On the function "sandwiched" between $\alpha(G)$ and $\overline{\chi}(G)$

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Abstract

A new function of a graph G is presented. Say that a matrix B that is indexed by vertices of G is feasible for G if it is real, symmetric and $I \leq B \leq I + A(G)$, where I is the identity matrix and $A(G)$ is the adjacency matrix of G. Let $\mathcal{B}(G)$ be the set of all feasible matrices for G, and let $\overline{\chi}(G)$ be the smallest number of cliques that cover the vertices of G. We show that

$$\alpha(G) \leq \min\{\,\mathrm{rank}(B)\,|B \in \mathcal{B}(G)\} \leq \overline{\chi}(G)$$

and that $\alpha(G) = \min\{\,\mathrm{rank}(B)\,|B \in \mathcal{B}(G)\}$ implies $\alpha(G) = \overline{\chi}(G)$.

The well known Lovász number $\vartheta(G)$ of a graph G [1] is "sandwiched" between the size of the largest stable set in G and the smallest number of cliques that cover the vertices of G

$$\alpha(G) \leq \vartheta(G) \leq \overline{\chi}(G).$$

Some alternative definitions of $\vartheta(G)$ are introduced in [2,3]. For example,

$$\begin{aligned}\vartheta(G) = \max\{\,\Lambda(B)\,|B \text{ is a real positive semidefinite matrix}\\ &\text{indexed by vertices of } G,\\ &B_{vv} = 1 \text{ for all } v \in V(G),\\ &B_{uv} = 0 \text{ whenever } u \text{ — } v \text{ in } G\},\end{aligned}$$

where $\Lambda(B)$ is the maximum eigenvalue of B, $V(G)$ — the set of vertices of G, u — v denotes the adjacency of vertices u and v.

Call the matrix B indexed by vertices of G feasible for G if

$$\begin{aligned}&B \text{ is real and symmetric},\\ &B_{vv} = 1 \text{ for all } v \in V(G),\\ &B_{uv} = 0 \text{ whenever } u \not\text{—}\, v \text{ in } G,\\ &0 \leq B_{uv} \leq 1 \text{ whenever } u \text{ — } v \text{ in } G.\end{aligned}$$

Let $\mathcal{B}(G)$ be the set of all feasible matrices for G. Then [4]

$$\overline{\chi}(G) = \min\{\,\mathrm{rank}(B)\,|B \in \mathcal{B}(G), B = C^TC, C \geq 0\},$$

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where the inequality denotes componentwise inequality.

The aim of this paper is to study a new function of graph G

$$\min\{\,\mathrm{rank}(B)\,|B \in \mathcal{B}(G)\}.$$

Clearly $\alpha(G) \leq \min\{\,\mathrm{rank}(B)\,|B \in \mathcal{B}(G)\}$, because every stable set in G with k vertices corresponds to an identity $k \times k$-submatrix of any matrix $B \in \mathcal{B}(G)$.

Theorem. *For all graphs G*

$$\alpha(G) = \min\{\,\mathrm{rank}(B)\,|B \in \mathcal{B}(G)\}$$

implies $\alpha(G) = \overline{\chi}(G)$.

Proof. Let $S = \{v_1, v_2, \ldots, v_{\alpha(G)}\}$ be the stable set of G, $\overline{S} = V(G) \setminus S$ and $B \in \mathcal{B}(G)$ is a matrix such that $\mathrm{rank}(B) = \alpha(G)$. We can assume that

$$B = \begin{pmatrix} I_{\alpha(G)} & X \\ X^T & Y \end{pmatrix},$$

where $I_{\alpha(G)}$ is the identity $\alpha(G) \times \alpha(G)$-matrix.

Applying block Gauss elimination, B reduces to the matrix

$$B' = \begin{pmatrix} I_{\alpha(G)} & X \\ 0 & Y - X^T X \end{pmatrix}.$$

We have

$$Y - X^T X = 0$$

or

$$Y_{uv} - \sum_{w \in S} X_{wu} X_{wv} = 0, \quad u, v \in \overline{S} \tag{1}$$

since $\mathrm{rank}(B') = \mathrm{rank}(B) = \alpha(G)$.

Equation (1) gives us further information about the graph G.

(i) If $v \in \overline{S}$ then exists $u \in S$ such that u — v. Indeed, $Y_{vv} = 1$ and $X_{wv} \geq 0$ for all $w \in S$. Hence, $\sum_{w \in S} X_{wv}^2 = 1$ and $X_{uv} > 0$ for some $u \in S$.

(ii) If $v', v'' \in \overline{S}$ and $X_{uv'} X_{uv''} > 0$ for some $u \in S$ then v' — v''. Indeed, if $\sum_{w \in S} X_{wv'} X_{wv''} > 0$ then $Y_{v'v''} > 0$.

Let

$$V_u = \{u\} \cup \{v | v \in \overline{S}, X_{uv} > 0\}$$

for all $u \in S$ and $G(V_u)$ be the subgraph induced from G by leaving out all vertices except vertices from V_u. We know from (i) and (ii) that $G(V_u)$ is a clique and $V(G) = \cup_{u \in S} V_u$. Hence, $\overline{\chi}(G) = \alpha(G)$. $\square$

Corollary. *If* $\overline{\chi}(G) \le \alpha(G) + 1$ *then*

$$\overline{\chi}(G) = \min\{\,\mathrm{rank}(B)\,|B \in \mathcal{B}(G)\}.$$

For example, consider the Petersen graph G. We have $\alpha(G) = \vartheta(G) = 4$, $\overline{\chi}(G) = 5$. Hence, $\min\{\,\mathrm{rank}(B)\,|B \in \mathcal{B}(G)\} = 5$.

There is a graph G such that

$$\alpha(G) < \min\{\,\mathrm{rank}(B)\,|B \in \mathcal{B}(G)\} < \overline{\chi}(G).$$

Let $V(G) = 2^{\{1,2,3,4,5,6\}}$ and u — v iff $2 \le |(u \setminus v) \cup (v \setminus u)| \le 5$ for all $u, v \in V(G)$. Then [5] $\chi(G) = 32$ and $\mathrm{rank}(A(G)) = 29$ where $A(G)$ is the adjacency matrix of G. Then $\overline{\chi}(\overline{G}) = 32$ and

$$\mathrm{rank}(I + A(\overline{G})) = \mathrm{rank}(J - A(G)) \le 1 + \mathrm{rank}(A(G)) = 30,$$

where J is all 1's.

Notice that $I + A(\overline{G}) \in \mathcal{B}(\overline{G})$. Therefore

$$\min\{\,\mathrm{rank}(B)\,|B \in \mathcal{B}(\overline{G})\} < \overline{\chi}(\overline{G})$$

and

$$\alpha(\overline{G}) < \min\{\,\mathrm{rank}(B)\,|B \in \mathcal{B}(\overline{G})\}$$

by theorem.

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