Shape Tiling

Kevin Keating & Jonathan L. King

keating@math.ufl.edu & squash@math.ufl.edu
University of Florida, Gainesville FL 32611-2082, USA
Submitted: 28 July 1996 Accepted: 21 November 1996

Abstract. Given a list $1{\times}1, 1{\times}a, 1{\times}b, \ldots, 1{\times}c$ of rectangles, with $a, b, \ldots, c$ non-negative, when can $1 \times t$ be tiled by positive and negative copies of rectangles which are similar (uniform scaling) to those in the list? We prove that such a tiling exists iff t is in the field $\mathbb{Q}(a, b, \ldots, c)$.

When can rectangle $1 \times t$ be packed by (finitely many) squares? Dehn 1903 gave the answer: *If and only if t is rational.* For irrational t he showed $1 \times t$ *not* packable by means of what we will call a "Dehn-functional". It is a map $\mathbf{D}$ from pairs of real numbers to $\mathbb{R}$ (or any abelian group) which satisfies:

$$\begin{aligned} \mathbf{D}\big([x + x'] \times y\big) &= \mathbf{D}(x \times y) + \mathbf{D}(x' \times y) \\ \mathbf{D}\big(x \times [y + y']\big) &= \mathbf{D}(x \times y) + \mathbf{D}(x \times y') \end{aligned}$$

It is straightforward to check that for a packing of a rectangle $c \times d$ by finitely-many others, $\mathbf{D}(c \times d)$ must equal the sum of the functional applied to each rectangle in the packing. (The analogous statement applies to tiling. See the **Definitions** section, below, for a formal definition of packing and tiling.)

Two recent papers by Freiling & Rinne 1994, and by Laczkovich & Szekeres 1995, turn the question around: *For which sidelengths, s, can the square be packed by rectangles similar to $1 \times s$ and $s \times 1$?* Employing a Dehn-functional and a theorem of Wall 1945, they give this astonishing answer: *Iff s is algebraic over $\mathbb{Q}$, and all of its conjugates in the complex plane have positive real part.* (We shall henceforth refer to such numbers s as ***Wall numbers***.)

Tilings. Every packing problem has an analogous problem using both positive and negative copies of the prototiles; we will call this operation "signed packing" or "tiling".

It turns out that Dehn's question has the same answer if tiling is allowed: $1 \times t$ *can be tiled by squares iff it can be packed by squares.* However, one sees readily that the [FR,LS] question has a larger answer if tiling is allowed, by considering the Golden Ratio

---

1991 *Mathematics Subject Classification.* 05B45. Secondary: 52C20 51M25 05B50 12E99.

Key words and phrases. Tiling, packing, Wall number, Dehn's theorem, fields.

$\lambda := \frac{1+\sqrt{5}}{2}$. The conjugate of the Golden Ratio is $\frac{1-\sqrt{5}}{2}$, which is negative. Thus the [FR,LS] theorem guarantees that no square can be packed by rectangles similar to $1{\times}\lambda$ and $\lambda{\times}1$. Nonetheless, there <u>is</u> a tiling:

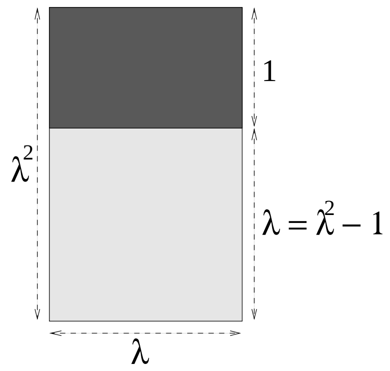

FIGURE 1 The dark rectangle, $\lambda\times 1$, is being subtracted from the top of the tall $\lambda\times\lambda^2$ rectangle. Since λ^2 equals $\lambda+1$, what remains after the subtraction is the $\lambda\times\lambda$ square.

The goal of our article is to establish a general tiling theorem for rectangles. A special case of the Tiling Theorem, below, is:

Rectangles with shapes $\{1\times s,\ s\times 1\}$ *can tile a square IFF* $s\in\mathbb{Q}(s^2)$.

Definitions. As usual, let $\mathbb{Q}(x)$ denote the field of rational functions of x, with coefficients in $\mathbb{Q}$. For ζ a complex number, $\mathbb{Q}(\zeta)$ is the smallest subfield of $\mathbb{C}$ containing ζ. Given a (finite or infinite) subset $S\subset\mathbb{C}$, let $\mathbb{Q}(S)$ be the smallest subfield of $\mathbb{C}$ which includes S.

Identify a rectangle $a\times b$ with a product of half-open intervals, the subset $[0,a)\times[0,b)$ of the plane. A translate, T, of $a\times b$ is a set of the form

$$[t_1, t_1+a) \;\times\; [t_2, t_2+b)$$

where $t_1, t_2\in\mathbb{R}$. Say that a collection $\mathbb{P}$ of rectangles ***packs*** $c\times d$ if we can find a (finite) collection, TRANS, of translates of copies of rectangles in $\mathbb{P}$ such that we have equality

$$\mathbf{1}_{c\times d} \;=\; \sum_{T\in\mathrm{TRANS}} \mathbf{1}_T$$

between indicator functions. (Indicator function $\mathbf{1}_T$ is 1 for each point (x,y) in T and is 0 for all other points in the plane.)

Say that collection $\mathbb{P}$ ***tiles*** (or "signed-packs") rectangle $B = b_1\times b_2$ if: A finite collection TRANS and coefficients $\alpha_T\in\{1,-1\}$ can be found so that

$$\mathbf{1}_B \;=\; \sum_{T\in\mathrm{TRANS}} \alpha_T\mathbf{1}_T\,. \tag{2}$$

(All of these definitions make sense in D-dimensional Euclidean space. For integer-sided D-dimensional polyominoes and bricks, this type of tiling question was studied by BARNES [B1,B2] and KING [Kin]. In particular, given a finite proto-set $\mathbb{P}$ of D-dimensional bricks there is an algorithm –which runs, as a function of the number of bits needed to describe a brick $B = b_1{\times}b_2{\times}\cdots{\times}b_D$, in linear time– to determine whether B is tilable by $\mathbb{P}$. There is also a computable number $\mathcal{M}=\mathcal{M}(\mathbb{P})$ so that if each sidelength $b_i\geq\mathcal{M}$, then B is $\mathbb{P}$-packable iff it is $\mathbb{P}$-tilable.)

Lastly, a tiling $\mathbf{1}_{c\times d} = \sum_{T\in\mathrm{TRANS}} \alpha_T \mathbf{1}_T$ is "horizontally splittable" if we can write $c = c^{(1)} + c^{(2)}$ and $\mathrm{TRANS} = \mathcal{C}^{(1)} \sqcup \mathcal{C}^{(2)}$, a disjoint union of non-empty sets, so that:

$$\mathbf{1}_{c^{(i)}\times d} \;=\; \sum_{T\in\mathcal{C}^{(i)}} \alpha_T \mathbf{1}_T\,,$$

for $i = 1, 2$. Define "vertically splittable" analogously.

Tiling (2) is ***completely-splittable*** if, either: TRANS is a singleton *or* –recursively– the tiling can be split, either horizontally or vertically, into two tilings each of which is completely-splittable.

Shapes. Uniformly scaling rectangle $a\times b$ by ***scale-factor*** u (a positive number) yields rectangle $au \times bu$. Let the ***shape*** $a \times b$ represent the set of all uniform-scalings of the rectangle. Consequently, say that $\mathbb{P}$ ***shape-packs*** $c \times d$ if the union

$$\bigcup_{a\times b\in\mathbb{P}} \bigl\{au \times bu \bigm| u > 0\bigr\} \qquad \text{packs} \qquad c \times d.$$

Define "$\mathbb{P}$ ***shape-tiles*** $c \times d$" analogously.

§2 Some Results

We start with a normalization. For each positive number v, a collection $\bigl\{1 \times s\bigr\}_{s\in S}$ shape-tiles $1 \times t$ iff $\bigl\{1 \times vs\bigr\}_{s\in S}$ shape-tiles $1 \times vt$. We can choose v so that some product vs is 1. Consequently, we can assume, gratis, that S contains 1.

Tiling Theorem, 3. *Suppose $1 \in S$, where S is a (finite or infinite) set of positive reals. Then rectangles $\mathbb{P} := \bigl\{1\times s \bigm| s \in S\bigr\}$ shape-tile $1\times t$ IFF t is in $\mathbb{Q}(S)$, and $t \geq 0$.*

Moreover, when $t \in \mathbb{Q}(S)$, there is a tiling which is completely-splittable and uses only scale-factors in the field $\mathbb{Q}(S)$.

Proof. For a tilable $1 \times t$, it will be temporarily convenient to say that $1 \times (-t)$ is tilable also. Definition (2) extends consistently to rectangles with negative sidelengths, if we identify $\mathbf{1}_{a\times(-b)}$ and $\mathbf{1}_{(-a)\times b}$ with $-\mathbf{1}_{a\times b}$. Thus we can freely remove the "$t \geq 0$" in the statement of the theorem.

We will make use of the field $K := \mathbb{Q}(S)$.

Establishing $(\Rightarrow)$. If $t \notin K$ then there exists[†] a K-linear functional $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(t) = 0$ and $f(1) = 1$. Thus

$$\mathbf{D}(x \times y) := x \cdot f(y)$$

is a Dehn-functional. For any $s \in S$ and real u,

$$\mathbf{D}(u \times su) \;=\; u \cdot f(su) \;=\; u \cdot s \cdot f(u) \;=\; \mathbf{D}(su \times u)\,.$$

[†]We can define the linear functional by picking a K-basis for $\mathbb{R}$. Or, we can avoid the Axiom of Choice, as follows. Let V be the K-vector-subspace of $\mathbb{R}$ spanned by the sidelengths of all the rectangles in the purported tiling. Extend the collection $\{t, 1\}$ to a K-basis for V, then define f on this basis to get the desired K-linear-functional $f\colon V \to \mathbb{R}$.

Thus the Dehn-functional $\mathbf{D}(y \times x) - \mathbf{D}(x \times y)$ is zero on every shape in the proto-set $\mathbb{P}$. Hence this Dehn-functional must be *zero* on each tilable rectangle. On the other hand, its value on $1 \times t$ is the difference $t \cdot 1 \;-\; 1 \cdot 0$, which is not zero.

Establishing ($\Leftarrow$). Let $\mathcal{G}$, the "good set", be the collection of numbers t such that $1 \times t$ is shape-tilable by the proto-set. Consider good numbers p and q. Then $1 \times (-p)$ is tilable and, by stacking $1 \times p$ on top of $1 \times q$, also $1 \times (p+q)$ is tilable. Thus

The good set is preserved under negation and addition.

What happens when we place $1 \times p$ and $1 \times q$ side-by-side? Scaling each appropriately gives rectangles $q \times qp$ and $p \times pq$. These tile $(p+q) \times pq$. So if $p + q \neq 0$, we conclude that $\frac{pq}{p+q}$ is good. Thus

The good set is preserved under "twisting"

where, for $p \neq -q$, we define the ***twist*** of p with q to be

$$p \bowtie q \;:=\; \frac{pq}{p+q}.$$

Notice that the operation of twisting rectangles $1 \times p$ and $1 \times q$ scales them by scale-factors $\frac{q}{p+q}$ and $\frac{p}{p+q}$, both of which are in K.

Lastly, since the operation of twisting (resp. addition) corresponds to building a tiling which splits horizontally (resp. vertically), the following Field Lemma will complete the proof of the theorem. $\Diamond$

FIELD LEMMA, 4. *Suppose $1 \in \mathcal{G}$, where $\mathcal{G}$ is a subset of $\mathbb{C}$ which is closed under negation, addition and twisting. Then $\mathcal{G}$ is a subfield of $\mathbb{C}$.*

PROOF. Suppose p is "good", that is, in $\mathcal{G}$. Then pn and p/n are good, for positive integers n; this follows by induction and using that goodness is preserved under addition and twist. In the following, p and q are assumed to be good.

Reciprocals are good: For $p \neq 0$, note that $(p-1) \bowtie 1 = \frac{p-1}{p}$ is good. Thus $\frac{1}{p}$, which equals $1 - \frac{p-1}{p}$, is good.

Squares are good: Since $(1 \pm p)$ is good, $(1-p) \bowtie (1+p)$ is good. Multiplying by -2 yields that $p^2 - 1$ is good, hence p^2.

Products are good: Since $(p+q)^2 - (p-q)^2$ is good, so is $4pq$ and thus pq. $\Diamond$

Addendum. Note that the lemma continues to hold with $\mathbb{C}$ replaced by any field whose characteristic is not two, i.e, $1 + 1 \neq 0$.

Question. By using a Dehn-functional, it is straightforward to see that if the tiling in Theorem 3 is actually a packing, then all the scale-factors must be in $\mathbb{Q}(S)$.

Does this same conclusion hold for all minimum-cardinality tilings? (I.e, those which minimize the cardinality of TRANS, the set of translates).

Closing remark. The [FR,LS] theorem suggests studying the following transitive relation $\Rrightarrow$ on the positive reals: $s \Rrightarrow t$ if $\{1{\times}s, s{\times}1\}$ shape-packs $1{\times}t$. Restating their result: $s \Rrightarrow 1$ iff s is a Wall number. Consequently, these numbers are hereditary; if $s \Rrightarrow t$, with t a Wall number, then s is too.

We currently have no understanding of the arrow relationship. Certainly if the minimal polynomial of s is unrelated to that of t, then there is no reason to expect $s \Rrightarrow t$. Our theorem can, of course, give no positive result. It does, however, give the negative result that even if s and t have the *same* minimal polynomial, neither need arrow the other—simply because neither tiles the other.

In the normalization of the Tiling Theorem, a collection $\{1{\times}s, s{\times}1\}$ shape-tiles $1{\times}t$ exactly when $st \in \mathbb{Q}(s^2)$. Now suppose lengths s and t have a common minimal polynomial $f(x) \in \mathbb{Z}[x]$ which is cubic with three positive roots. Certainly $st \notin \mathbb{Q}(s^2)$ occurs if $\mathbb{Q}(s)$ *fails* to contain all three roots. And this will be the case if the discriminant of f is not a perfect square. (See definition and corollary of [Jac, p. 258].) Indeed, we only need find such an f with 3 real roots since, for a sufficiently large integer T, the translated polynomial $x \mapsto f(x-T)$ will have all roots positive.

An example is provided by $f(x) := x^3 - 6x + 2$, which has 3 real roots and, by the Eisenstein Criterion [H, Thm. 3.10.2], is irreducible. The discriminant of f equals $-4\cdot(-6)^3 - 27\cdot 2^2 = 6^2\cdot 3\cdot 7$, which is not a perfect square.

References

[B1] F.W. Barnes, *Algebraic theory of brick packing, I*, Discrete Math. **42** (1982), 7–26.

[B2] F.W. Barnes, *Algebraic theory of brick packing, II*, Discrete Math. **42** (1982), 129–144.

[Deh] M. Dehn, *Über die Zerlegung von Rechtecken in Rechtecke*, Math. Ann. **57** (1903), 314–332.

[FR] C. Freiling & D. Rinne, *Tiling a Square with Similar Rectangles*, Math. Research Letters **1** (1994), 547–558.

[H] I.N. Herstein, *Topics in Algebra*, Wiley & Sons, 1975.

[Jac] Nathan Jacobson, *Basic Algebra I, 2nd ed.*, W.H. Freeman, 1974.

[Kin] J.L. King, *Brick Tiling and Monotone Boolean Functions*, Preprint available at webpage `http://www.math.ufl.edu/~squash/`

[LS] M. Laczkovich & G. Szekeres, *Tilings of the Square with Similar Rectangles*, Discrete Comput. Geom. **13** (1995), 569–572.