A Plethysm Formula for $p_\mu(\underline{x}) \circ h_\lambda(\underline{x})$

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Abstract

This paper gives a new formula for the plethysm of power-sum symmetric functions and complete symmetric functions. The form of the main result is that for $\mu \vdash b$ and $\lambda \vdash a$ with length t, then

$$p_\mu(\underline{x}) \circ h_\lambda(\underline{x}) = \sum_T \underline{\omega}^{\mathrm{maj}_{\mu^t}(T)} s_{\mathrm{sh}(T)}(\underline{x})$$

where the sum is over semistandard tableaux of weight $\lambda_1^b \lambda_2^b \ldots \lambda_t^b$ and $\underline{\omega}^{\mathrm{maj}_{\mu^t}(T)}$ is a root of unity which depends on μ, t, and T.

1. Introduction

This paper gives a formula for the plethysm of power-sum symmetric functions and complete symmetric functions. In Section 1, some tableaux and symmetric function notation is given. In Section 2, the work of [**D**] is reviewed. The key result in this section is that for $\mu \vdash b$,

$$p_\mu(\underline{x}) \circ h_a(\underline{x}) = \sum_T \underline{\omega}^{\mathrm{maj}_\mu(T)} s_{\mathrm{sh}(T)}(\underline{x})$$

where the sum is over all semistandard tableaux T of weight a^b and $\underline{\omega}^{\mathrm{maj}_\mu(T)}$ is a root of unity which depends on T and μ. In Section 3, a technical result which is needed later is proven. In Section 4, the main result of the paper is proven. The form of this result is that for $\mu \vdash b$ and $\lambda \vdash a$ with length t

$$p_\mu(\underline{x}) \circ h_\lambda(\underline{x}) = \sum_T \underline{\omega}^{\mathrm{maj}_{\mu^t}(T)} s_{\mathrm{sh}(T)}(\underline{x})$$

where the sum is over semistandard tableaux T of weight $\lambda_1^b \lambda_2^b \ldots \lambda_t^b$ and $\underline{\omega}^{\mathrm{maj}_{\mu^t}(T)}$ is a root of unity which depends on μ, t, and T.

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1.1 Tableaux

A *partition* of n is a weakly decreasing sequence $\lambda = (\lambda_1 \geq \cdots \geq \lambda_l)$ of positive integers which sum to n. Either $|\lambda| = n$ or $\lambda \vdash n$ is used to denote that λ is a partition of n. The value l is called the *length* of λ and is denoted $l(\lambda)$. Let $[\lambda] = \{(i,j)\colon 1 \leq i \leq l(\lambda) \text{ and } 1 \leq j \leq \lambda_i\} \subset \mathbf{Z}^2$. The set $[\lambda]$ is the *Ferrers diagram* of λ and is thought of as a collection of boxes arranged using matrix coordinates.

A *composition* of n is a sequence $\alpha = (\alpha_1, \ldots, \alpha_l)$ of positive integers which sum to n. For notational purposes, let $a_1^{b_1} a_2^{b_2} \ldots a_l^{b_l}$ denote the composition

$$(\underbrace{a_1, \ldots, a_1}_{b_1 \text{ times}}, \underbrace{a_2, \ldots, a_2}_{b_2 \text{ times}}, \ldots, \underbrace{a_l, \ldots, a_l}_{b_l \text{ times}}).$$

Also, given a composition $\alpha = (\alpha_1, \ldots, \alpha_l)$, let α^t denote the composition

$$(\alpha_1, \ldots, \alpha_l, \ldots, \alpha_1, \ldots, \alpha_l)$$

where α is repeated t times.

A *tableau* of *shape* λ and *weight* (or *content*) $\alpha = (\alpha_1, \ldots, \alpha_l)$ is a filling of the Ferrers diagram of λ with positive integers such that i appears α_i times. A tableau is *semistandard* if its entries are weakly increasing from left to right in each row and strictly increasing down each column. The *Kostka number* $K_{\lambda,\mu}$ equals the number of semistandard tableaux of shape λ and weight μ. If $[\nu] \subseteq [\lambda]$, let $[\lambda/\nu]$ denote the *skew-shape* $[\lambda] \setminus [\nu]$. A filling of $[\lambda/\nu]$ with α_i i's is a *skew-tableau* of shape λ/ν and weight α. A semistandard skew-tableau is defined similarly.

1.2 Symmetric Functions

The symmetric function notation in this papers closely follows that of Chapter 1 in Macdonald **[M]**. Let Λ denote the ring of symmetric functions with rational coefficients in the variables $\{x_1, x_2, \ldots\}$. Let $s_\lambda(\underline{x})$, $p_\lambda(\underline{x})$, and $h_\lambda(\underline{x})$ denote the Schur symmetric functions, power-sum symmetric functions, and complete symmetric functions respectively. The ring Λ has a bilinear, symmetric, positive definite scalar product given by $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}$.

When two Schur symmetric functions are multiplied together and expanded in terms of Schur symmetric functions,

$$s_\mu(\underline{x}) s_\nu(\underline{x}) = \sum_{\lambda \vdash |\mu|+|\nu|} c_{\mu,\nu}^{\lambda} s_\lambda(\underline{x}),$$

the resulting multiplication coefficients $c_{\mu,\nu}^{\lambda}$ are nonnegative integers. These coefficients are called *Littlewood-Richardson coefficients*. See either Section I.9 of **[M]** or Section 4.9 of **[S]** for details.

Let $f(\underline{x})$ and $g(\underline{x})$ be symmetric functions. The *plethysm* of $f(\underline{x})$ and $g(\underline{x})$ is denoted $f(\underline{x}) \circ g(\underline{x})$. A definition of plethysm is given in Section I.8 of [**M**]. The plethysm results in this paper use a plethysm formula in [**D**] which is recalled in Section 2. Plethysm has the property of being algebraic in the first coordinate.

Proposition 1.1. *For all symmetric functions $f_1(\underline{x})$, $f_2(\underline{x})$, and $g(\underline{x})$ the following two identities hold.*
(a) $(f_1(\underline{x}) + f_2(\underline{x})) \circ g(\underline{x}) = (f_1(\underline{x}) \circ g(\underline{x})) + (f_2(\underline{x}) \circ g(\underline{x}))$.
(b) $(f_1(\underline{x})f_2(\underline{x})) \circ g(\underline{x}) = (f_1(\underline{x}) \circ g(\underline{x}))(f_2(\underline{x}) \circ g(\underline{x}))$.

While plethysm is not in general symmetric, the following is true.

Proposition 1.2. *For any symmetric function $f(\underline{x})$ and positive integer n, $p_n(\underline{x}) \circ f(\underline{x}) = f(\underline{x}) \circ p_n(\underline{x})$.*

Proofs of these propositions are given in Section I.8 of [**M**].

2. Formula for $p_\mu(\underline{x}) \circ h_a(\underline{x})$

This section reviews some results in [**D**].

Definition: Given a semistandard tableau or semistandard skew-tableau T, i is a *descent* with *multiplicity* k if there exists k disjoint pairs $\{(x_1, y_1), \ldots, (x_k, y_k)\}$ of boxes in the Ferrers diagram of T such that the entry in each x_j is i, the entry in each y_j is $i+1$, y_j is in a lower row than x_j for all j, and there does not exist a set of $k+1$ pairs of boxes which satisfy these conditions. Let $m_i(T)$ denote the multiplicity of i as a descent in T. Finally, the *Major index* of T, denoted $\mathrm{maj}(T)$, is sum of the descents with multiplicity. That is, $\mathrm{maj}(T) = \sum i\, m_i(T)$.

Given a semistandard tableau T whose weight has length b and a composition $\alpha = (\alpha_1, \ldots, \alpha_l)$ of b, decompose T into a sequence of semistandard skew-tableaux $(T_1, \ldots, T_l)$ by letting the shape of T_1 be those positions in T which contain 1 through α_1, T_2 be those positions in T which contain $\alpha_1 + 1$ through α_2, and so on. The actual entries in T_i are the entries of T but reindexed so they run from 1 to α_i.

Definition: Following the notation of the previous paragraph, define the root of unity

$$\underline{\omega}^{\mathrm{maj}_\alpha(T)} = \prod_{i=1}^{l} \omega_{\alpha_i}^{\mathrm{maj}(T_i)}$$

where the $\omega_k = e^{2\pi i/k}$.

Example 1: Consider

$$T = \begin{array}{ccccc} 1 & 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 & 4 \\ 3 & 4 & 5 & & \end{array}.$$

T is a semistandard tableau of shape $(5,5,3)$ and weight $(3,4,3,2,1)$. The descent multiplicities are $m_1(T)=3$, $m_2(T)=2$, $m_3(T)=2$, $m_4(T)=1$. Its Major index is 17. Take $\alpha=(2,3)$. Then the decomposition of T is

$$T_1 = \begin{array}{cccc} 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & \end{array} \qquad \text{and} \qquad T_2 = \begin{array}{ccccc} & & & & 1 \\ & & & 1 & 2 \\ 1 & 2 & 3 & & \end{array}$$

The Major indices are $\mathrm{maj}(T_1)=3$ and $\mathrm{maj}(T_2)=4$. Finally, $\underline{\omega}^{\mathrm{maj}_\alpha(T)} = \omega_2^3\omega_3^4 = -\omega_3$.

The main result of [**D**] is the following.

Theorem 2.1. *Let $\mu \vdash b$, then*

$$p_\mu(\underline{x}) \circ h_a(\underline{x}) = \sum_{\substack{\mathrm{SST}\ T \\ \mathrm{wt}(T)=a^b}} \underline{\omega}^{\mathrm{maj}_\mu(T)} s_{\mathrm{sh}(T)}(\underline{x}).$$

Here is a quick application of this formula. It is a generalization of Example 9(b) in Section I.8 of [**M**].

Proposition 2.2. *If b is prime power p^n, then $\langle p_b(\underline{x}) \circ h_a(\underline{x}), s_\lambda(\underline{x})\rangle \equiv K_{\lambda,a^b}$ (mod p).*

Proof: By Theorem 2.1, $\langle p_b(\underline{x}) \circ h_a(\underline{x}), s_\lambda(\underline{x})\rangle$ is a sum of K_{λ,a^b} many terms where each term is a p^n-th root of unity. $\square$

One last result needed (Theorem 2.1 of [**D**]). This concerns an algorithm for selecting the (x_j, y_j) pairs which contribute to the statistic $m_i(T)$ for a given T. Here is the algorithm.

Select an ordering σ of the α many boxes which contain an i.
Set $j=0$.
For $k=1$ to α Do
  Let $x=\sigma(k)$.
  If there are any boxes below x which contain $i+1$ and have not already been selected as an y_j Then
    Increment j by 1.
    Let $x_j = x$.
    Let y_j be the right-most of these unselected boxes containing $i+1$.
  End If
End Do

Algorithm 1.

This algorithm clearly works when the ordering σ is to consider the i's from right to left in T. The surprising result is that this algorithm works for any choice of σ.

Theorem 2.3. *The above algorithm for determining $m_i(T)$ works for all choices of the ordering σ of the i's.*

It should be noted that the choice of pairs (x_j, y_j) may differ based on the choice of σ, but the number of pairs does not differ.

3. A technical result

Definition: Given a semistandard skew-tableau T, define $J(T)$ to be the semistandard tableau obtained by the following algorithm, called *Jeu de Taquin*.

```
While T has skew-shape Do
    Let λ/μ be the shape of T.
    Let ★ be any outer corner of μ.
    While ★ is not an outer corner of λ Do
        If value of T to right of ★ is greater than or equal to the value below ★
            Swap ★ with the value below it.
        Else
            Swap ★ with the value to its right.
        End If
    End Do
End Do
```

Algorithm 2.

Example 2: Let

$$T = \begin{array}{cccccc} &&&1&2&3\\ &1&2&2&& \\ 1&2&3&&& \end{array}.$$

One iteration of the inner loop is

$$\begin{array}{cccccc} &&\bigstar&1&2&3\\ &1&2&2&& \\ 1&2&3&&& \end{array}, \quad \begin{array}{cccccc} &&1&\bigstar&2&3\\ &1&2&2&& \\ 1&2&3&&& \end{array}, \quad \begin{array}{cccccc} &&1&2&2&3\\ &1&2&\bigstar&& \\ 1&2&3&&& \end{array}.$$

Here are the tableaux after each application of the inner loop along with the choice of $\bigstar$ for the next iteration:

$$\begin{array}{cccccc} &\bigstar&1&2&2&3\\ &1&2&&& \\ 1&2&3&&& \end{array}, \quad \begin{array}{cccccc} &1&1&2&2&3\\ \bigstar&2&2&&& \\ 1&3&&&& \end{array},$$

$$\begin{array}{cccccc} \bigstar&1&1&2&2&3\\ 1&2&2&&& \\ 3&&&&& \end{array}, \quad \begin{array}{cccccc} 1&1&1&2&2&3\\ 2&2&&&& \\ 3&&&&& \end{array}.$$

It is clear that $J(T)$ is semistandard and that $\mathrm{wt}(J(T)) = \mathrm{wt}(T)$. However it is not clear from the definition that this operation is well-defined. The choices for the initial position of $\bigstar$ in each iteration might effect the outcome of the algorithm. But, it has been shown (see Section 3.9 of **[S]** for a proof) that for all choices of the $\bigstar$'s, the resulting $J(T)$ is the same. The next result gives an important property of the Jeu de Taquin operation. A proof is given in the proof of Theorem 4.9.4 of **[S]**.

Lemma 3.1. *Given a semistandard tableau Q of shape η, the number of semistandard tableaux T of shape λ/ν such that $J(T) = Q$ equals the Littlewood-Richardson coefficient $c^{\lambda}_{\nu,\eta}$.*

Another fact about Jeu-de-Taquin is that preserves descent multiplicities.

Lemma 3.2. *Let T be a semistandard tableau whose weight has length b and let α be a composition of b. Then $m_i(T) = m_i(J(T))$ for all i.*

Proof: Suppose $T \mapsto T'$ in one application of the inner loop of the Jeu de Taquin algorithm. We want to show that $m_i(T) = m_i(T')$ for all i. If the swap of this kind

$$\begin{matrix} \bigstar & j \\ k & \end{matrix} \rightarrow \begin{matrix} j & \bigstar \\ k & \end{matrix},$$

where $k > j$, then since no element has changed which row it occupies, it is clear that all $m_i(T)$ are unchanged. However, if the swap is of the other kind

$$\begin{matrix} \bigstar & j \\ k & \end{matrix} \rightarrow \begin{matrix} k & j \\ \bigstar & \end{matrix}$$

where $k \leq j$, then there are two possible problems: some $(k, k+1)$ pairs may have been created, or some $(k-1, k)$ pairs may have been destroyed. Either of these "problems" might alter the value of $m_k(T)$ or $m_{k-1}(T)$, respectively.

First, we show that any new $(k, k+1)$ pairs do not effect the value of $m_k(T)$. Let r be the row containing $\bigstar$ in T. So, the possible new $(k, k+1)$ pairs in T' are from the k which just moved into row r and the $k+1$'s in row $r+1$. Since T and T' are semistandard, each of these $k+1$ must have a k directly above them in both T and T'. The key is apply Theorem 2.3 with the properly chosen ordering σ of the k's. The proper choice is to place the k's in row r which have a $k+1$ directly below them (say from right to left) at the beginning of σ, then fill in all the other k's (again say from right to left). Now apply the algorithm to determine $m_k(T)$ and $m_k(T')$. In both cases, the first group of k's in row r with a $k+1$ below them get matched with the $k+1$ directly below them. Thus, all of the new $(k, k+1)$ pairs in T' do effect the outcome of the algorithm since all of the $k+1$'s get selected before the algorithm can consider using one of these pairs. Therefore, $m_k(T) = m_k(T')$.

The proof that $m_{k-1}(T) = m_{k-1}(T')$ is similar. The $(k-1, k)$ pairs that get destroyed in T' are from the k which moved into row r and the $k-1$'s in row r. Since T is semistandard, each of the $k-1$'s in row r must have a k directly below them in both T and T'. Order the $k-1$'s by placing the $k-1$'s in row r first and other $k-1$'s later. The algorithm for determining $m_{k-1}(T)$ and $m_{k-1}(T')$ give the same result. $\square$

Finally, these two lemmas are put together to prove a key technical result used in the next section.

Theorem 3.3. *Let $|\lambda/\nu| = n$, β be a composition of n with l parts, and α be a composition of l. Then*

$$\sum_{\substack{\text{SSST } T \\ \text{sh}(T)=\lambda/\nu, \text{wt}(T)=\beta}} \underline{\omega}^{\text{maj}_\alpha(T)} = \sum_{\eta \vdash n} \left(c^{\lambda}_{\nu,\eta} \sum_{\substack{\text{SST } T \\ \text{sh}(T)=\eta, \text{wt}(T)=\beta}} \underline{\omega}^{\text{maj}_\alpha(T)} \right).$$

Proof: By Lemma 3.1, Jeu-de-Taquin provides the bijection between the two sides. By Lemma 3.2, $\underline{\omega}^{\text{maj}_\alpha(T)}$ is preserved under this bijection. $\square$

4. Formula for $p_\mu(\underline{x}) \circ h_\lambda(\underline{x})$

Theorem 4.1. *Given $\mu \vdash b$ and $\lambda \vdash a$ where the length of μ is l and the length of λ is t, let $\alpha = \mu^t$ and $\beta = \lambda_1^b \lambda_2^b \ldots \lambda_t^b$. So, β is a composition of ab with tb parts, and α is a composition of tb. Then*

$$p_\mu(\underline{x}) \circ h_\lambda(\underline{x}) = \sum_{\substack{\text{SST } T \\ \text{wt}(T)=\beta}} \underline{\omega}^{\text{maj}_\alpha(T)} s_{\text{sh}(T)}(\underline{x}).$$

Proof: Proof by induction on t. The case $t = 1$ is Theorem 2.1. So assume the theorem is valid when μ has length less than t. Let $\lambda^* = (\lambda_1, \ldots, \lambda_{t-1})$. Now a computation gives the desired result. Explanations of the steps are given after the computation.

$$p_\mu(\underline{x}) \circ h_\lambda(\underline{x}) = \prod_i p_{\mu_i}(\underline{x}) \circ h_\lambda(\underline{x}) \tag{1}$$

$$= \prod_i h_\lambda(\underline{x}) \circ p_{\mu_i}(\underline{x}) \tag{2}$$

$$= \prod_i \left(h_{\lambda^*}(\underline{x}) \circ p_{\mu_i}(\underline{x})\right)\left(h_{\lambda_t}(\underline{x}) \circ p_{\mu_i}(\underline{x})\right) \tag{3}$$

$$= \prod_i \left(p_{\mu_i}(\underline{x}) \circ h_{\lambda^*}(\underline{x})\right) \prod_i \left(p_{\mu_i}(\underline{x}) \circ h_{\lambda_t}(\underline{x})\right) \tag{4}$$

$$= \left(\sum_{\substack{\mathrm{SST}\ T_1 \\ \mathrm{wt}(T_1)=\lambda_1^b \ldots \lambda_{t-1}^b}} \underline{\omega}^{\mathrm{maj}_{\mu^{t-1}}(T_1)} s_{\mathrm{sh}(T_1)}(\underline{x}) \right) \left(\sum_{\substack{\mathrm{SST}\ T_2 \\ \mathrm{wt}(T_2)=\lambda_t^b}} \underline{\omega}^{\mathrm{maj}_\mu(T_2)} s_{\mathrm{sh}(T_2)}(\underline{x}) \right) \tag{5}$$

$$= \sum_{\nu \vdash ab} s_\nu(\underline{x}) \left[\sum_{\substack{\mathrm{SST}\ T_1 \\ \mathrm{wt}(T_1)=\lambda_1^b \ldots \lambda_{t-1}^b}} \left(\underline{\omega}^{\mathrm{maj}_{\mu^{t-1}}(T_1)} \left(c^\nu_{\mathrm{sh}(T_1),\eta} \sum_{\substack{\mathrm{SST}\ T_2 \\ \mathrm{sh}(T_2)=\eta, \mathrm{wt}(T_2)=\lambda_t^b}} \underline{\omega}^{\mathrm{maj}_\mu(T_2)} \right)\right)\right] \tag{6}$$

$$= \sum_{\nu \vdash ab} s_\nu(\underline{x}) \left[\sum_{\substack{\mathrm{SST}\ T_1 \\ \mathrm{wt}(T_1)=\lambda_1^b \ldots \lambda_{t-1}^b}} \left(\underline{\omega}^{\mathrm{maj}_{\mu^{t-1}}(T_1)} \sum_{\substack{\mathrm{SST}\ T_2 \\ \mathrm{sh}(T_2)=\nu/\mathrm{sh}(T_1), \mathrm{wt}(T_2)=\lambda_t^b}} \underline{\omega}^{\mathrm{maj}_\mu(T_2)} \right)\right] \tag{7}$$

$$= \sum_{\nu \vdash ab} s_\nu(\underline{x}) \left[\sum_{\substack{\mathrm{SST}\ T_3 \\ \mathrm{sh}(T_3)=\nu, \mathrm{wt}(T_3)=\beta}} \underline{\omega}^{\mathrm{maj}_\alpha(T_3)} \right] \tag{8}$$

(1) Apply Proposition 1.1(b). (2) Apply Proposition 1.2. (3) Apply Proposition 1.1(b). (4) Apply Proposition 1.2. (5) Apply the induction hypothesis. (6)

Multiply the Schur symmetric functions and collect the terms which contribute to $s_\nu(\underline{x})$. (7) Apply Theorem 3.3. (8) Combine T_1 and T_2 to form T_3 by first reindexing the values in T_2 to run from $b(\lambda_1+\cdots+\lambda_{t-1})+1$ to $b(\lambda_1+\cdots+\lambda_t)$ and then place T_1 inside the reindexed T_2. Notice that under this bijection $\underline{\omega}^{\mathrm{maj}_{\mu^{t-1}}(T_1)}\,\underline{\omega}^{\mathrm{maj}_\mu(T_2)} = \underline{\omega}^{\mathrm{maj}_\alpha(T_3)}$. $\square$

Example 3: Let $\mu = (2)$ and $\lambda = (2,1)$. Then $\alpha = (2,2)$ and $\beta = (2,2,1,1)$. The semistandard tableaux T with weight β and $\underline{\omega}^{\mathrm{maj}_\alpha(T)}$ are given in Table 1. So, $p_{(2)}(\underline{x}) \circ h_{(2,1)}(\underline{x}) = s_{(6)}(\underline{x}) - s_{(5,1)}(\underline{x}) + 2s_{(4,2)}(\underline{x}) - 2s_{(3,3)}(\underline{x}) - s_{(4,1,1)}(\underline{x}) + s_{(2,2,2)}(\underline{x}) + s_{(3,1,1,1)}(\underline{x}) - s_{(2,2,1,1)}(\underline{x})$.

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T	$\underline{\omega}^{\mathrm{maj}_{(2,2)}(T)}$	T	$\underline{\omega}^{\mathrm{maj}_{(2,2)}(T)}$
1 1 2 2 3 4	1	1 1 2 2 3 4	-1
1 1 2 2 4 3	1	1 1 2 4 2 3	-1
1 1 2 2 3 4	-1	1 1 2 3 2 4	1
1 1 2 3 4 2	-1	1 1 2 2 3 4	1
1 1 2 2 3 4	1	1 1 2 2 4 3	-1
1 1 2 4 2 3	-1	1 1 4 2 2 3	1
1 1 2 3 2 4	1	1 1 3 2 2 4	-1
1 1 3 4 2 2	1	1 1 2 2 3 4	1
1 1 2 2 3 4	1	1 1 2 2 3 4	1
1 1 3 2 2 4	1	1 1 2 2 3 4	-1

Table 1: Semistandard Tableaux of Weight $(2,2,1,1)$