2.

The next theorem characterizes the lower 2-cycled graphs. The proof is simple and is omitted.

**Theorem 3** *A graph $G$ is lower 2-cycled if and only if $G$ has no bridges and every chordless cycle $C$ of $G$ satisfies at least one of the following conditions:*

1. *For each edge $e = uv$ of $C$, $G - V(C)$ has a connected component $H$ such that $N(H) \cap V(C) = \{u, v\}$;*

2. *$G - V(C)$ has a connected component $H$ such that $N(H)$ contains a pair of non-consecutive vertices of $C$;*

3. *$C$ is a triangle, and $G - V(C)$ has a connected component $H$ such that $V(C) \subseteq N(H)$.*

# 4 Proof of the Main Theorem

We only need to prove the "only if" part of the Main Theorem. We do so by establishing a series of properties that a 2-connected 2-cycled graph $G$ must possess.

By Corollary 2, we have the following:

**Property 1** *$G$ is planar.*

**Property 2** *For each edge $ab$ of $G$, $G - \{a, b\}$ has at most two connected components.*

Indeed, otherwise there would be three chordless cycles containing $ab$.

We call an edge $ab$ *critical* if $G - \{a, b\}$ has exactly two connected components.

Let $F$ be any face of a plane drawing of $G$. The boundary of $F$ is a cycle $C$ by 2-connectivity, and we call it a *face-cycle*. If $F$ is the outer face of a plane drawing $D$ of $G$, we call $C$ the *outer cycle* of $D$. Suppose the outer cycle $C$ has a critical edge $ab$. We denote by $H$ the connected component of $G - \{a, b\}$ not containing the other vertices of $C$. By 2-connectivity we have $N(H) \cap V(C) = \{a, b\}$. We can find another plane drawing $D'$ of $G$ by moving $H$ outside of $C$. If $C'$ denotes the new outer cycle, then $V(C') \supset V(C)$, and $ab$ is a chord of $C'$, rather than an edge of it. We call the operation transforming $D$ into $D'$ *flipping*. If $C'$ still has critical edges, we repeat this operation. In a finite number of steps, we obtain a plane drawing of $G$ whose outer cycle $C^*$ has no critical edges. We now assert that $C^*$ is chordless. If not, a chord $ab$ would spilt the cycle $C^*$ into two cycles $C'$ and $C''$, and we may assume that $C'$ is chordless. There must be a vertex $u$ within $C'$, for otherwise there would be no other chordless cycles containing an edge from $C' - ab$. Let $H$ be the connected component of $G - V(C')$ containing $u$. The set $N(H) \cap V(C')$ cannot be the two ends of an edge from $C' - ab$, because by Property 2 such an edge would be a critical edge on the outer cycle $C^*$. Thus $C'$ does not satisfy Condition 1 of Theorem 3, and it must satisfy Condition 2 or 3. We can therefore find a chordless path $P$ connecting two non-consecutive vertices of the path $C' - ab$ such that all the internal vertices of $P$ are from $H$, as illustrated in Figure 8. It is easy to see that $C^* \cup \{ab\} \cup P$ can be contracted to $K_2 \oplus 3K_1$, contradicting Corollary 1. This proves the assertion.

Recall that the flipping operation adds a new chord of the new outer cycle. Hence by the assertion, no flipping ever takes place, and the outer cycle $C$ of the arbitrary drawing $D$ is chordless. Since each face of a plane drawing can be drawn as the outer face, we have established the following two properties:

**Property 3** *$G$ has no critical edge.*

**Property 4** *In each plane drawing of $G$, each face-cycle is chordless.*

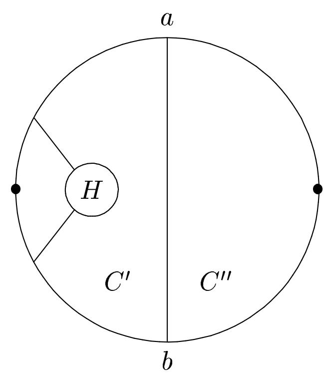


Figure 8: An illustration for the proof of the assertion.

In each plane drawing, each edge $e$ belongs to two face-cycles by 2-connectivity. The latter are chordless by Property 4, and must be the only chordless cycles containing $e$ since $G$ is 2-cycled. We therefore conclude the following:

**Property 5** *In each plane drawing of $G$, each chordless cycle is a face-cycle.*

Another property of $G$ is given below.

**Property 6** *At least one of the face-cycles is not a triangle if $G \neq K_4$.*

Suppose to the contrary that every face-cycle of $G$ is a triangle. Let $C$ be the outer cycle with vertices $a, b$ and $c$. Without loss of generality, let $k \geq 3$ be the degree of $a$, and let $ax_1, ax_2, \ldots, ax_k$ be the edges incident with $a$ in counterclockwise order, where $b = x_1$ and $c = x_k$. Then by our assumption, $x_1x_2, x_2x_3, \ldots, x_{k-1}x_k$ must be edges of $G$, as illustrated in Figure 9.

If $k \geq 4$, there is an edge $e = x_ix_j$, $j - i > 1$, $e \neq x_1x_k$, for otherwise $\{x_1, \ldots, x_k\}$ is a chordless cycle, hence a face-cycle by Property 5, but it is not a triangle. But then the triangle $\{a, x_i, x_j\}$ is not a face-cycle, contradicting Property 5. Therefore $k$ must be 3, and for each $\{p, q, r\} \subseteq \{a, x_1, x_2, x_3\}$, the triangle $\{p, q, r\}$ is a face-cycle by Property 5. Therefore $G$ has no other vertices, and so $G = K_4$.

By Property 6, we may assume that we have chosen a plane drawing of $G$ whose outer cycle $C$ has length at least 4. We make this assumption for the rest of the proof, and use the notations $N_C(u) = N(u) \cap V(C)$ and $N_C(S) = N(S) \cap V(C)$ for a vertex $u$ and a vertex-subset $S$.

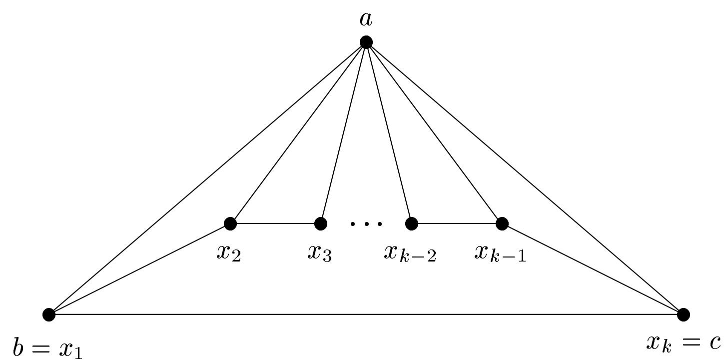


Figure 9: An illustration for the proof of Property 6.

**Property 7** *For each connected component $H$ of $G-V(C)$, $N_C(H)$ contains a pair of non-consecutive vertices along $C$.*

In fact, $N_C(H)$ cannot be empty or a single vertex by 2-connectivity. Neither can it be the two ends of an edge $e$ of $C$, since otherwise $e$ would be a critical edge by Property 2, contrary to Property 3. Therefore $N_C(H)$ contains a pair of non-consecutive vertices along $C$.

**Property 8** $G - V(C)$ *is connected.*

If not, $G - V(C)$ would have at least two connected components $H_1$ and $H_2$. For $i = 1, 2$, $N_C(H_i)$ contains a pair of non-consecutive vertices $a_i, b_i$ on $C$ by Property 7. we can find a path $P_i$ connecting $a_i$ to $b_i$ all of whose internal vertices are from $H_i$. By planarity, $P_1$ and $P_2$ do not intersect except possibly at the ends. Therefore the minor $C \cup P_1 \cup P_2$ can be contracted to $K_2 \oplus 3K_1$, contrary to Corollary 1.

**Property 9** $G - V(C)$ *contains no cycle.*

If $G-V(C)$ contains a cycle, it must contain a chordless cycle $C'$. There exists vertex-disjoint paths $P_1$ and $P_2$ between $C$ and $C'$ (this can be seen by adding a new vertex $s$ adjacent to every vertex of $C$ and another new vertex $t$ adjacent to every vertex of $C'$ without destroying 2-connectivity, and then applying Menger's Theorem to $s$ and $t$). Let $x_i$ and $y_i$ be the ends of $P_i$ on $C$ and $C'$, respectively.

If $x_1$ and $x_2$ are not consecutive along $C$, let $y'$ be a third vertex on $C'$, and let $e$ be any edge of the subpath of $C'$ from $y_1$ to $y_2$ that avoids $y'$, as

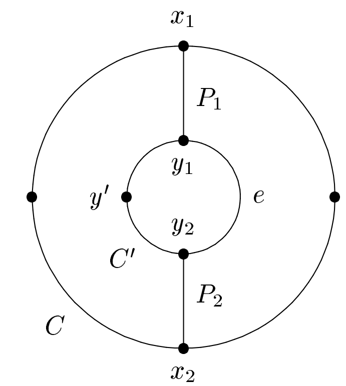


Figure 10: An illustration for the proof of Property 9.

illustrated in Figure 10. Then $e$ belongs to three chordless cycles of the minor $C \cup C' \cup P_1 \cup P_2$, contrary to Lemma 1.

Therefore we may assume that $x_1$ and $x_2$ are consecutive along $C$, and similarly $y_1$ and $y_2$ are consecutive along $C'$. Then since the edge $x_1x_2$ of $C$ is not critical by Property 3, $G - \{x_1, x_2\}$ has a shortest path $P_3$ from $C$ to $C'$. Let $x_3$ and $y_3$ be the ends of $P_3$ on $C$ and $C'$, respectively. If $P_3$ and $P_1 \cup P_2$ are disjoint, then since $C$ has at least four vertices, we can forget $P_1$ or $P_2$ and then we are back to the previous case. Otherwise, let $z$ be the first vertex of $P_3$ that belongs to $P_1 \cup P_2$. We may assume without loss of generality that $z$ is on $P_1$, as illustrated in Figure 11.

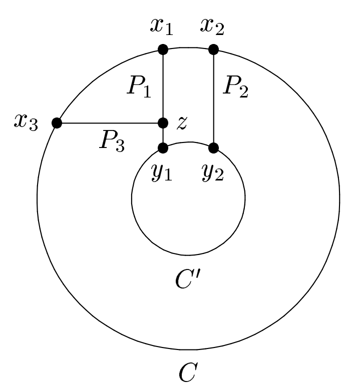


Figure 11: An illustration for the proof of Property 9.

Consider the minor $M = C \cup C' \cup P_1 \cup P_2 \cup P_3[x_3, z]$ of $G$. The edge $y_1y_2$ is on three chordless cycles of $M$, namely $C'$, $P_1 \cup P_2 \cup \{x_1x_2, y_1y_2\}$, and

$P_3[x_3, z] \cup P_1[z, y_1] \cup \{y_1 y_2\} \cup P_2 \cup P'$, where $P'$ is the subpath of $C$ from $x_2$ to $x_3$ that avoids $x_1$. This contradicts Lemma 1, thereby proving Property 9.

By Property 8 and Property 9, $G - V(C)$ is a tree $T$.

**Property 10** *$T$ must be a path.*

If $T$ is not a path, it has a vertex $v$ such that $\deg_T(v) \geq 3$. By Property 7, $N_C(T)$ has two non-consecutive vertices $a$ and $b$ along $C$. The forest $T - v$ has connected components $T_1$ and $T_2$ (possibly identical) such that $\{a\} \subseteq N_C(T_1) \cup N_C(v)$ and $\{b\} \subseteq N_C(T_2) \cup N_C(v)$. Let $T_3$ be a connected component of $T - v$ distinct from $T_1$ and $T_2$. Since $v$ is not a cut vertex of $G$ by 2-connectivity, there exists a vertex $c \in N_C(T_3)$. Let $P$ be a path from $v$ to $c$ via $T_3$, as illustrated in Figure 12.

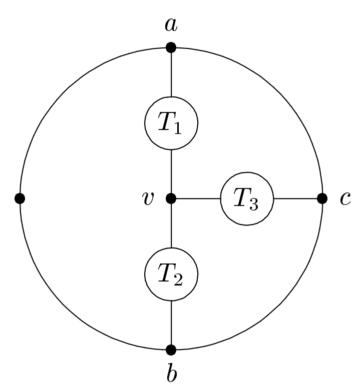


Figure 12: An illustration for the proof of Property 10.

We contract $T - T_3$ to a single vertex $w$, which becomes an end of $P$. Consider the minor $M = C \cup P \cup \{wa, wb\}$ of $G$. If $c \neq a, b$, then, as illustrated in Figure 13 (a), $M$ is a bad subdivision of $K_4$, which is not upper 2-cycled. Otherwise we may assume that $c = a$, as illustrated in Figure 13 (b), and we contract the edge $wb$ of $M$ to obtain a subdivision of $K_2 \oplus 3K_1$, which is not upper 2-cycled. In both cases, Lemma 1 is contradicted.

**Property 11** *If $u$ and $v$ are the two ends of the path $T$ and $u \neq v$, then $N_C(u)$ and $N_C(v)$ are nonempty, and each of $N_C(T - u)$ and $N_C(T - v)$ consists of either a single vertex or a pair of consecutive vertices along $C$.*

The sets $N_C(u)$ and $N_C(v)$ (and hence also $N_C(T - u)$ and $N_C(T - v)$) are nonempty by 2-connectivity. Assume that $N_C(T - v)$ contains non-consecutive vertices $a$ and $b$ along $C$. We contract $T - v$ to a vertex $w$,

$c$



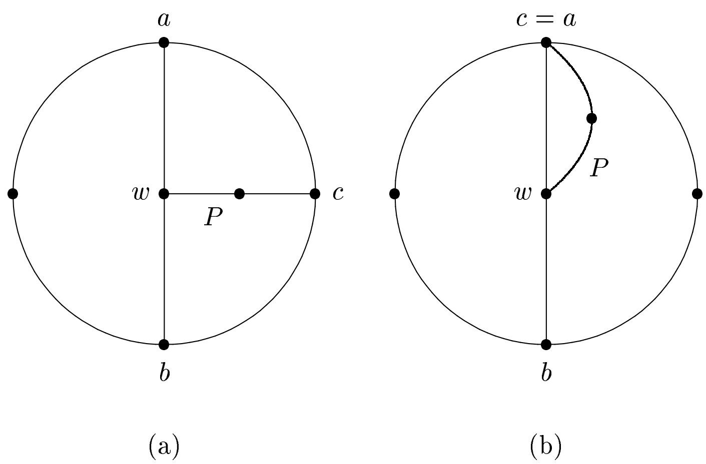


Figure 13: Illustrations for the proof of Property 10. (a): $c \neq a, b$; (b): $c = a$.

and let $P$ be a path from $w$ to $C$ via $v$. Then we argue about the minor $M = C \cup P \cup \{wa, wb\}$ as in Property 10. Similarly, $N_C(T-u)$ has no non-consecutive vertices along $C$.

Now we are ready to list all the possible 2-connected 2-cycled graphs, and thereby prove Theorem 1, by considering all possibilities for $T$.

*Case 1:* $T$ is a single vertex $v$. Then $G$ is a rim-subdivision of a wheel $W_k$ with $k \geq 3$ if $\deg_G(v) \geq 3$, and by Property 7 $G$ is a subdivision of $K_{2,3}$ if $\deg_G(v) = 2$.

*Case 2:* $V(T) = \{u, v\}$. If each of $N_C(u)$ and $N_C(v)$ has only one vertex, then the two vertices are distinct and non-consecutive by Property 7, so $G$ is a subdivision of $K_{2,3}$. If one of $N_C(u)$ and $N_C(v)$ has one vertex and the other has two (necessarily consecutive by Property 11), then $N_C(u) \cap N_C(v) = \emptyset$ by Property 7 and so $G$ is a star-subdivision of $K_4$. If both $N_C(u)$ and $N_C(v)$ (distinct by Property 7) have two vertices (necessarily consecutive by Property 11), then $G$ is a triangles-joining or a rim-subdivision of the wheel $W_4$ depending on whether $N_C(u) \cap N_C(v)$ is empty or not.

*Case 3:* $T$ is a path of length at least 2, with ends $u, v$. Neither $N_C(T-u)$ nor $N_C(T-v)$ contains a pair of non-consecutive vertices of $C$ by Property 11, whereas $N_C(T)$ does by Property 7. So $N_C(u) \cup N_C(v)$ must contain a pair $a, b$ of non-consecutive vertices, with $a \in N_C(u)$ and $b \in N_C(v)$. Moreover, $N_C(x)$ does not meet $\{a, b\}$ for any internal vertex $x$ of $T$. If $y \in N_C(x)$ for some internal vertex $x$ of $T$, then by the above and Property 11, $y$ must be different from and adjacent to both $a$ and $b$. For the same reason, $N_C(u) \subseteq \{y, a\}$ and $N_C(v) \subseteq \{y, b\}$, and $N_C(z) \subseteq \{b, y\} \cap \{a, y\} = \{y\}$ for all internal vertices $z$ of $T$. Therefore $G$ must be a rim-subdivision of a wheel with center $y$. If $N_C(x)$ is empty for every internal vertex $x$ of $T$, then the argument is similar to the one of Case 2. ■

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