Frankl-Füredi Type Inequalities for Polynomial Semi-lattices

Jin Qian and Dijen K. Ray-Chaudhuri[1]

Department of Mathematics

The Ohio State University

Submitted: April 2, 1997; Accepted: October 20, 1997

Abstract

Let X be an n-set and L a set of nonnegative integers. $\mathcal{F}$, a set of subsets of X, is said to be an L -intersection family if and only if for all $E \neq F \in \mathcal{F}$, $|E \cap F| \in L$. A special case of a conjecture of Frankl and Füredi [4] states that if $L = \{1, 2, \ldots, k\}$, k a positive integer, then $|\mathcal{F}| \leq \sum_{i=0}^{k} \binom{n-1}{i}$.

Here $|\mathcal{F}|$ denotes the number of elements in $\mathcal{F}$.

Recently Ramanan proved this conjecture in [6]. We extend his method to polynomial semi-lattices and we also study some special L-intersection families on polynomial semi-lattices.

Finally we prove two modular versions of Ray-Chaudhuri-Wilson inequality for polynomial semi-lattices.

§1. Introduction

Throughout the paper, we assume $k, n \in \mathbb{N}$, $I_n = \{1, 2, \ldots, n\} \subset \mathbb{N}$, where $\mathbb{N}$ denotes the set of positive integers.

In this part, we briefly review the concept of polynomial semi-lattice introduced by Ray-Chaudhuri and Zhu in [8]. The definition of polynomial semi-lattice given here is equivalent to but simpler than that in [8]. For the convenience of the reader, we also include various examples of polynomial semi-lattices.

Let $(X, \leq)$ be a finite nonempty partially ordered set having the property that $(X, \leq)$ is a semi-lattice, i.e., for every $x, y \in X$ there is a unique greatest lower bound of x and y denoted by $x \wedge y$. If $x \leq y$ and $x \neq y$, we write $x < y$. We

[1]e-mail addresses: `<qian@math.ohio-state.edu>`, `<dijen@math.ohio-state.edu>`

also assume that $(X, \leq)$ has a height function $l(x)$, where $l(x)+1$ is the number of terms in a maximal chain from the least element 0 to the element x including the end elements in the count. Let n be the maximum of $l(x)$ for all the x in X. Define $X_i = \{x \in X|\ l(x) = i\}$, $0 \leq i \leq n$ and $X_0 = \{0\}$. Then $X = \cup_{i=0}^{n} X_i$ is a partition and the subsets X_i's are called fibres. The integer n is said to be the height of $(X, \leq)$.

$(X, \leq)$ is called a polynomial semi-lattice, if for each fibre X_i there is a size number $m_i \in \mathbb{N} \cup \{0\}$ and a polynomial $f_i(w) \in \mathbb{Q}[w]$, where $\mathbb{Q}$ is the set of rational numbers such that

a) $m_0 < m_1 < \ldots < m_n$,

b) $f_i(w) = a_i(w-m_0)(w-m_1)\ldots(w-m_{i-1})$ for some positive rational number a_i for $i > 0$, and $f_0(w) = 1$,

c) For any i, j, k, $0 \leq k \leq i \leq j \leq n$, $x \in X_k$, $y \in X_j$ and $x \leq y$, $|\{z \mid z \in X_i, x \leq z \leq y\}| = f_{i-k}(m_{j-k})$.

Remarks.

1) Taking $k = 0$ in c), we have $|\{z \mid z \in X_i, z \leq y\}| = f_i(m_j)$ for every $y \in X_j$.

2) For any $x \in X$ we define $|x|$ to be m_i if $x \in X_i$. Specializing $y = E \wedge F$ in remark 1), we have $|\{I \in X_i \mid I \leq E \wedge F\}| = f_i(|E \wedge F|)$, where $E, F \in X$. This result is going to be used later.

3) Taking $i = j$ in remark 1), we have $f_i(m_i) = 1$ since $\{z \mid z \in X_i, z \leq y\} = \{y\}$. From this we can solve for a_i:

$$a_i = \frac{1}{(m_i - m_0)(m_i - m_1)\ldots(m_i - m_{i-1})}$$

for $i = 1, 2, \ldots, n$.

4) From remark 3), we get

$$f_i(w) = \frac{(w-m_0)(w-m_1)\ldots(w-m_{i-1})}{(m_i - m_0)(m_i - m_1)\ldots(m_i - m_{i-1})}.$$

For $j > i$, we have $m_j > m_i$ and therefore $f_i(m_j) > 1$.

In the following examples we let $s \in \mathbb{N}$, q be a prime power, and

$$[w, i]_q = \frac{(w-1)(w-q)\cdots(w-q^{i-1})}{(q^i-1)(q^i-q)\cdots(q^i-q^{i-1})}.$$

Examples:

1) *Johnson Scheme.* Let V be an n-element set and X_i be the set of all i-element subsets of $V, 0 \leq i \leq n$. Then $X = \cup_{i=0}^n X_i$, with inclusion as the partial order, is a semi-lattice. Let $m_i = i, f_i(w) = \binom{w}{i}$. It is easy to see that $(X, \leq)$ is a polynomial semi-lattice.

2) *q-analogue of Johnson Scheme.* Let V be an n-dimensional vector space over a finite field $GF(q)$, X_i be the set of all i-dimensional subspaces of V, $0 \leq i \leq n$. Let $m_i = q^i, f_i(w) = [w, i]_q$ (defined after remark 4). Then $X = \cup_{i=0}^n X_i$ is a polynomial semi-lattice with inclusion as the partial order.

3) *Hamming Scheme.* Let W be an s-element set. We define $X_i = \{(L, h) \mid L \subseteq \{1, 2, \ldots, n\},\ |L| = i,\ h : L \to W \quad \text{a map } \}, 1 \leq i \leq n,\ X_0 = \{0\}$, where 0 is taken to be the *least element*, and $X = \cup_{i=0}^n X_i$. $(L_1, h_1) \leq (L_2, h_2)$ if and only if $L_1 \subseteq L_2$, and $h_2|_{L_1} = h_1$. Then $(X, \leq)$ is a polynomial semi-lattice, with $m_i = i, f_i(w) = \binom{w}{i}$.

4) *q-analogue of Hamming Scheme.* Let V be an s-dimensional vector space over a finite field $GF(q)$ and W be an n-dimensional vector space over a finite field $GF(q)$. Define $X_i = \{(U, h) \mid U \subseteq W, dim(U) = i, \quad h : U \to V, \text{ a linear transformation}\}$, $0 \leq i \leq n$. Let $X = \cup_{i=0}^n X_i$. $\forall (U_1, h_1), (U_2, h_2) \in X$, define $(U_1, h_1) \leq (U_2, h_2)$ if and only if $U_1 \subseteq U_2$ and $h_2|_{U_1} = h_1$. Then $(X, \leq)$ is a polynomial semi-lattice, with $m_i = q^i, f_i(w) = [w, i]_q$.

5) *Ordered Design.* Let W be an s-element set and V be an n-element set with $n \leq s$. We define $X_i = \{(L, h) \mid L \subseteq \{1, 2, \ldots, n\}, |L| = i, \quad h : L \to W \text{ an injection}\}$, $1 \leq i \leq n$, $X_0 = \{0\}$, where 0 is taken as the *least element*, and $X = \cup_{i=0}^n X_i$. $\forall (L_1, h_1), (L_2, h_2) \in X$, define $(L_1, h_1) \leq (L_2, h_2)$ if and only if $L_1 \subseteq L_2$ and $h_2|_{L_1} = h_1$. Then $(X, \leq)$ is a polynomial semi-lattice, with $m_i = i, f_i(w) = \binom{w}{i}$.

6) *q-analogue of Ordered Design.* Let W be an s-dimensional vector space and V be an n-dimensional vector space over a finite field $GF(q)$ with $n \leq s$. Define

$X_i = \{(U, h) \mid U \subseteq V, dim(U) = i,\ h : U \rightarrow W$, a nonsingular linear transformation $\}$, $0 \leq i \leq n$. Let $X = \cup_{i=0}^{n} X_i$. $\forall (U_1, h_1), (U_2, h_2) \in X$, define $(U_1, h_1) \leq (U_2, h_2)$ if and only if $U_1 \subseteq U_2$ and $h_2|_{U_1} = h_1$. Then $(X, \leq)$ is a polynomial semilattice, with $m_i = q^i, f_i(w) = [w, i]_q$.

§2. **Statement of Results**

Let $(X, \leq)$ be a polynomial semi-lattice of height n, i.e $X = \cup_{i=0}^{n} X_i$ and L be a k-subset of $I_n \cup \{0\}$, where $k \leq n$ is a natural number. We call $\mathcal{F} \subseteq X$ an L-intersection family if and only if $\forall E \neq F \in \mathcal{F}$, $E \wedge F \in \cup_{l \in L} X_l$. If $\mathcal{F}$ is empty or contains only one element, it is vacuously an L-intersection family and all the theorems below are trivially true. So in the rest of this paper, we assume that $\mathcal{F}$ has at least two elements.

Ray-Chaudhuri and Zhu extended the well-known Ray-Chaudhuri-Wilson theorem to the polynomial semi-lattice and they have [8]:

Theorem 1. Let $(X, \leq)$ be a polynomial semi-lattice. If $\mathcal{F} \subseteq X$ is an L-intersection family, then $|\mathcal{F}| \leq \sum_{i=0}^{k} |X_i|$.

For the special case $L = \{l, l+1, \ldots, l+k-1\}$, we extend the method in Ramanan [6] to polynomial semi-lattices, and we have:

Theorem 2. Let $(X, \leq)$ be a semi-lattice of height n, $l, k \in \mathbb{N}$, $l + k - 1 \leq n$ and $\mathcal{F}$ be an $\{l, l+1, \ldots, l+k-1\}$-intersection family. Then

$|\mathcal{F}| \leq |X_k| + |X_{k-2}| + \cdots + |X_{k-[k/2]2}|$.

Here $[x]$ means the greatest integer less than or equal to x.

The above result for the set case was raised by Ramanan [6] as an interesting problem.

In the case of Johnson scheme where $|X_n| = \binom{n}{i}$, $0 \leq i \leq n$, we have the

Corollary. Let X be an n-set. If $\mathcal{F}$ is a family of subsets of X such that $\forall E \neq F \in \mathcal{F}$, $|E \cap F| \in \{1, 2, \ldots, k\}$, then $|\mathcal{F}| \leq \sum_{i=0}^{k} \binom{n-1}{i}$.

This follows by specializing $l = 1$ in Theorem 2 and the easy observation that

$$\sum_{i=0}^{[k/2]} \binom{n}{k-2i} = \sum_{i=0}^{k} \binom{n-1}{k-i}.$$

This is a special case of a conjecture of Frankl and Füredi which was recently proved by G. V. Ramanan [6]. Indeed, Frankl and Füredi conjectured a more general result

Conjecture 1. Let $k \in \mathbb{N}$, $l \in \mathbb{N} \cup \{0\}$, $k > 2l + 1$, $n > n_0(k)$, X be an n-set, $L = \{0, 1, 2, \cdots, k\} - \{l\}$. If $\mathcal{F}$ is an L-intersection family of subsets of X, then

$$|\mathcal{F}| \leq \sum_{i \leq l-1} \binom{n}{i} + \sum_{i=l+1}^{k+1} \binom{n-l-1}{i-l-1}.$$

Ramanan proved the special case of Frankl-Füredi conjecture when $l = 0$. The general case is still open.

We also studied the special case of Theorem 1 when $L = \{0, 1, \ldots, k-1\}$ and got a simpler proof of the inequality as well as a necessary and sufficient condition under which the equality holds.

Theorem 3. Let $(X, \leq)$ be a polynomial semi-lattice. If $\mathcal{F}$ is an L-intersection family for $L = \{0, 1, \ldots, k-1\}$, then $|\mathcal{F}| \leq \sum_{i=0}^{k} |X_i|$.

The equality holds if and only if $\mathcal{F} = \cup_{i=0}^{k} X_i$.

In the direction of Theorem 1, Snevily [9] studied the case $L = \{0, 1, \cdots, k-1\}$, and $\forall E \in \mathcal{F}, |E| \geq k$ and he obtained a better upper-bound. We show that it can be generalized to polynomial semi-lattices (Theorem 4 below) and we give a simpler proof of the inequality as well as a necessary and sufficient condition under which the equality holds.

Theorem 4. Let $(X, \leq)$ be a polynomial semi-lattice of height n, $k \in \mathbb{N}$, $\mathcal{F}$ an L-intersection family for $L = \{0, 1, \ldots, k-1\}$ and $\mathcal{F} \subseteq \cup_{i=k}^{n} X_i$. Then $|\mathcal{F}| \leq |X_k|$.

The equality holds if and only if $\mathcal{F} = X_k$.

Next, we show that some modular versions of Ray-Chaudhuri-Wilson Theorem [7] also extend to polynomial semi-lattices.

First the uniform case (Frankl and Wilson's modular version [5]):

Theorem 5. Let $(X, \leq)$ be a polynomial semi-lattice of height n, s, $k \in \mathbb{N}$ with $s \leq k$, $L \subseteq I_n \cup \{0\}$ and $\mathcal{F} \subseteq X_k$ an L-intersection family. Suppose $\mu_0, \mu_1, \cdots, \mu_s$ are distinct residues modulo a prime p such that $m_k \equiv \mu_0 \pmod p$ and $\forall l \in L, m_l \equiv \mu_i \pmod p$ for some $i, 1 \leq i \leq s$. Further suppose that for every μ_i, $\exists l_i \in L$, such that $m_{l_i} \equiv \mu_i\ (mod\ p)$, for $i = 1, 2, \cdots, s$. Then $|\mathcal{F}| \leq |X_s|$.

Then the nonuniform case (Deza, Frankl and Singhi's modular version [3]):

Theorem 6. Let $(X, \leq)$ be a polynomial semi-lattice of height n and $\mathcal{F} \subseteq X_{k_1} \cup X_{k_2} \cup \cdots \cup X_{k_\nu}$ be an L-intersection family, where $L \subseteq I_n \cup \{0\}$ and $k_1, k_2, \cdots, k_\nu$ are integers in $I_n \cup \{0\}$. Suppose $\mu_1, \mu_2, \cdots, \mu_s$ are distinct residues modulo a prime p such that $\forall l \in L, m_l \equiv \mu_i \pmod p$ for some $i, 1 \leq i \leq s$ and m_{k_i} is not congruent to any one of $\mu_1, \mu_2, \cdots, \mu_s$ modulo p for $i = 1, 2, \cdots, \nu$. Then $|\mathcal{F}| \leq \sum_{i=0}^{s} |X_i|$.

§3. The Proof of Theorem 2

Convention: Empty product is defined to be 1.

First we prove two lemmas.

Lemma 1. Let $k, \in \mathbb{N}$ and $l_1 < l_2 < \cdots < l_k$ be k positive integers in I_n. There exist $k+1$ positive real numbers $b_0, b_1, \ldots, b_k$ such that

$$\sum_{i=0}^{k} (-1)^i b_i f_i(x) = (-1)^k (x - m_{l_1})(x - m_{l_2}) \ldots (x - m_{l_k}).$$

Proof. Recall that $f_i(x) = a_i(x-m_0)(x-m_1)\ldots(x-m_{i-1})$ and $m_0 < m_1 < \cdots < m_n$, where a_i's are positive. So it is enough to prove that there exist positive real numbers $c_0, c_1, \cdots, c_k$ such that

$\sum_{i=0}^{k}(-1)^i c_i (x-m_0)(x-m_1)\cdots(x-m_{i-1}) = (-1)^k (x-m_{l_1})(x-m_{l_2})\ldots(x-m_{l_k})$.

The above result follows from the following more general statement:

Claim. For any j such that $0 \le j < l_1$, there exist positive real numbers $d_0, d_1, \cdots, d_k$ such that

$$\sum_{i=0}^{k}(-1)^i d_i(x-m_j)(x-m_{j+1})\cdots(x-m_{j+i-1}) = (-1)^k(x-m_{l_1})(x-m_{l_2})\cdots(x-m_{l_k}).$$

Proof of the claim. When $k=1$, it is trivially true. Suppose it is true for k. Now we want to prove that it holds for $k+1$.

$$\begin{aligned}
&(-1)^{k+1}(x-m_{l_1})(x-m_{l_2})\cdots(x-m_{l_{k+1}})\\
=&(-1)(x-m_{l_1})[(-1)^k(x-m_{l_2})\cdots(x-m_{l_{k+1}})]\\
=&(-1)[(x-m_j)-(m_{l_1}-m_j)][(-1)^k(x-m_{l_2})\cdots(x-m_{l_{k+1}})]\\
=&(-1)(x-m_j)[(-1)^k(x-m_{l_2})\cdots(x-m_{l_{k+1}})]+\\
&(m_{l_1}-m_j)[(-1)^k(x-m_{l_2})\cdots(x-m_{l_{k+1}})] \qquad (1)
\end{aligned}$$

Since $j+1 < l_1+1$, we can apply the induction hypothesis to the first term of (1) (denoted by **I**) and we have

$$\begin{aligned}
\mathbf{I} &= (-1)(x-m_j)\sum_{i=0}^{k}(-1)^i u_i(x-m_{j+1})\cdots(x-m_{j+1+i-1})\\
&= \sum_{i=0}^{k}(-1)^{i+1}u_i(x-m_j)(x-m_{j+1})\cdots(x-m_{j+1+i-1})
\end{aligned}$$

for some positive real numbers $u_k, u_{k-1}, \cdots, u_0$. Then we use the induction hypothesis on the second term (denoted by **II**) and we have

$$\begin{aligned}
\mathbf{II} &= (m_{l_1}-m_j)\sum_{i=0}^{k}(-1)^i v_i(x-m_j)\cdots(x-m_{j+i-1})\\
&= \sum_{i=0}^{k}(-1)^i(m_{l_1}-m_j)v_i(x-m_j)\cdots(x-m_{j+i-1})
\end{aligned}$$

for some positive real numbers $v_k, v_{k-1}, \cdots, v_0$. Now add up **I** and **II** and we have

$$\begin{aligned} & (-1)(x-m_j)[(-1)^k(x-m_{l_2})\cdots(x-m_{l_{k+1}})] + \\ & (m_{l_1}-m_j)[(-1)^k(x-m_{l_2})\cdots(x-m_{l_{k+1}})] \\ = \quad & \sum_{i=0}^{k+1}(-1)^i d_i((x-m_j)\cdots(x-m_{j+i-1}) \end{aligned}$$

where

$$\begin{aligned} d_{k+1} &= u_k, \\ d_k &= u_{k-1} + (m_{l_1}-m_j)v_k, \\ d_{k-1} &= u_{k-2} + (m_{l_1}-m_j)v_{k-1}, \\ & \cdots \\ d_0 &= (m_{l_1}-m_j)v_0. \end{aligned}$$

so $d_{k+1}, d_k, \cdots, d_0$ are positive, which proves the claim and therefore the lemma. $\square$

Remark. In the rest of the paper, we will only use Lemma 1 in its special case where $l_1 = l$, $l_2 = l+1, \cdots, l_k = l+k-1$. Let's denote $(x-m_l)(x-m_{l+1})\ldots(x-m_{l+k-1})$ by $g(x)$. Since $\mathcal{F}$ is an $\{l, l+1, \cdots, l+k-1\}$-intersection family, $|E| \geq m_l$ for all $E \in \mathcal{F}$. So it is clear that $g(|E|) \geq 0$ for all $E \in \mathcal{F}$ and $g(|E|) > 0$ if $|E| > m_{l+k-1}$.

To each $E \in \mathcal{F}$ we associate a variable x_E. For each $I \in X$, we define a linear form L_I as follows:

$$L_I := \sum_{E\in\mathcal{F}, I\leq E} x_E.$$

Lemma 2. With the same notation as in Lemma 1 and further we assume that $l \in \mathbb{N}$, $l+k-1 \leq n$, $l_1 = l$, $l_2 = l+1, \cdots, l_k = l+k-1$. We have

$$\sum_{i=0}^{k}(-1)^i b_i \sum_{I\in X_i} L_I^2 = \sum_{E\in\mathcal{F}}(-1)^k g(|E|)x_E^2 \, . \tag{2}$$

Proof. We regard both sides as quadratic forms on x_E's, where $E \in \mathcal{F}$ and try to show that the corresponding coefficients are equal.

For example, for $E \neq F \in \mathcal{F}$, the term L_I^2 contributes a term $2x_E x_F$ if and only if $I \leq E \wedge F$. Therefore the coefficient of $x_E x_F$ in the L. H. S of (2) is $2\sum_{i=0}^{k}(-1)^i b_i f_i(|E \wedge F|)$ (see remark 2 in the introduction) which is equal to

$$2(-1)^k(|E \wedge F| - m_l)((|E \wedge F| - m_{l+1}) \ldots (|E \wedge F| - m_{l+k-1})$$

by Lemma 1. Since $\mathcal{F}$ is an L-intersection family with $L = \{l, l+1, \ldots, l+k-1\}$, $|E \wedge F| \in \{m_l, m_{l+1}, \ldots, m_{l+k-1}\}$ and so the product in the previous sentence is 0. Obviously the coefficient of $x_E x_F$ in the R.H.S is also 0. So the coefficient of $x_E x_F$ in the L.H.S is equal to that in the R.H.S.

Similarly the coefficient of x_E^2 in the L.H.S is $\sum_{i=0}^{k}(-1)^i b_i f(|E|)$ for the same reason as above. By Lemma 1 it is equal to $(-1)^k g(|E|)$ which is the coefficient of x_E^2 in the R.H.S. Here $g(x)$ is as defined in the remark immediately after the proof of Lemma 1. $\square$

We define the real vector space W to be $\mathbb{R}^{|\mathcal{F}|}$ whose coordinates are indexed by elements of $\mathcal{F}$, $\mathcal{L}$ to be the set of linear forms $\{L_I \,:\, |I| \in \{m_k, m_{k-2}, \ldots, m_{k-[k/2]2}\}\}$ and $W_0 \subseteq W$ to be the space of common solutions of the set of equations $L_I = 0$, $L_I \in \mathcal{L}$. Clearly $|\mathcal{L}| = \sum_{i=0}^{[k/2]} |X_{k-2i}|$. An element of W_0 will be written as $(v_E, E \in \mathcal{F}) = (v_E)$ (for short).

If we can show that W_0 consists of the zero vector only, then by linear algebra, $rank(\mathcal{L})$ = the number of variables = $|\mathcal{F}|$ and therefore $|\mathcal{F}| = rank(\mathcal{L}) \leq |\mathcal{L}| = \sum_{i=0}^{[k/2]} |X_{k-2i}|$, which finishes the proof of Theorem 2.

So it is enough to prove

Lemma 3. $W_0 = \{(0, 0, \ldots, 0)\}$.

Proof. Suppose W_0 contains (v_E). It suffices to show $v_E = (0, 0, \ldots, 0)$.

By Lemma 2, we have

$$\sum_{i=0}^{k}(-1)^i b_i \sum_{I \in X_i} L_I^2 = (-1)^k \sum_{E \in \mathcal{F}} g(|E|) x_E^2.$$

Specializing $x_E = v_E, \forall E \in \mathcal{F}$, we have

$$\sum_{i=0}^{k}(-1)^i b_i \sum_{I\in X_i} L_I^2((v_E)) = (-1)^k \sum_{E\in\mathcal{F}} g(|E|)v_E^2.$$

Since $L_I((v_E)) = 0$, for all $L_I \in \mathcal{L}$, we have $L_I((v_E)) = 0$ for $k-i$ even and thus

$$\sum_{i\in\{0,1,\ldots,k\},k-i\ is\ odd} (-1)^i b_i \sum_{I\in X_i} L_I^2((v_E)) = (-1)^k \sum_{E\in\mathcal{F}} g(|E|)v_E^2.$$

We divide both sides by $(-1)^k$ and move the L.H.S to the R.H.S. So we have

$$0 = \sum_{i\in\{0,1,\ldots,k\},k-i\ is\ odd} b_i \sum_{I\in X_i} L_I^2((v_E)) + \sum_{E\in\mathcal{F}} g(|E|)v_E^2. \tag{3}$$

Since b_i's are positive by Lemma 1 and $g(|E|) \geq 0$ by the remark immediately after the proof of Lemma 1 in this section, the R.H.S is a sum of nonnegative terms. So obviously if $l(E) > l+k-1$, *i.e.* $|E| > m_{l+k-1}$, then $g(|E|) > 0$, which implies $v_E = 0$. Here $l(.)$ is the height function of X defined in §1. The equation (3) also implies that $L_I((v_E)) = 0$ for $I \in X_i$, $i = k-1, k-3, \cdots$. So $L_I((v_E)) = 0$ for all $I \in X_0 \cup X_1 \cup \cdots \cup X_k$. In particular, $L_0((v_E)) = 0$ where 0 is the least element of X.

To show that $v_E = 0$ for all $E \in \mathcal{F}$, we assume the contrary. Define $J = \{l(E)|E \in \mathcal{F}, v_E \neq 0\}$. Let j_0 be the largest number of J. By the results in the previous paragraph and the remark after the proof of Lemma 1, we have $l \leq j_0 \leq k+l-1$.

In the following, we distinguish 2 cases:

Case 1. Suppose $j_0 = l$, then there exists an $E \in \mathcal{F}$ with $l(E) = l$ and $v_E \neq 0$.

Since $\mathcal{F}$ is an $\{l, l+1, \cdots, l+k-1\}$-intersection family, $l(E \wedge F) \geq l = l(E)$ for $\forall F \in \mathcal{F}$, so either $F > E$ or $F = E$. If $F \neq E$, then $F > E$ and $l(F) > l(E)$. Therefore $v_F = 0$ by the definition of J and j_0. Further because $0 = L_0((v_F)) = \sum_{F\in\mathcal{F}} v_F = v_E$, we have $v_E = 0$, a contradiction.

Case 2. Suppose $l < j_0 \leq l+k-1$ and there exists an $E \in \mathcal{F}$ such that $l(E) = j_0$ and $v_E \neq 0$. We fix such an E.

Since $f_0, f_1, \cdots, f_{j_0-l}$ form a base of the vector space of polynomials of degree $\leq j_0 - l$, there exist real numbers $c_0, c_1, \cdots, c_{j_0-l}$ such that

$$\sum_{i=0}^{j_0-l} c_i f_i(x) = (x-m_l)\cdots(x-m_{j_0-1}).$$

In the following we let $h(x) = (x - m_l)\cdots(x - m_{j_0-1})$ and define $\lambda_I(E) = 1$ if $I \leq E$ and 0 otherwise.

As in the proof of Lemma 2, we have

$$\begin{aligned}
&\sum_{i=0}^{j_0-l} c_i \sum_{I\in X_i} \lambda_I(E)L_I \\
=&\sum_{i=0}^{j_0-l} c_i \sum_{F\in\mathcal{F}} x_F|\{I|I \in X_i, I \leq F, I \leq E\}| \\
=&\sum_{i=0}^{j_0-l} c_i \sum_{F\in\mathcal{F}} x_F f_i(|F\wedge E|) \\
=&\sum_{F\in\mathcal{F}} x_F \sum_{i=0}^{j_0-l} c_i f_i(|F\wedge E|) \\
=&\sum_{F\in\mathcal{F}} x_F h(|E\wedge F|).
\end{aligned}$$

Specializing $x_F = v_F$, $\forall F \in \mathcal{F}$, we have

$$\sum_{i=0}^{j_0-l} c_i \sum_{I\in X_i} \lambda_I(E)L_I((v_F)) = \sum_{F\in\mathcal{F}} v_F h(|E\wedge F|) \tag{$*$}$$

Since $L_I((v_F)) = 0, \forall I \in X_0 \cup X_1 \cup \cdots \cup X_k$, the left hand side is equal to 0. We know if $F \not\geq E$, then $l(E \wedge F) < l(E) = j_0$ and so $|E \wedge F| \in \{m_l, m_{l+1}, \cdots, m_{j_0-1}\}$ which implies $h(|E \wedge F|) = 0$; if $F \geq E$ and $F \neq E$, then $l(F) > l(E) = j_0$ and by the definition of J and j_0, $v_F = 0$. So the right hand side of $(*)$ is equal to $h(|E \wedge E|)v_E = h(|E|)v_E$. Since $h(|E|) \neq 0$, we get $v_E = 0$, a contradiction. This proves Lemma 3 and therefore completes the proof of Theorem 2.

§4. The Proof of Theorem 3

Let $l = |\cup_{i=0}^k X_i| = \sum_{i=0}^k |X_i|$. We consider the $|\mathcal{F}|$ by l 0-1 matrix M whose rows are indexed by elements of $\mathcal{F}$ and whose columns are indexed by elements of Y_k, where $Y_k := \cup_{i=0}^k X_i$ for $k = 0, 1, \ldots, n$. For $F \in \mathcal{F}, S \in Y_k$, the (F, S)-entry of M is defined to be 1 if either $F \in Y_k$ and $S = F$ or $F \in X - Y_k$, $S \in X_k$ and $S \leq F$. It is defined to be 0 otherwise.

Observations. It is clear from the above definition of M that

(1) if the (F,S)-entry of M is 1 then $S \leq F$,

(2) each row has at least one nonzero entry, and

(3) if $F \in \mathcal{F}$ and $F \in X_u, u > k$, then the row corresponding to F has $f_k(m_u) > 1$ nonzero entries (see remark 4 in the introduction).

Claim. For $F \neq E \in \mathcal{F}$, the (F,E)-entry in MM^T is 0.

Proof of the claim. Suppose the (F,E)-entry of MM^T is ≥ 1. Then there exists an $S \in Y_k$ such that both the (F,S)-entry and the (E,S)-entry of M are 1. By observation (1), $S \leq F \wedge E$, so $S \in \cup_{i=0}^{k-1} X_i = Y_{k-1}$ since $\mathcal{F}$ is a $\{0,1,\ldots,k-1\}$-intersection family. But from the definition of M, for such an S, the (F,S)-entry of M is 1 if and only if $F = S$. The same is true for the (E,S)-entry. So $F = S = E$, which is a contradiction. This proves the claim.

From the above claim, it is clear that MM^T is a diagonal matrix, and it is also clear that the diagonal entries are nonzero by observation (2) above. So MM^T is a nonsingular $|\mathcal{F}|$ by $|\mathcal{F}|$ matrix.

Therefore $Rank(M) = Rank(MM^T) = |\mathcal{F}|$. Since M is an $|\mathcal{F}|$ by l matrix, we must have $l \geq |\mathcal{F}|$.

Now suppose $|\mathcal{F}| = \sum_{i=0}^{k} |X_i|$, so M is a square matrix. It is clear that each column can contain at most one nonzero entry, otherwise MM^T would not be a diagonal matrix. So the total number of 1's in M is $\leq |\mathcal{F}|$. Therefore by observation (2) the total number of 1's in M is $|\mathcal{F}|$. So each row of M should contain exactly one nonzero entry by the above observations. This means that $F \in Y_k$ for any $F \in \mathcal{F}$. So $\mathcal{F} \subseteq Y_k = \cup_{i=0}^{k} X_i$. But $|\mathcal{F}| = \sum_{i=0}^{k} |X_i|$, so $\mathcal{F} = \cup_{i=0}^{k} X_i$. $\square$

Remark . For the proof of Theorem 4, we consider the 0-1 incidence matrix M_k whose rows and columns are indexed by the elements of $\mathcal{F}$ and X_k respectively. The rest of the proof is similar to but simpler than that of Theorem 3 above and hence omitted.

§5. **The Proof of Theorem 5**

Define $g_i(x) = (x - m_0)(x - m_1)\cdots(x - m_{i-1})$ for $i = 1, 2, \cdots, s$ and $g_0(x) = f_0(x) = 1$. Since $g_i(x)$'s are monic, there exist $s+1$ integers $b_0, b_1, \cdots, b_s$ such that $(x - m_{l_1})(x - m_{l_2})\cdots(x - m_{l_s}) = \sum_{i=0}^{s} b_i g_i(x)$. So

$$(x - m_{l_1})(x - m_{l_2})\cdots(x - m_{l_s}) = \sum_{i=0}^{s} c_i f_i(x) \tag{4}$$

where $c_i = b_i(m_i - m_0)(m_i - m_2)\cdots(m_i - m_{i-1})$ are integers for $i = 1, 2, \cdots, s$ and $c_0 = b_0$, since by remark 4 in the introduction $f_i(x) = g_i(x)/(m_i - m_0)(m_i - m_2)\cdots(m_i - m_{i-1})$ for $i = 1, 2, \cdots, s$.

For $0 \le l \le k$ we define $M_{k,l}$ to be an incidence matrix whose rows and columns are indexed by elements of X_k and X_l respectively. For $A \in X_k, S \in X_l$, the (A, S)-entry is 1 if $S \le A$ and 0 otherwise.

The (A, B)-entry of $M_{k,s}M_{s,i}$ is easily seen to be the number of elements S such that $A \ge S \ge B$ where $A \in X_k, S \in X_s, B \in X_i$. Therefore from the definition of polynomial semi-lattice it follows that this number is $f_{s-i}(m_{k-i})$ and hence

$M_{k,s}M_{s,i} = f_{s-i}(m_{k-i})M_{k,i}$ for $i \le s \le k$.

From this we see that the column space of $M_{k,i}$ is contained in that of $M_{k,s}$ for $i = 0, 1, \ldots, s$.

Then the column space of the integer matrix $M := \sum_{i=0}^{s} c_i M_{k,i} M_{k,i}^T$ is contained in that of $M_{k,s}$. So $rank(M) \le rank(M_{k,s}) \le |X_s|$.

Define $M_{\mathcal{F}}$ to be the submatrix of M, whose rows and columns are indexed by elements of $\mathcal{F}$. Similarly as in the proof of Lemma 2 of §3, we easily check that the (A, B)-entry of $M_{\mathcal{F}}$ is $\sum_{i=0}^{s} c_i f_i(|A \wedge B|)$ which by (4) above is equal to

$$(|A \wedge B| - m_{l_1})(|A \wedge B| - m_{l_2})\cdots(|A \wedge B| - m_{l_s}).$$

So the (A, B)-entry of $M_{\mathcal{F}}$ is $\equiv (|A \wedge B| - \mu_1)(|A \wedge B| - \mu_2)\cdots(|A \wedge B| - \mu_s) \pmod p$.

Now if $A \ne B$, then $|A \wedge B| \in L$ and therefore $|A \wedge B|$ is congruent to one of $\mu_1, \mu_2, \cdots, \mu_s$ by the condition of Theorem 5, so the (A, B)-entry of $M_{\mathcal{F}}$ is $\equiv 0 \pmod p$; if $A = B$, then $|A \wedge B| = |A| = m_k$ which is not congruent to any one of $\mu_1, \mu_2, \cdots, \mu_s$, so the (A, A)-entry of $M_{\mathcal{F}}$ is $\not\equiv 0 \pmod p$.

In summation we have: (A, B)-entry of $M_{\mathcal{F}}$ is $\equiv 0 \pmod p$ if $A \neq B$ but $\not\equiv 0 \pmod p$ if $A = B$.

So $M_{\mathcal{F}}$, considered as a matrix over $\mathbb{F}_p$, the finite field of order p, is a square matrtix whose diagonal entries are nonzero and whose nondiagonal entries are 0. So $det(M_{\mathcal{F}}) \not\equiv 0$ $(mod\ p)$. This implies that $det(M_{\mathcal{F}})$ is a nonzero rational integer and therefore $M_{\mathcal{F}}$ is nonsingular.

Therefore $|\mathcal{F}| = rank(M_{\mathcal{F}})$ and hence $|\mathcal{F}| = rank(M_{\mathcal{F}}) \leq rank(M) \leq |X_s|$. □

§6. The Proof of Theorem 6

We keep the notations as in the proof of Theorem 5 in §5. In particular, by the argument in §5, we have the following fact:

there exist $s+1$ *integers* $c_0, c_1, \cdots, c_s$ *such that* $(x-\mu_1)(x-\mu_2)\cdots(x-\mu_s) \equiv \sum_{i=0}^{s} c_i f_i(x)$ $(mod\ p)$.

Define M_i to be the 0-1 incidence matrix of $\mathcal{F}$ and X_i whose rows and columns are indexed by the elements of $\mathcal{F}$ and X_i respectively. The (F, S)-entry of M_i is 1 if and only if $F \geq S$, 0 otherwise. It is not hard to see that the (F, E)-entry of $M_i M_i^T$ is $f_i(|F \wedge E|)$.

Define $M_{\mathcal{F}}$ to be an integer matrix whose rows and columns are indexed by the elements of $\mathcal{F}$. The (F, E)-entry of $M_{\mathcal{F}}$ is $\sum_{i=0}^{s} c_i f_i(|F \wedge E|)$. It is clear that each row of $M_{\mathcal{F}}$ is a linear combination of rows of $M_0 M_0^T, M_1 M_1^T, \cdots, M_s M_s^T$ and so

$rank(M_{\mathcal{F}}) \leq \sum_{i=0}^{s} rank(M_i M_i^T) \leq \sum_{i=0}^{s} |X_i|$.

By a similar argument as in the proof of Theorem 5, we can show that $M_{\mathcal{F}}$, considered as a matrix over $\mathbb{F}_p$, is such that all the nondiagonal entries are 0 and all the diagonal entries are nonzero. So $det(M_{\mathcal{F}}) \not\equiv 0$ $(mod\ p)$, which implies that $det(M) \neq 0$ in $\mathbb{Z}$. So M is nonsingular and therefore $|\mathcal{F}| = rank(M) \leq \sum_{i=0}^{s} |X_i|$. □

References

[1] N. Alon, L. Babai and H. Suzuki, "Multilinear Polynomials and Frankl-Ray-Chaudhuri-Wilson Type Theorems," *Journal of Combinatorial Theory Ser(A),* 58: 165-180, 1991.

[2] L.Babai and P. Frankl, "Linear Algebra Methods in Combinatorics," (Preliminary version 2) Department of Computer Science, University of Chicago.

[3] M.Deza, P.Frankl and N.M.Singhi, "On functions of strength t," *Combinatorica* 3 (1983) 331-339.

[4] P. Frankl and Z. Füredi, "Families of Finite sets with missing intersections," *Colloquia Mathematica Societatis Janos Bolyai* 37:305-320, 1981.

[5] P. Frankl and R. M. Wilson, "Intersection Theorem with Geometric Consequences," *Combinatorica* 1(4) (1981) 357-368.

[6] G. V. Ramanan, "Proof of a conjecture of Frankl and Füredi," To appear.

[7] D. K Ray-Chaudhuri and R. M Wilson, "On t-designs," *Osaka J. Math* 12(1975), 737-744.

[8] D. K. Ray-Chaudhuri and Tianbao Zhu, "S-Intersection Families and Tight Designs," *Coding Theory, Design Theory, Group Theory* Proceeding of Marshall Hall Conference. Page: 67-75 John-Wiley & Sons 1993.

[9] H. S Snevily, "A generalization of the Ray-Chaudhuri-Wilson theorem," *J. Comb. Des.* 3(1995) No. 5 349-352.