A SYMMETRIC FUNCTIONS APPROACH TO STOCKHAUSEN'S PROBLEM

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Abstract. We consider problems in sequence enumeration suggested by Stockhausen's problem, and derive a generating series for the number of sequences of length k on n available symbols such that adjacent symbols are distinct, the terminal symbol occurs exactly r times, and all other symbols occur at most $r-1$ times. The analysis makes extensive use of techniques from the theory of symmetric functions. Each algebraic step is examined to obtain information for formulating a direct combinatorial construction for such sequences.

1. Introduction

The score of the piano work *nr. 7 Klavierstück XI* by Karlheinz Stockhausen (1957) [S] consists of 19 fragments of music. The performer is instructed to choose at random one of these fragments, and play it; then to choose another, different, fragment and play that, and so on. If a fragment is chosen that has already been played twice, the performance ends. We can state a more general problem as follows: Denote each fragment of music by a symbol. Then an *r-Stockhausen sequence* on n symbols is a sequence such that

(1) adjacent symbols are distinct,
(2) the terminal symbol occurs exactly r times,
(3) no symbol occurs more than r times,
(4) exactly one symbol occurs r times,
(5) the symbol alphabet is $\mathcal{N}_n = \{1, 2, \ldots, n\}$.

The original problem posed by Stockhausen is then the case when $r = 3$ and $n = 19$. We shall refer to 3-Stockhausen sequences simply as *Stockhausen sequences.* We let $c_r(n,k)$ be the number of r-Stockhausen sequences of length k on n symbols, and

$$s_r(n) := \sum_{k=2r-1}^{(r-1)n+1} c_r(n,k),$$

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Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX for the minimum (resp. maximum) length of an r-Stockhausen sequence is $2r-1$ (resp. $(r-1)n+1$). The expected length of a performance under the assumption that each fragment is equally likely is given in [RY].

The method of generating series is a major technique in enumeration. In this case, the determination of the exact number of Stockhausen sequences on n symbols is an enumerative question that can be approached using the theory of symmetric functions for coefficient extraction in the generating series approach. A generalization of the problem leads to a combinatorial construction for the Stockhausen sequences. Although only known techniques [G] are used, these are sophisticated, and the problem serves as a useful study of these techniques, and the algebraic analysis gives partial information about the formulation of a bijective proof.

The paper is organized as follows. In Section 2, we define a few classical symmetric functions, and state some properties they satisfy. Section 3 contains the derivation of the generating series for the number of Stockhausen sequences. In Section 4 we generalize the method given in Section 3 and give an analogous derivation for the generating series for the number of r-Stockhausen sequences. By examining the generating series for r-Stockhausen sequences, we give in Section 5 a series of combinatorial constructions that leads to r-Stockhausen sequences.

2. Symmetric Functions

We review some classical symmetric functions, and recall some properties of the ring of symmetric functions. The reader is directed to [M] for a fuller account.

A function $f(x_1, x_2, \dots)$ in infinitely many indeterminates is a *symmetric function* if f is invariant under any permutation of any finite number of the variables. We shall consider such symmetric functions over the rationals. The following symmetric functions are used in the derivation of the generating series for $c_r(n,k)$.

The *complete* symmetric function h_n is defined by

$$h_n(x_1, x_2, \dots) = \sum_{i_1 \le i_2 \le \cdots \le i_n} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

Moreover, $h_0 = 1$. If $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ is a partition, i. e. a non-increasing sequence of positive integers, we let h_λ denote $h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_k}$. The weight of λ is $|\lambda| := \sum_i \lambda_i$.

The *monomial* symmetric function m_λ is the sum of all distinct monomials of the form $x_{i_1}^{\lambda_1} \cdots x_{i_k}^{\lambda_k}$, where $i_1, \dots, i_k$ are distinct.

The *power sum* symmetric function p_n is defined by

$$p_n(x_1, x_2, \dots) = \sum_i x_i^n.$$

Similar to h_λ, p_λ is defined to be $p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_k}$.

It is known that [M] each of the sets $\{h_\lambda\}$, $\{m_\lambda\}$, and $\{p_\lambda\}$, where λ ranges over all partitions of n, is a basis of the vector space over $\mathbb{Q}$ of symmetric functions homogeneous of degree n.

Let $m_j(\lambda)$ denote the number of j's in the partition λ. The complete symmetric function h_n can be expressed in terms of the power sum symmetric functions as follows:

$$h_n = \sum_{\lambda \vdash n} z^{-1}(\lambda) p_\lambda,$$

where

$$z(\lambda) = \Big(\prod_{j \geq 1} j^{m_j(\lambda)} m_j(\lambda)!\Big)$$

and $\lambda \vdash n$ indicates that λ is a partition of n. For instance, $h_1 = p_1$, $h_2 = (p_1^2 + p_2)/2$, and $h_3 = (p_1^3 + 3p_2p_1 + 2p_3)/6$.

There is [M] an inner product $\langle \cdot, \cdot \rangle$ defined on the vector space of symmetric functions such that $\langle m_\lambda, h_\mu \rangle = \delta_{\lambda,\mu}$, so $\langle p_\lambda, p_\mu \rangle = z(\lambda)\delta_{\lambda,\mu}$ for any two partitions λ and μ, where $\delta_{\lambda,\mu}$ is 1 if $\lambda = \mu$ and zero otherwise. It follows that if f and g are symmetric functions, then $\langle p_n f, g \rangle = \langle f, n\frac{\partial g}{\partial p_n} \rangle$, so the action adjoint to multiplication by p_n is $n\frac{\partial}{\partial p_n}$.

For any partition λ, let $l(\lambda)$ denote the number of parts of λ. Then the number of monomials on n $(\geq l(\lambda))$ symbols of the form $x_{i_1}^{\lambda_1} \cdots x_{i_{l(\lambda)}}^{\lambda_{l(\lambda)}}$, where $i_1, \ldots, i_{l(\lambda)}$ are distinct, is

(2.1)
$$\frac{n!}{(n - l(\lambda))!\, m_1(\lambda)!\, m_2(\lambda)! \cdots m_{\lambda_1}(\lambda)!} = m_\lambda(\underbrace{1, \ldots, 1}_{n \text{ 1's}}, 0, \ldots) = \frac{(n)_{l(\lambda)}}{m_1(\lambda)!\, m_2(\lambda)! \cdots m_{\lambda_1}(\lambda)!},$$

where $(n)_j = n(n-1)\cdots(n-j+1)$ denotes the jth falling factorial.

The following proposition is needed and can be regarded as the adjoint form of Taylor's Theorem on the ring of symmetric functions.

Proposition 2.1. *Let f and g be symmetric functions and u a real number, and suppose that $f = f(p_1, p_2, \ldots)$. Then*

$$\langle f(p_1, p_2, \ldots), e^{up_j} g \rangle = \langle f(p_1, p_2, \ldots, p_{j-1}, p_j + ju, p_{j+1}, p_{j+2}, \ldots), g \rangle.$$

Proof. We use the adjoint action to multiplication by p_j,

$$\begin{aligned}
\langle f(p_1, p_2, \ldots), e^{up_j} g \rangle &= \sum_{k \geq 0} \frac{u^k}{k!} \left\langle j^k \frac{\partial^k}{\partial p_j^k} f(p_1, p_2, \ldots), g \right\rangle \\
&= \left\langle \sum_{k \geq 0} \frac{(uj)^k}{k!} \frac{\partial^k}{\partial p_j^k} f(p_1, p_2, \ldots), g \right\rangle \\
&= \left\langle \exp\left(ju \frac{\partial}{\partial p_j} \right) f(p_1, p_2, \ldots), g \right\rangle \\
&= \langle f(p_1, p_2, \ldots, p_j + ju, \ldots), g \rangle
\end{aligned}$$

using the formal version of Taylor's Theorem. $\square$

3. A solution for the original problem

We derive a generating series for the number $c_3(n,k)$ of Stockhausen sequences on n symbols of length k. To reach this goal, we begin with the generating series for sequences satisfying condition (1) for Stockhausen sequences (Lemma 3.1), then we derive the generating series of sequences satisfying conditions (1) and (2) for $r = 3$ (Proposition 3.2); and finally we address the generating series for Stockhausen sequences by imposing appropriate restrictions in order to satisfy conditions (3) and (4).

Lemma 3.1. *The generating series for all strings on n symbols such that adjacent symbols are different is*

$$D(z, x_1, \ldots, x_n) = \frac{1}{1 - \sum_{i=1}^{n} \frac{zx_i}{1+zx_i}},$$

where z is an ordinary marker for the length of the string and x_i marks the occurrence of symbol i.

See [RY,§2] for a proof.

Proposition 3.2. *Let $D_n(z, x_1, \ldots, x_n)$ be the generating series of sequences on the symbols $\{1, 2, \ldots, n\}$ (marked by $x_1, \ldots, x_n$) of length k (marked by z) such that the terminal symbol occurs exactly three times (non-terminal symbols occur arbitrarily many times) and adjacent symbols are distinct. Then*

$$D_n(z, x_1, \ldots, x_n) = z^3 \left(\sum_{j=1}^{n} x_j^3 \frac{\left(\sum_{\substack{i=1 \\ i \neq j}}^{n} \frac{zx_i}{1+zx_i} \right)^2}{\left(1 - \sum_{\substack{i=1 \\ i \neq j}}^{n} \frac{zx_i}{1+zx_i} \right)^3} \right).$$

Proof. Consider such a sequence which terminates with the symbol j. The sequence decomposes into $AjBjB'j$, where A, B, and B' are strings on the symbols $\{1, 2, \ldots, n\} \setminus \{j\}$ with distinct adjacent elements, A possibly empty but B and B' non-empty. By Lemma 3.1, the generating series for all such sequences is

$$z^3 x_j^3 L_w \frac{1}{1 - w \sum_{\substack{i=1 \\ i \neq j}}^{n} \frac{zx_i}{1+zx_i}},$$

where L_w is the linear transform defined by

$$L_w f(w) = \left. \frac{1}{2!} \frac{\partial^2 f}{\partial w^2} \right|_{w=1}.$$

Although we introduce L_w here to simplify the expression for the generating series, it will be seen later that it has combinatorial significance. To get the sequences described in the statement, we take the union over j, and so, summing the above generating series over j we obtain

$$D_n = z^3 \sum_{j=1}^{n} x_j^3 L_w \frac{1}{1 - w \sum_{\substack{i=1 \\ i \neq j}}^{n} \frac{zx_i}{1+zx_i}}. \quad \square$$

We remark that a very easy direct proof is possible, but L_w is introduced in order to illustrate how its counterpart, $L_w^{(r)}$, will be used in Section 4, where for $r > 3$, this linear transform helps in carrying out the calculations with symmetric functions.

To restrict the frequency of occurrence of symbols, we let $\mathcal{P}_3$ be the set of all partitions with exactly one 3 as the largest part. For α in $\mathcal{P}_3$, let x^α denote $x_1^{\alpha_1} x_2^{\alpha_2} \cdots$ and let $[x^\alpha]f$ denote the coefficient of x^α in f; then

$$[x^\alpha]D_n = [x^\alpha]z^3 p_3 L_w \frac{1}{1 - w(zp_1 - z^2 p_2)}$$

for the power sums p_1, p_2, p_3 in $x_1, x_2, \dots, x_n$.

Let

$$G(z, x_1, \dots, x_n) = z^3 p_3 L_w \frac{1}{1 - w(zp_1 - z^2 p_2)}, \tag{3.1}$$

where p_1, p_2, and p_3 are power sum symmetric functions in infinitely many indeterminates. Then

$$[x^\alpha]D_n = [x^\alpha]G(z, x_1, x_2, \dots, x_n, 0, 0, \dots).$$

Therefore the generating series for the set of all Stockhausen sequences on n symbols is

$$F_n(z, x_1, \dots, x_n) = \sum_{\alpha \in \mathcal{P}_3} \sum_{\beta} [x^\beta] G(z, x_1, x_2, \dots, x_n, 0, 0, \dots),$$

where the inner sum is over all distinct permutations β of α.

The sequences defined in the statement of Proposition 3.2 satisfy conditions (1) and (2). Conditions (3) and (4) are imposed by the restriction that all partitions α are in $\mathcal{P}_3$. The last condition is imposed by evaluating G with $x_{n+1} = x_{n+2} = \cdots = 0$ to exclude $n+1, n+2, \dots$ from the alphabet, and by the sum over β so that each element of the alphabet $\mathcal{N}_n$ is permitted to occur.

Since G is a symmetric function in the x_i's, we may expand it in terms of the monomial symmetric functions:

$$G = \sum_{\theta \in \mathcal{P}_3} m_\theta(x_1, x_2, \dots) a_\theta z^{|\theta|},$$

where the a_θ's are scalars.

Then another way of expressing F_n is

$$F_n = \sum_{\alpha \in \mathcal{P}_3} m_\alpha(\underbrace{1, 1, \dots, 1}_{n}, 0, 0, \dots)[m_\alpha]G = \sum_{\alpha \in \mathcal{P}_3} \frac{(n)_{l(\alpha)}}{m_1(\alpha)!\, m_2(\alpha)!} a_\alpha z^{|\alpha|},$$

because from (2.1) $m_\alpha(\underbrace{1, 1, \dots, 1}_{n}, 0, 0, \dots)$ is the number of terms in the monomial symmetric function m_α on $x_1, x_2, \dots, x_n$, which is equal to the number of sequences over $\{0, 1, 2, 3\}$ of length n with $n - l(\alpha)$ occurrences of 0, $m_1(\alpha)$ occurrences of 1, $m_2(\alpha)$ occurrences of 2 and one occurrence of 3 (as specified by $\mathcal{P}_3$).

We use the property $\langle m_\alpha, h_\theta \rangle = \delta_{\alpha,\theta}$ of the inner product to extract the coefficient of m_α in G. In order to have $m_1(\alpha)!$ and $m_2(\alpha)!$ in the denominator and one part of size 3 in α, we consider

$$uh_3 e^{u(h_1+h_2)} = \sum_{\alpha \in \mathcal{P}_3} h_\alpha \frac{u^{l(\alpha)}}{m_1(\alpha)!\, m_2(\alpha)!}.$$

Clearly,

$$\left\langle G, uh_3 e^{u(h_1+h_2)} \right\rangle = \sum_{\alpha \in \mathcal{P}_3} \frac{u^{l(\alpha)}}{m_1(\alpha)!\, m_2(\alpha)!} a_\alpha z^{|\alpha|}.$$

With the help of the mapping $\Xi_n \colon u^j \mapsto (n)_j$, for $j = 0, 1, 2, \ldots$, extended linearly to the power series ring in u, we obtain

$$F_n = \sum_{k \geq 0} c_3(n,k) z^k = \Xi_n \left\langle G, uh_3 e^{u(h_1+h_2)} \right\rangle.$$

We are now in a position to work on $\Phi_3(z,u) := \sum_n F_n(z) u^n/n!$, our goal for this section.

Lemma 3.3. *Let c be independent of u. Then $\Xi_n e^{uc} u^k = (n)_k (1+c)^{n-k}$. If, moreover, $g(u) = 1 + g_1 u + g_2 u^2 + \cdots$ is a power series in u, then $\Xi_n e^{uc} g(u) = \left[\frac{u^n}{n!}\right] e^{u(1+c)} g(u)$.*

Proof. The first statement follows from direct computation. Now

$$\Xi_n e^{uc} g(u) = \Xi_n \sum_{k \geq 0} e^{uc} g_k u^k = \sum_{k \geq 0} g_k \cdot (n)_k (1+c)^{n-k} = \sum_{k=0}^{n} g_k \tfrac{n!}{(n-k)!} (1+c)^{n-k},$$

and the second statement follows. Note that $(n)_k = 0$ if $n < k$. $\square$

It now follows that the generating series for $c_3(n,k)$ is

Corollary 3.4.

$$\Phi_3(z,u) := \sum_{k,n \geq 0} c_3(n,k) z^k \tfrac{u^n}{n!} = e^u \langle G, uh_3 e^{u(h_1+h_2)} \rangle.$$

Proof. We know that

$$\sum_{k \geq 0} c_3(n,k) z^k = \langle G, \Xi_n uh_3 e^{u(h_1+h_2)} \rangle = \langle G, \left[\frac{u^n}{n!}\right] e^u uh_3 e^{u(h_1+h_2)} \rangle$$

by Lemma 3.3. Since G is independent of u, the last expression is

$$\left[\frac{u^n}{n!}\right] \langle G, e^u uh_3 e^{u(h_1+h_2)} \rangle = \left[\frac{u^n}{n!}\right] e^u \langle G, uh_3 e^{u(h_1+h_2)} \rangle.$$

It follows that

$$\Phi_3(z,u) = e^u \langle G, uh_3 e^{u(h_1+h_2)} \rangle.$$

To evaluate the inner product, we expand G and $uh_3e^{u(h_1+h_2)}$ in terms of the power sums, and use the property $\langle p_\lambda, p_\mu \rangle = z(\lambda)\delta_{\lambda,\mu}$ of the inner product, the adjoint action to multiplication by p_j, and Taylor's Theorem on the ring of symmetric functions. We know that upon substitution of G using (3.1)

$$\Phi_3(z,u) = e^u \left\langle L_w \frac{z^3 p_3}{1-w(zp_1 - z^2p_2)}, \frac{u}{6}(p_1^3 + 3p_2p_1 + 2p_3)e^{u(p_1+(p_1^2+p_2)/2)} \right\rangle$$

by linearity of $\langle , \rangle$ and ignoring terms in the second argument that are independent of p_3, we get

$$= L_w e^u \left\langle \frac{z^3 p_3}{1-w(zp_1 - z^2p_2)}, \frac{u}{3} p_3 e^{u(p_1+(p_1^2+p_2)/2)} \right\rangle$$

by adjoint action of p_3, we have

$$= L_w e^u \left\langle \frac{z^3}{1-w(zp_1 - z^2p_2)}, u e^{u(p_1+(p_1^2+p_2)/2)} \right\rangle$$

using Proposition 2.1 we obtain

$$= L_w e^u \left\langle \frac{z^3}{1-w(z(p_1+u) - z^2(p_2+u))}, u e^{up_1^2/2} \right\rangle$$

again by the property of the inner product we ignore terms in the first argument independent of p_1 and reach

$$= L_w e^u \left\langle \frac{z^3}{1-wzu+wz^2u-wzp_1}, u e^{up_1^2/2} \right\rangle.$$

Since

$$\frac{1}{1-wzu+wz^2u-wzp_1} = \sum_{j\geq 0} \frac{(wzp_1)^j}{(1-wzu+wz^2u)^{j+1}},$$

we have that

$$\begin{aligned}\Phi_3(z,u) &= z^3 u e^u L_w \sum_{j\geq 0} \frac{(wz)^j}{(1-wzu+wz^2u)^{j+1}} \left\langle p_1^j, e^{up_1^2/2} \right\rangle \\ &= z^3 u e^u L_w \sum_{j\geq 0} \frac{(wz)^{2j}}{(1-wzu+wz^2u)^{2j+1}} \left\langle p_1^{2j}, e^{up_1^2/2} \right\rangle \\ &= z^3 u e^u L_w \sum_{j\geq 0} \frac{(wz)^{2j}}{(1-wzu+wz^2u)^{2j+1}} \left\langle p_1^{2j}, u^j \frac{p_1^{2j}}{2^j j!} \right\rangle \\ &= z^3 u e^u L_w \sum_{j\geq 0} \frac{u^j (wz)^{2j}}{(1-wzu+wz^2u)^{2j+1}} \frac{(2j)!}{2^j j!}.\end{aligned}$$

We summarize the result in the following.

Proposition 3.5. *Let* $L_w^{(r)}(f(w)) = \frac{1}{r!} \left.\frac{\partial^r f}{\partial w^r}\right|_{w=1}$, *then*

$$\Phi_3(z,u) = z^3 u e^u L_w \sum_{j\geq 0} \frac{u^j (wz)^{2j}}{(1-wzu+wz^2u)^{2j+1}} \frac{(2j)!}{2^j j!}.$$

We remark that the original Stockhausen number $s_3(n)$ is

$$\left[\frac{u^n}{n!}\right] \Phi_3(1,u) = \left[\frac{u^n}{n!}\right] \tfrac{1}{2} u e^u \sum_{j\geq 0} (2j)(2j-1) \frac{(2j)!}{2^j j!} u^j,$$

so

$$s_3(n) = n \sum_{j=0}^{n-1} \binom{n-1}{j} \binom{2j}{2} \frac{(2j)!}{2^j}.$$

A direct combinatorial proof of this expression is given in [RY].

4. A generating series for r-Stockhausen sequences

We use the method suggested by the previous section to derive a generating series for r-Stockhausen sequences.

Theorem 4.1. *Let*

$$\Phi_r(z,u) := \sum_{n,k\geq 0} c_r(n,k) z^k \frac{u^n}{n!}$$

be the generating series for the number $c_r(n,k)$ of sequences of length k on n symbols such that adjacent symbols are distinct, the terminal symbol occurs exactly r times, and all other symbols occur at most $r-1$ times. Then

$$\Phi_r(z,u) = u z^r L_w^{(r-1)} \Theta_w \exp\left(u[t^{r-1}] \frac{e^{wzt/(1+zt)}}{1-t} \right),$$

where z is the ordinary marker for the length of the sequence, u is the exponential marker for the number of available symbols, $L_w^{(r)}(f(w)) = \frac{1}{r!}\frac{\partial^r f}{\partial w^r}\Big|_{w=1}$ and Θ_w is the inverse Laplace transform $\Theta_w\colon w^j \mapsto j!\,w^j$.

We may take advantage of the analysis of the original problem (when $r=3$).

Proof. Let

$$F_{n,r} := z^r p_r \frac{\left(\sum_{j\ge1}(-1)^{j-1}z^j p_j\right)^{r-1}}{\left(1-\sum_{j\ge1}(-1)^{j-1}z^j p_j\right)^{r}}.$$

Then

$$c_r(n,k) = [z^k]\Xi_n\left\langle F_{n,r}, uh_r e^{u(h_1+h_2+\cdots+h_{r-1})}\right\rangle.$$

Now

$$F_{n,r} = z^r L_w^{(r-1)} p_r\Big(1-w\sum_{j\ge1}(-1)^{j-1}z^j p_j\Big)^{-1},$$

and

$$\begin{aligned}\sum_{k\ge0} c_r(n,k)z^k &= \Xi_n z^r L_w^{(r-1)}\left\langle p_r\Big(1-w\sum_{j\ge1}(-1)^{j-1}z^j p_j\Big)^{-1}, uh_r e^{u(h_1+h_2+\cdots+h_{r-1})}\right\rangle\\ &= \left[\frac{u^n}{n!}\right] z^r L_w^{(r-1)}\left\langle p_r\Big(1-w\sum_{j\ge1}(-1)^{j-1}z^j p_j\Big)^{-1}, uh_r e^{u(1+h_1+h_2+\cdots+h_{r-1})}\right\rangle\\ &= \left[\frac{u^n}{n!}\right]\Phi_r(z,u),\end{aligned}$$

by definition. So

$$\Phi_r(z,u) = uz^r L_w^{(r-1)}\left\langle p_r\Big(1-w\sum_{j\ge1}(-1)^{j-1}z^j p_j\Big)^{-1}, h_r e^{u(1+h_1+h_2+\cdots+h_{r-1})}\right\rangle. \tag{4.1}$$

In order to extract the coefficients from (4.1), we perform some technical maneuvers using the properties of symmetric functions stated in Section 2.

It is convenient for expository purposes to isolate these into a series of steps.

Step 1. Apply the adjoint action to multiplication by p_j to remove p_j from the first argument of the inner product.

Since $h_r = \sum_{\alpha\in\mathcal{P}} z^{-1}(\alpha)p_\alpha$, where $z(\alpha) = 1^{m_1(\alpha)}m_1(\alpha)!\,2^{m_2(\alpha)}m_2(\alpha)!\cdots$, it follows that $\frac{\partial h_r}{\partial p_r} = \frac{1}{r}$. Also, $e^{u(1+h_1+\cdots+h_{r-1})}$ expanded in terms of the power sums using the formula above is independent of p_r, so

$$\begin{aligned}\Phi_r &= uz^r L_w^{(r-1)}\left\langle \left(1-w\sum_{j\ge1}(-1)^{j-1}z^j p_j\right)^{-1}, e^{u(1+h_1+\cdots+h_{r-1})}\right\rangle\\ &= uz^r L_w^{(r-1)}\left\langle 1, \left(1-w\sum_{j\ge1}(-1)^{j-1}z^j j\frac{\partial}{\partial p_j}\right)^{-1} e^{u(1+h_1+\cdots+h_{r-1})}\right\rangle.\end{aligned}$$

Step 2. Introduce the mapping $\Theta_w\colon w^j \mapsto j!\,w^j$ to express

$$\left(1 - w\sum_{j\geq 1}(-1)^{j-1}z^j j\frac{\partial}{\partial p_j}\right)^{-1}$$

as an exponential function, and h_j's in terms of the power sums so that the ring theoretic Taylor's Theorem can be used.

Since $\left(1 - w\sum_{j\geq 1}(-1)^{j-1}z^j p_j\right)^{-1} = \Theta_w e^{w\sum_{j\geq 1}(-1)^{j-1}z^j p_j}$,

$$\Phi_r = uz^r L_w^{(r-1)}\Theta_w\left\langle 1, e^{w\sum_{j\geq 1}(-1)^{j-1}z^j j\frac{\partial}{\partial p_j}} e^{u(1+h_1+\cdots+h_{r-1})}\right\rangle$$

The generating series for the complete symmetric functions is $\sum_{k\geq 0} h_k t^k = \prod_{i=1}^{\infty}(1 - tx_i)^{-1}$, so

$$\begin{aligned}
1 + h_1 + \cdots + h_{r-1} &= [t^{r-1}]\frac{1}{1-t}\prod_{i=0}^{\infty}(1 - tx_i)^{-1}\\
&= [t^{r-1}]\frac{1}{1-t}\exp\sum_{i\geq 1}\ln(1 - tx_i)^{-1}\\
&= [t^{r-1}]\frac{1}{1-t}e^{p_1 t + p_2 t^2 + \cdots}.
\end{aligned}$$

Thus $e^{u(1+h_1+\cdots+h_{r-1})} = \exp\left(u[t^{r-1}]\frac{e^{p_1 t + p_2\frac{t^2}{2}+\cdots}}{1-t}\right)$.

Step 3. Apply Taylor's Theorem to get

$$\begin{aligned}
\Phi_r &= uz^r L_w^{(r-1)}\Theta_w\left\langle 1, e^{w\sum_{j\geq 1}(-1)^{j-1}z^j j\frac{\partial}{\partial p_j}}\exp\left(u[t^{r-1}]\frac{e^{p_1 t + p_2\frac{t^2}{2}+\cdots}}{1-t}\right)\right\rangle\\
&= uz^r L_w^{(r-1)}\Theta_w\left\langle 1, \exp\left(u[t^{r-1}]\frac{e^{(wt+p_1)+(-2wz^2+p_2)\frac{t^2}{2}+\cdots}}{1-t}\right)\right\rangle\\
&= uz^r L_w^{(r-1)}\Theta_w\left\langle 1, \exp\left(u[t^{r-1}]\frac{e^{wzt - wz^2t^2+\cdots}}{1-t}\right)\right\rangle
\end{aligned}$$

since the inner product is with 1, all p_i, $i\geq 1$, are set to 0 in the second argument of the inner product.

We conclude that $\Phi_r = uz^r L_w^{(r-1)}\Theta_w\exp\left(u[t^{r-1}]\frac{e^{wzt/(1+zt)}}{1-t}\right)$. $\square$

We remark that the exponential $e^{wzt/(1+zt)}$ is the generating function for Laguerre polynomials. A short proof of the r-Stockhausen problem that is inspired by this analysis is given in [RY, §5]. An explicit formula derived from this expression for the number of r-Stockhausen sequences and a direct combinatorial proof are given in [Y].

5. Combinatorial constructions

We have shown that the generating series for generalized Stockhausen sequences is

$$\begin{aligned}\Phi_{r+1}(z,u) &:= \sum_{n,k\geq 0} c_{r+1}(n,k) z^k \frac{u^n}{n!} \\ &= uz^{r+1} L_w^{(r)} \Theta_w \exp\left(u[t^r]\frac{1}{1-t}\exp\frac{wzt}{1+zt}\right),\end{aligned}$$

where

$$L_w^{(r)}(f) = \frac{1}{r!}\left.\frac{w^r \partial^r f}{\partial w^r}\right|_{w=1}$$

and $\Theta_w : w^k \to k! w^k$.

The combinatorial nature of the transformations leads one to ask whether combinatorial objects and constructions exist for each expression so that one can obtain generalized Stockhausen sequences directly. In this section, we present a series of constructions that has generalized Stockhausen sequences as the final consequence. The set-wise actions of the constructions are suggested by the algebraic analysis, although the element-wise actions (the combinatorial content) have to be supplied. Therefore we define first the set $\mathcal{A}$ generated by

$$A = \frac{1}{1 - w(\sum_{i=1}^{n} \frac{zt_i}{1+zt_i})},$$

which is a fundamental structure in the series of constructions. Next, we convert the univariate series in t into a multivariate series in $t_1, t_2, \ldots, t_n$ to obtain n non-terminal symbols in the Stockhausen sequences. To this end, we exhibit a bijection between sets which are enumerated by

$$\left[\frac{u^n}{n!}\right] \Theta_w \exp\left(u[t^r]\frac{1}{1-t}\exp\frac{wzt}{1+zt}\right)$$

and

$$[t_1^r t_2^r \ldots t_n^r] \frac{1}{1 - w(\sum_{i=1}^{n} \frac{zt_i}{1+zt_i})} \prod_{i=1}^{n} \frac{1}{1-t_i}.$$

Once we have all n non-terminal symbols, we give a combinatorial interpretation for the transformation $L_w^{(r)}$. After the application of $L_w^{(r)}$ to A, the last step is the placement of $r+1$ copies of the terminal symbol to make $(r+1)$-Stockhausen sequences.

Step 1. Signed Divided Strings. To construct strings with distinct adjacent elements, an argument involving the principle of inclusion–exclusion can be applied. However, one can also consider *signed sets* as described in [SW]. The set of *signed divided strings* is a signed set of strings with *dividers* such that

(1) different symbols are separated by dividers,
(2) the same symbol appearing in a block may be separated by dividers,

(3) no dividers are adjacent, and no string ends with a divider,
(4) the sign of the string is

$$(-1)^{\#\text{ (undivided even sub-strings)}}.$$

An example of a typical signed divided string is $+t_1t_1wt_1wt_2t_2wt_3wt_1$, where w marks the divider and t_1, t_2 and t_3 are symbols.

Let $\mathcal{A}$ be the set of strings on n symbols such that a non-empty string begins with a divider. Let the symbols be $zt_1, zt_2, \ldots, zt_n$, and let w be an ordinary marker for dividers. Then the ordinary generating series for $\mathcal{A}$ is

$$A(w, z, t_1, \ldots, t_n) = \frac{1}{1 - w(\sum_{i=1}^{n} \frac{zt_i}{1+zt_i})}.$$

Step 2. A Bijection. We define sets generated by

$$[t_1^r t_2^r \ldots t_n^r] \frac{1}{1 - w(\sum_{i=1}^{n} \frac{zt_i}{1+zt_i})} \prod_{i=1}^{n} \frac{1}{1-t_i},$$

and

$$\left[\frac{u^n}{n!}\right] \Theta_w \exp\left(u[t^r]\frac{1}{1-t}\exp\frac{wzt}{1+zt}\right),$$

and show a bijection between the sets. From the previous section, the series

$$A(w, z, t_1, \ldots, t_n) = \frac{1}{1 - w(\sum_{i=1}^{n} \frac{zt_i}{1+zt_i})}$$

generates all, possibly empty, signed divided strings where a non-empty string begins with a divider marked by w. The multiplication of A by $\prod_{i=1}^{n} \frac{1}{1-t_i}$ and the extraction of the coefficient of $t_1^r t_2^r \cdots t_n^r$ in the result corresponds at the set theoretic level to the concatenation of the signed divided strings in $\mathcal{A}$ with $t_1^{i_1} t_2^{i_2} \cdots t_n^{i_n}$ for $0 \leq i_1, i_2, \ldots, i_n$ and the restriction to only those concatenated strings with exactly r t_i's, $i = 1, 2, \ldots, n$. But this is equivalent to considering the strings in $\mathcal{A}$ with no more than r t_i's for $i = 1, 2, \ldots, n$. Thus we conclude that

$$[t_1^r t_2^r \ldots t_n^r] \frac{1}{1 - w(\sum_{i=1}^{n} \frac{zt_i}{1+zt_i})} \prod_{i=1}^{n} \frac{1}{1-t_i}$$

generates all signed divided strings ($\subset \mathcal{A}$) with no more than r t_i's ($i = 1, \ldots, n$) in each string.

For

$$\left[\frac{u^n}{n!}\right] \Theta_w \exp\left(u[t^r]\frac{1}{1-t}\exp\frac{wzt}{1+zt}\right),$$

or equivalently,

$$\Theta_w \left([t^r] \frac{1}{1-t} \exp \frac{wzt}{1+zt} \right)^n ,$$

where Θ_w is an inverse Laplace transform on w, i. e. $\Theta_w \colon w^k \mapsto k!\, w^k$, we describe a set generated by the inner expression

$$\left([t^r] \frac{1}{1-t} \exp \frac{wzt}{1+zt} \right)^n$$

first, and then apply Θ_w combinatorially to the set.

Consider a labelled ordered n-tuple $(S_1, S_2, \ldots, S_n)$ of possibly empty, ordered sets of all possible sizes. That is, given a label set ($\subset \mathcal{N} := \{1, 2, \ldots\}$) of size n, label S_1 with the smallest label, S_2 with the next label, etc, and S_n with the largest one. Every coordinate in S_i, $i \in \mathcal{N}_n$, is also labelled from a label set $\subset \mathcal{N}$ in the same way. A typical entry of S_i has the form $(-1)^{k+1} w_x (zt)^k$ $(k \geq 1)$, where x is the label associated with that entry of S_i. Let w be an exponential marker for the labels w_x's. When we restrict the number of t's in each S_i to no more than r, then the generating series for S_i is

$$[t^r] \frac{1}{1-t} \frac{w^{|S_i|}}{|S_i|!} \left(\frac{zt}{1+zt} \right)^{|S_i|} .$$

Summing over all $|S_i|$ from 0 to ∞, we get

$$\left([t^r] \frac{1}{1-t} \exp \frac{wzt}{1+zt} \right)^n$$

as the generating series for the n tuple of tuples. To apply Θ_w, or inverse Laplace transform, i. e. removal of labels marked by w, we form signed divided strings in the following way. Take the entry in some S_x that has the label w_1, and put down the entire entry including its sign as the first segment, $w_1 z t_i z t_i \cdots z t_i$ writing t_i instead of t if the set S_x has the symbol i assigned to it. In the second segment, find some set S_y that contains w_2, and write $w_2 z t_j \ldots z t_j$ if S_y has symbol j assigned to it. Continue in a similar manner until the w's are exhausted. Now combine the signs multiplicatively and erase the subscripts on w to get a signed divided string on n available symbols because there are n S_x's, such that every t_i appears no more than r times (since we start out with r or fewer t's in each S_x).

Step 3. Application of $L_w^{(r)}$. Let $\mathcal{B}$ be the set of non-empty strings on n symbols such that a non-empty string begins without a divider. Let the symbols be $zx_1, zx_2, \ldots, zx_n$, and let w be an ordinary marker for dividers. Then the ordinary generating series for $\mathcal{B}$ is

$$B(w, z, x_1, \ldots, x_n) = \frac{\sum_{i=1}^n \frac{zx_i}{1+zx_i}}{1 - w(\sum_{i=1}^n \frac{zx_i}{1+zx_i})} .$$

Let ω be a divider. Define $\omega^r \times (\frac{\partial}{\partial \omega})^r / \Sigma_r$ (see [GJ, p.36]) to be an operation on the elements of the sets $\mathcal{A}$ (as defined in Step 1) and $\mathcal{B}$ that marks r distinct dividers in all possible ways regardless of the order in which dividers are marked. (Here Σ_r denotes the symmetric group on r elements.) Then

$$(\omega^r \times (\frac{\partial}{\partial \omega})^r/\Sigma_r)\mathcal{A} \xrightarrow{\sim} \mathcal{A}\hat{\omega}\,\underbrace{\mathcal{B}\hat{\omega}\mathcal{B}\hat{\omega}\cdots\hat{w}\mathcal{B}}_{r\ \mathcal{B}\text{'s}}$$

by induction on r, since

$$\frac{\omega\partial}{\partial\omega}\mathcal{A} \xrightarrow{\sim} \mathcal{A}\hat{\omega}\mathcal{B} \quad \text{and} \quad \frac{\omega\partial}{\partial\omega}\mathcal{B} \xrightarrow{\sim} \mathcal{B}\hat{\omega}\mathcal{B}.$$

Notice that $A(1, zx_1, \ldots, zx_n)$ (respectively $B(1, zx_1, \ldots, zx_n)$) is the ordinary generating series for possibly empty (respectively non-empty) strings on n symbols $zx_1, \ldots, zx_n$ such that adjacent symbols are distinct. Therefore, the operator $L_w^{(r)}$ applied to A gives the generating series for $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \ldots, \tilde{\mathcal{B}})$ with r $\tilde{\mathcal{B}}$'s, where $\tilde{\mathcal{A}}$ (respectively $\tilde{\mathcal{B}}$) is the result of ignoring dividers in the set $\mathcal{A}$ (respectively $\mathcal{B}$). This is an $(r+1)$-tuple of strings such that adjacent symbols are distinct, and the first string (but no others) may be empty.

In short, the purpose of $L_w^{(r)}$ is first to mark ω's and secondly get rid of strings that have at least one undivided sub-string of length greater than or equal to 2 (by setting $w = 1$ algebraically or ignoring dividers, ω's, set theoretically).

Step 4. Placement of the terminal symbol. Given $n+1$ symbols such that the terminal symbol occurs $r+1$ times, let u be an exponential marker for the available symbols, z be an ordinary marker for the length of the string, and $t_1, \ldots, t_n$ be ordinary markers for the occurrence of the non-terminal symbols, then

$$uz^{r+1}L_w^{(r)}\Theta_w \exp\left(u[t^r]\frac{1}{1-t}\exp\frac{wzt}{1+zt}\right)$$

is the generating series for $\mathcal{T} * \mathcal{S}$ (see [GJ, p. 163] for the definition of $*$), where $\mathcal{T}$ is the set containing $r+1$ copies of the terminal symbol t, and $\mathcal{S}$ is $(\mathcal{A}|_{w=1}, \mathcal{B}|_{w=1}, \ldots, \mathcal{B}|_{w=1})$ with r $\mathcal{B}$'s. The $*$ product is used because the symbols $t, t_1, \ldots, t_n$ form the label set $\{1, 2, \ldots, n, n+1\}$. To obtain Stockhausen sequences from $\mathcal{T} * \mathcal{S}$, place zt in front of each $\mathcal{B}$, and at the final position, then remove all commas, and let z commute with the t's. Therefore, z marks the length of the string, and the string is faithfully represented by the t's.

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