Escher's Combinatorial Patterns

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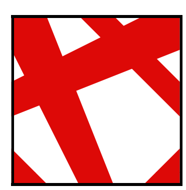
ABSTRACT: It is a little-known fact that M. C. Escher posed and answered some combinatorial questions about patterns produced in an algorithmic way. We report on his explorations, indicate how close he came to the correct solutions, and pose an analogous problem in 3 dimensions.

In the years 1938-1942, the Dutch graphic artist M. C. Escher developed what he called his "layman's theory" on regular division of the plane by congruent shapes. During this time he also experimented with making repeating patterns with decorated squares by using combinatorial algorithms. The general scheme is easy to describe. Take a square and place inside it some design; we call such a one-square design a *motif*. Then put together four copies of the decorated square to form a 2x2 square array. The individual decorated squares in the array can be in any *aspect*, that is, each can be any rotated or reflected copy of the original square. Finally, take the 2x2 array (which we call a *translation block*) and translate it repeatedly in the directions perpendicular to the sides of the squares to fill the plane with a pattern.

The process can be easily carried out. In his article "Potato Printing, a Game for Winter Evenings," Escher's eldest son George describes how this can be a pleasurable game with children or grandchildren. (He and his brothers played the game with his father.) Two pieces of cut potato can serve as the medium on which to carve the motif and its reflected image, and then these potato stamps are inked and used to produce a pattern according to the rules of the game. Escher himself used various means to produce patterns in this algorithmic way. He made quick sketches of square arrays of patterns in his copybooks, he stamped out patterns with carved wooden stamps, and he decorated small square wooden tiles (like Scrabble pieces) and then assembled them into patterns.

Escher's sketchbooks show his attempts to design a suitable motif to use for such a pattern—a single design that was uncomplicated, yet whose repeated copies would produce interesting patterns of ribbons that would connect and weave together. The first motif he chose was very simple, yet effective. In it, three bands cross each other in a square. Two of them connnect a corner to the midpoint of the opposite side and the third crosses these, connecting midpoints of two adjacent sides. Small pieces of bands occupy the two remaining corners. Every corner and every midpoint of the square is touched by this motif.



Escher carved two wooden stamps with this motif, mirror images of each other, and used them to experiment, stamping out patches of patterns. His sketchbooks are splotched with these, filling blank spaces on pages alongside rough ideas and preliminary drawings for some of his graphic works and periodic drawings. His many experimental stamped pattterns show no particular methodical approach—no doubt he was at first interested only in seeing the visual effects of various choices for the 2x2 translation block. At some point Escher asked himself the question:

How many different patterns can be made with a single motif, following the rules of the game?

In order to try to answer the question, he restricted the rules of choice for the four aspects of the motif that make up the 2x2 translation block. (Definition: Two motifs have the same aspect if and only if they are congruent under a translation.) He considered two separate cases:

(1) The four choices that make up the translation block are each a direct (translated or rotated) image of the original motif. Only one wooden stamp is needed to produce the pattern.

(2) Two of the choices for the translation block are direct images of the original motif and two are opposite (reflected) images. Additionally, one of the following restrictions also applies:

(2A) the two direct images have the same aspect and the two reflected images have the same aspect

(2B) the two direct images have different aspects and the two reflected images have different aspects.

Escher set out in his usual methodical manner to answer his question. Each pattern could be associated to a translation block that generated it. In order to codify his findings, he represented each of these 2x2 blocks by a square array of four numbers—each number represented the aspect of the motif in the corresponding square of the translation block. The square array of four numbers provided a *signature* for the pattern generated by that translation block. The four rotation aspects of the motif gotten by turning it 90° three successive times were represented by the numbers 1, 2, 3, 4 and the reflections of these (across a horizontal line) were $\underline{1}$, $\underline{2}$, $\underline{3}$, $\underline{4}$. Sometimes Escher chose his basic 90° rotation to be clockwise, sometimes counterclockwise.

Figure 1 shows three different motifs that Escher used to generate patterns according to his rules, together with one particular translation block and the patterns generated by that block for each of the three motifs. The first motif is just a segment that joins a vertex of the square to a midpoint of an opposite side, while the second is a **V** of two segments that join the center of the square to the midpoint and a vertex of one side. These could be quickly drawn to sketch up patterns. For each of these motifs, Escher used a clockwise turn to obtain the successive rotated aspects. The third motif was stamped from a carved wooden block and the patterns hand-colored. This motif was turned counterclockwise to obtain the successive rotated aspects. In our figures, we represent the four rotation aspects of each motif by A, B, C, D instead of Escher's 1, 2, 3, 4.

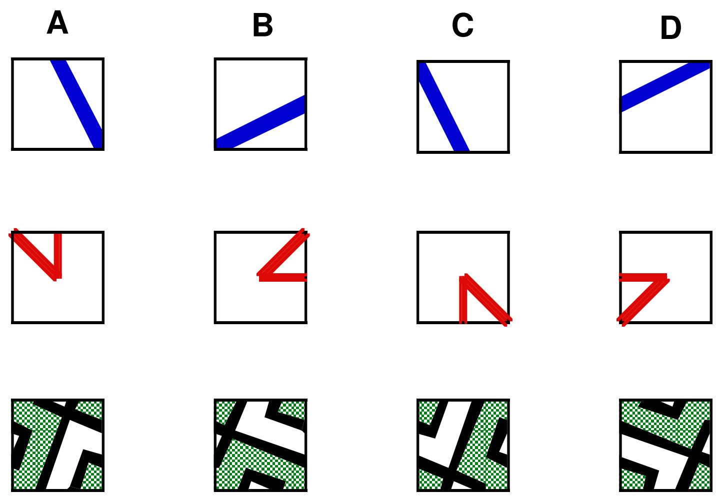
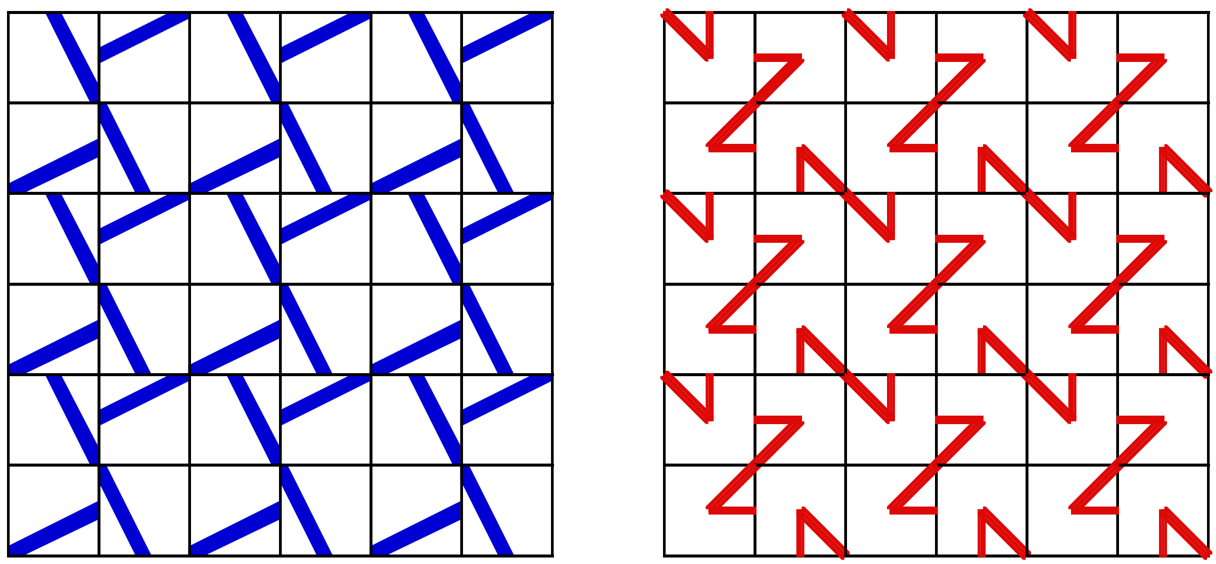
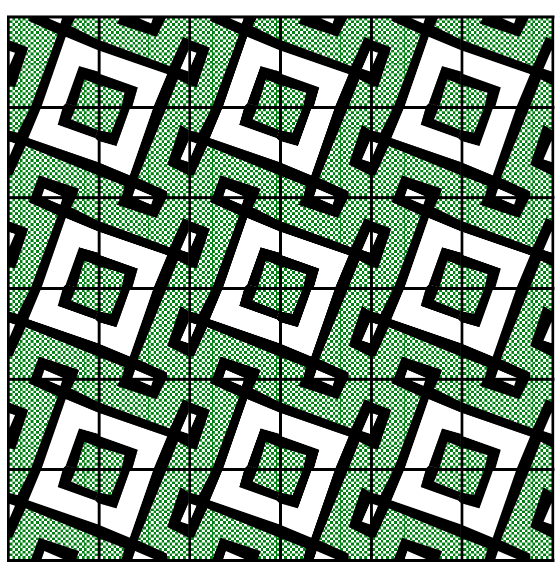


FIGURE 1. A, B, C, D name the four rotated aspects of each of three motifs used by Escher. The 2x2 translation block below produces the patterns shown on this page.

A	D
B	C





At first it may seem as if Escher's question (how many patterns are there?) can be answered by simply multiplying the number of possibilities for each square in the translation block. Yet symmetries relate the different aspects of the motif in a translation block and each pattern has additional periodic symmetry induced by the repeated horizontal and vertical translations of the translation block. These symmetries add a geometric layer of complexity to the combinatorial scheme.

Escher's Case (1)

We first consider Escher's case (1), in which the four choices that make up the translation block are each a direct image of the original motif. Here there are four possible rotation aspects of the motif for each of the four squares in the translation block, so there are $4^4 = 256$ different signatures for patterns that can be produced. Each square array of four letters that is a signature will be represented as a string of four letters by listing the letters from left to right as they appear in clockwise order in the square array, beginning with the upper left corner. Thus the signature for the square array at the right (and in Figure 1) is ADCB.

$$\boxed{\begin{array}{cc} A & D \\ B & C \end{array}}$$

We will say that two signatures are *equivalent* if they produce the same pattern. (Two patterns are the same if one can be made to coincide with the other by an isometry.) Since patterns are not changed by rotation, repeated 90˚ rotations of the translation block of a pattern produces four translation blocks for that pattern, and the four corresponding signatures are equivalent. When the translation block is rotated 90˚, each motif in it changes its aspect as it is moved to the next position in the block. In our example in Figure 1, a 90˚ clockwise rotation of either of the first two motifs (or a 90˚ counterclockwise rotation of the third motif) sends A to B, B to C, C to D, and D to A. Thus under successive 90˚ clockwise rotations of the block, the signature ADCB for the first pattern is equivalent to the signatures CBAD, ADCB, and CBAD. The fact that the second two signatures are repeats of the first two reflects the fact that this translation block has 180˚ (2-fold) rotation symmetry. A translation block with 90˚ (4-fold) rotation symmetry will have only one signature under rotation (for example, ABCD). A translation block with no rotation symmetry will always have four equivalent signatures produced by rotating the block (for example, the block with signature AABB has equivalent signatures CBBC, DDCC and DAAD). But there is still more to consider.

If a pattern is held in a fixed position, there are four distinct translation blocks that produce it (their signatures may or may not differ). This is most easily seen by looking at a pattern of letters generated according to Escher's rule of translating the 2x2 block. The translation block with signature PQSR produces a pattern with alternating rows P Q P Q . . . and R S R S . . . as shown below. The same pattern can be generated by a translation block whose upper left corner is P, or Q, or R, or S:

```
P Q P Q P Q P Q P Q P Q P Q
R S R S R S R S R S R S R S
P Q P Q P Q P Q P Q P Q P Q
R S R S R S R S R S R S R S
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For Escher's patterns, the letters P, Q, R, S in the above array are replaced by various rotated aspects of the motif, represented by the letters A, B, C, D. In this case, some of the four translation blocks outlined may be the same, depending on whether or not there are repeated aspects of the motif that are interchanged by the permutations that correspond to moving the block to a new position. Moving the translation block horizontally one motif unit corresponds to the permutation that interchanges the columns of that block; thus it also rearranges the order of the letters in the signature string by the permutation (12)(34). Moving the block vertically one unit interchanges rows of the block, which corresponds to reordering the signature string by the permutation (14)(23). Moving the block diagonally (a composition of moving vertically one unit and horizontally one unit) interchanges the pairs of diagonal elements of the block, which corresponds to reordering the signature string by the permutation (13)(24). It is easy to see that the four possible translation blocks for a pattern gotten by these moves may all have the same signature (eg., AAAA), or there may be two signatures (e.g., AABB, BBAA), or four signatures (e.g., AAAB, AABA, ABAA, BAAA).

Each of Escher's patterns has at least one signature that begins with the letter A, since rotating and translating the translation block will always give at least one block with its upper left corner occupied by a motif with aspect A. Since there are four aspects of the motif possible for each of the other three squares in the block, there are at most $4^3 = 64$ different patterns. But we know, in fact, that there are far fewer than 64 since many patterns will have as many as four signatures that begin with the letter A. So the final answer to the question "How many different patterns are there?," even in case (1), is not obvious.

The correct answer is *23 different patterns*, and Escher found the answer by a process of methodical checking. He filled pages of his sketchbooks with quickly-drawn patterns of simple motifs generated by various signatures. Each time he found a pattern that had already been drawn, he crossed it out and noted the additional signature for it. In 1942 he made a chart summarizing his results and accompanied it by a display of sketches of all 23 patterns for the first two motifs in Figure 1. In Figure 2, we display all 23 patterns made with Escher's simple line segment motif.

Next to each pattern are all its signatures that begin with the letter A. Note that the signatures are positioned around each pattern so that in order to see a corresponding translation block with a particular signature, you must turn the page so that the letters are upright. This display gives a visual proof that there are 23 different patterns, since all 64 signatures that begin with the letter A are accounted for.

In addition to his inventory of pencil-sketched patterns, Escher made stamped, hand-colored patterns of all 23 types for the third motif of Figure 1 and collected these in a small binder that is dated V-'42.

FIGURE 2(a). The segment motif is rotated clockwise 90˚ three times successively to obtain its four rotated aspects A, B, C, D. Figure 2(b) shows that exactly 23 different patterns are possible according to Escher's case (1) scheme. Each pattern is determined by one or more translation blocks of the type shown below, in which aspect A is in the upper left corner. Each different translation block corresponds to a signature of the form AXYZ, in which X, Y, Z are chosen from A, B, C, D (with repetitions allowed). In the sample pattern below, which has four equivalent signatures, each of the four different translation blocks that generate it are displayed; they are also outlined in the pattern. Turn the page so that signatures are upright to view the translation block with A in the upper left corner. In the display in Figure 2(b), each translation block has been repeated 3 times horizontally and 3 times vertically to produce the patches of patterns.

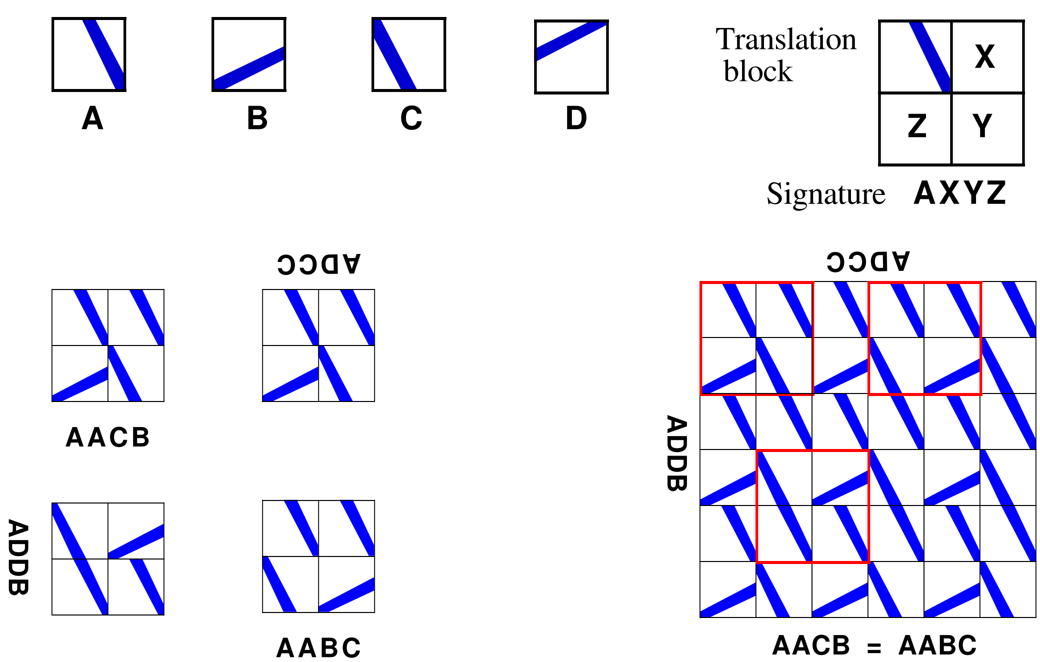


FIGURE 2(b) . The 23 pattern types for Escher's scheme with direct images only.

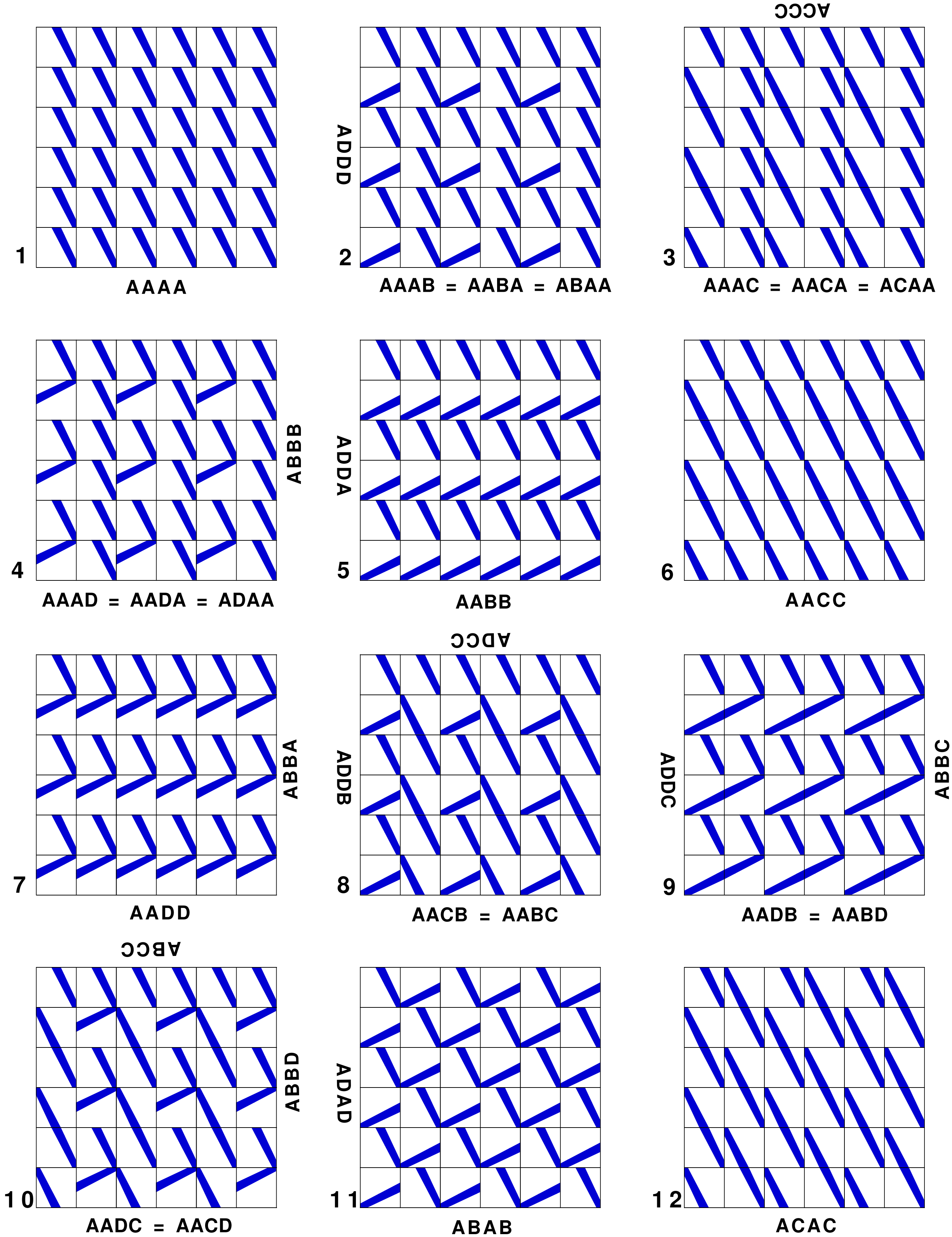
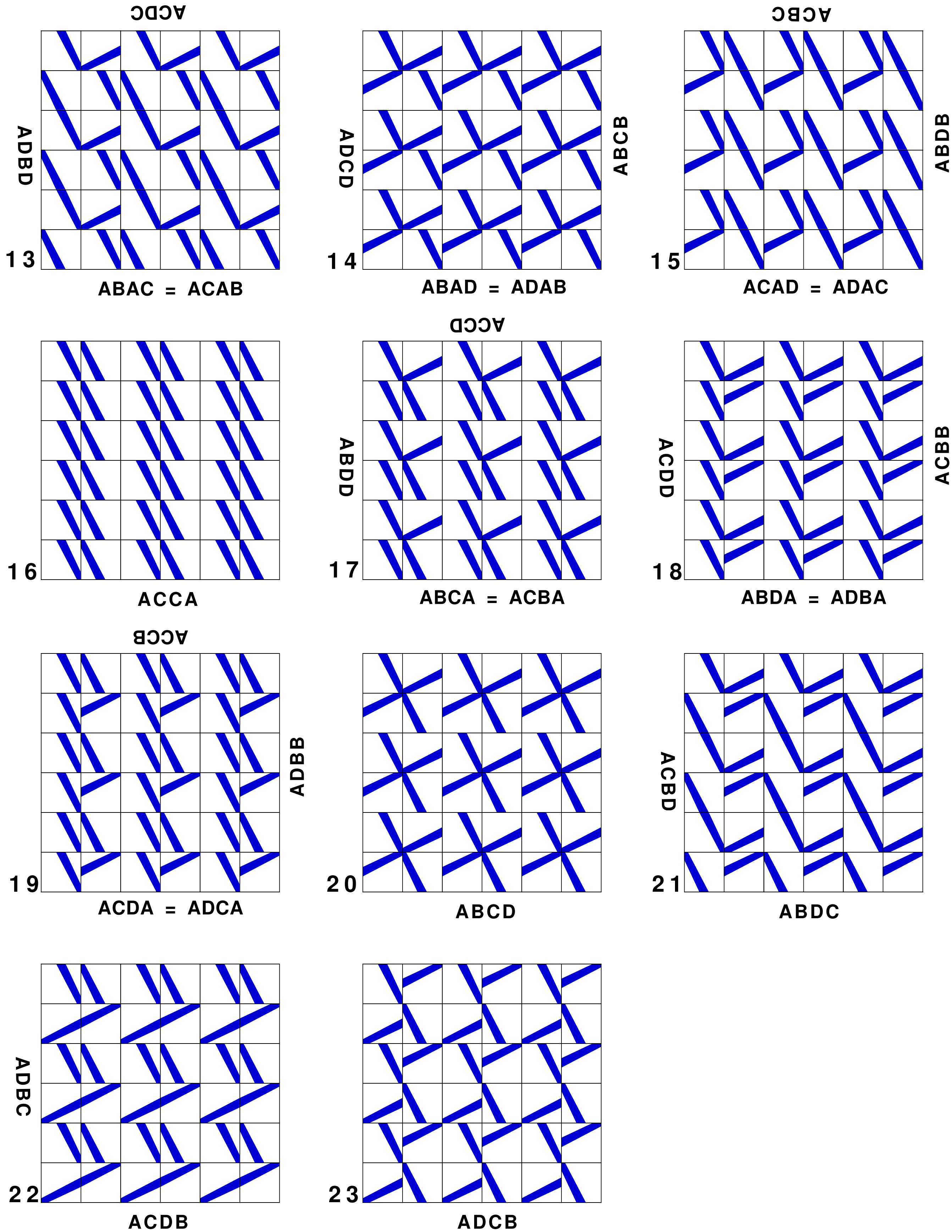


FIGURE 2(b), continued. The 23 pattern types for Escher's scheme with direct images only.



Our display and signatures in Figure 2 are not exactly as Escher made them; we have drawn these so that every pattern has in its upper left corner a motif in aspect A. It is perhaps interesting to see how Escher methodically recorded his combinatorial considerations (which he calls his "Scheme") that gives his evidence that there are exactly 23 patterns. His scheme considers four cases for the translation block in which the four copies of the motif can have various aspects:

case **A**) motif in one aspect only,
case **B)** motif in two aspects,
case **C**) motif in three aspects,
case **D**) motif in four aspects.

Recall that he labeled the four rotated aspects of a motif as 1, 2, 3, 4 (whereas we have used A, B, C, D; these letters should not be confused with his use of the letters to label his cases). For each case, there are subcases, according to which aspects are used. For example, in case **Aa** he lists the signature 1111, and records its pattern as number 1 (of the 23 patterns); he does not bother to record the other equivalent signatures for this case. In Figure 3 we replicate Escher's summary chart that indicates what cases he considered and those signatures that he found to be superfluous. He drew a line through any signatures that produced an earlier pattern, and until he apparently grew tired at the middle of case **Cb**, he identified the equivalent pattern by its number. Case **Ba** consists of all signatures that use aspects l and 2, case **Bb** those that use aspects 1 and 3, case **Bc** those that use aspects 1 and 4, case **Bd** those that use aspects 2 and 3, and case **Be** those that use aspects 3 and 4. Escher omits the case that uses aspects 2 and 4; it is most likely that he realized that this case would be redundant with case **Bb**, just as cases **Bd** and **Be** are redundant with case **Ba**, with the equivalence induced by rotations of the translation block. Case **Ca** consists of all signatures that use aspects 1, 2, and 3, case **Cb** consists of those that use aspects 2, 3, and 4, and for cases **Cc** and **Cd** (presumably those signatures that use aspects 1, 3, and 4 or aspects 1, 2, and 4), he simply writes "none." Having noticed the redundancy of case **Cb** with **Ca**, he no doubt realized the remaining cases were also redundant.

We need to note that Escher's signatures in Figure 3 record the aspects of the motifs in a translation block in the following order: top left, top right, bottom left, bottom right. (This differs from our signature convention of recording aspects in clockwise order, beginning with the top left corner.)

FIGURE 3. Escher's scheme that found the 23 patterns for his case (1).

Case	signature	pattern no°	Case	signature	pattern no°	Case	signature	pattern no°
Aa	1 1 1 1	1	**Bd**	~~2 2 2 3~~	= 2	**Ca**	3 3 1 2	17
Ba	1 1 1 2	2		~~2 2 3 3~~	= 4		~~3 3 2 1~~	= 17
	~~1 1 2 1~~	= 2		~~2 3 2 3~~	= 3		3 1 2 3	18
	~~1 2 1 1~~	= 2		~~2 3 3 1~~	= 5		3 1 3 2	19
	~~2 1 1 1~~	= 2		~~3 3 3 2~~	= 6		~~3 2 1 3~~	= 18
	1 1 2 2	3	**Be**	~~3 3 3 4~~	= 2		~~3 2 3 1~~	= 19
	1 2 1 2	4		~~3 3 4 4~~	= 3	**Cb**	~~2 2 3 4~~	= 12
	1 2 2 1	5		~~3 4 3 4~~	= 4		~~2 3 2 4~~	= 11
	2 2 2 1	6		~~3 4 4 3~~	= 5		~~2 3 4 2~~	= 13
	~~2 2 1 2~~	= 6		~~4 4 4 3~~	= 6		~~3 3 2 4~~	= 15
	~~2 1 2 2~~	= 6	**Ca**	1 1 2 3	11		~~3 2 3 4~~	= 14
	~~1 2 2 2~~	= 6		~~1 1 3 2~~	= 11		~~3 2 4 3~~	
Bb	1 1 1 3	7		1 2 1 3	12		~~4 4 2 3~~	
	1 1 3 3	8		1 2 3 1	13		~~4 2 3 4~~	
	1 3 1 3	9		~~1 3 1 2~~	= 12		~~4 2 4 3~~	
	1 3 3 1	10		~~1 3 2 1~~	= 12	**Cc**	none	
	~~3 3 3 1~~	= 7		2 2 1 3	14	**Cd**	none	
Bc	~~1 1 1 4~~	= 6		~~2 2 3 1~~	= 14	**Da**	1 2 3 4	20
	~~1 1 4 4~~	= 4		2 1 2 3	15		1 2 4 3	21
	~~1 4 1 4~~	= 3		2 1 3 2	16		1 3 2 4	22
	~~1 4 4 1~~	= 5		~~2 3 1 2~~	= 16		1 4 2 3	23
	~~4 4 4 1~~	= 2		~~2 3 2 1~~	= 15			

For case (1), although there are a large number of signatures to consider, an exhaustive search by hand such as that done by Escher is feasible and should lead to the correct answer of 23 distinct patterns. But this problem, as well as Escher's case (2) and more general problems of this nature, are more easily handled by a clever application of counting such as Burnside's Lemma (or Pólya counting) that takes into account the action of a group that induces the equivalence classes of signatures for the patterns (see [deB64]).

We have already discussed for case (1) the rotation and translation symmetries that can produce equivalent signatures for a given pattern. We denote by C_4 the group generated by the cyclic permutation r that changes each letter in a signature by the permutation (ABCD) and then moves the new letter one position to the right (and the last letter to first); the permutations in this group are induced by rotations of the translation block. Thus $C_4 = \{r, r^2, r^3, r^4 = e\}$. We denote by K_4 the group of products of disjoint transpositions of the set $\{1, 2, 3, 4\}$; these permutations correspond to the horizontal, vertical, and diagonal translations of the translation block that generates a given pattern. Thus the elements of K_4 are $k_0 = e$, $k_1 = (12)(34)$, $k_2 = (14)(23)$, and $k_3 = (13)(24)$. Products of elements in C_4 and K_4 generate a group H that acts on signatures to produce equivalent signatures. Although in general, elements of C_4 do not commute with those of K_4, it is straightforward to show that C_4 normalizes K_4. Since $K_4 \cap C_4 = e$, H is the semidirect product K_4C_4 and has order 16. If we think of a signature as four ordered cells, each occupied by a letter chosen from the set {A, B, C, D}, then an element $k_jr^i \in H$ acts on the signature as follows: r^i transforms the letter in each cell to a new letter and moves it i cells clockwise, then k_j permutes the ordering of the occupied cells, not changing the letters in them.

To compute the number of equivalence classes of signatures using Burnside's lemma, we first need to determine how many signatures are fixed by the permutations in H. If X denotes an aspect of a motif in a translation block, let X', X'', X''' denote the successive aspects of the motif after a clockwise rotation of 90˚, 180˚, 270˚, respectively. We demonstrate how to find signatures fixed by the element k_3r of H. First, $k_3r(PQRS) = k_3(S'P'Q'R') = Q'R'S'P'$, so if the signature is to be fixed, then $S = P'$, $R = S' = P''$, $Q = R' = P'''$. Thus k_3r fixes only the signature $PP'''P''P'$.
The following chart summarizes all the signatures fixed by non-identity elements of H and those elements (other than e) that fix them:

SIGNATURE	IS FIXED BY ELEMENT(S) OF H
$PPQQ$	k_1
$PQQP$	k_2
$PQPQ$	k_3
$PP'P''P'''$	r
$PQP''Q''$	r^2
$PP'''P''P'$	r^3, k_3r, k_3r^3
$PQQ''P''$	k_1r^2
$PP''Q''Q$	k_2r^2

From this list, since there are four choices for each distinct letter in a fixed signature $PQRS$, we have the following summary of numbers of signatures fixed by elements of H:

ELEMENT OF H	e	r	r^2	r^3	k_1	k_2	k_3	k_3r	k_1r^2	k_2r^2	k_3r^3
NO. FIXED SIGNATURES	256	4	16	4	16	16	16	4	16	16	4

If, for each $h \in H$, $\psi(h)$ denotes the number of signatures that h fixes, then Burnside's lemma gives the number of equivalence classes of signatures as $\frac{1}{|H|}\sum_{h\in H} \psi(h)$. Thus the number of equivalence classes (and so the number of different patterns) for Escher's case (1) is

$$\frac{1}{16}(256 + 6\cdot 16 + 4\cdot 4) = \frac{1}{16}(368) = 23.$$

Escher's Case (2)

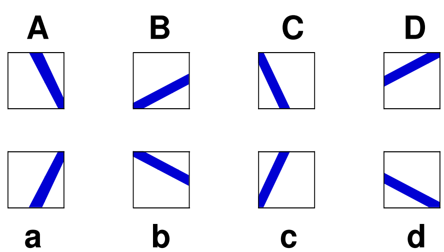
The Burnside counting technique can also be employed to determine the number of equivalence classes of signatures for Escher's case (2). For this case, letters in a signature for a translation block can represent any of the eight direct and reflected aspects of a motif (with Escher's restrictions) and the group G that produces equvalent signatures is generated by the elements of H together with permutations that are induced by a reflection of the pattern. G will be the semidirect product K_4D_4, where D_4 is the symmetry group of the square; $|G| = 32$. This technique of counting gives an answer to the question "how many patterns are there?", but does not produce a list of the signatures in each class. [Remark: A referee for this paper has indicated that it might be interesting to see if there is a Pólya-type pattern inventory approach that can be taken that will produce a list of signatures, sorted into equivalence classes. The paper [deB64] develops a theory of two-part permutations, but that theory does not seem to directly apply here.] To actually produce a list of signatures in each equivalence class, a computer program that performs permutations on the signatures and sorts them into equivalence classes is most helpful. Also, computer programs can be written to produce the representative patterns for each equivalence class.

At least two persons who read my brief description of Escher's combinatorial pattern game in *Visions of Symmetry* [Sch90] wrote computer programs to calculate all the equivalence classes of signatures in cases (1) and (2) (both with and without Escher's restrictions). In January 1990, Eric Hanson, then a graduate student at the University of Wisconsin, sent me the results of his computer program that sorted into equivalence classes all signatures for an unrestricted version of Escher's case (2), in which the translation block contains two direct and two reflected aspects of the motif. For this case, there are $6\cdot(4\cdot4)(4\cdot4) = 1536$ different signatures (6 ways to place two direct and two reflected motifs in a translation block, and 4 choices for each motif) and he found 67 different equivalence classes. With Escher's additional restrictions, there are 49 different equivalence classes. At the San Antonio MAA-AMS meeting in January 1993, Dan Davis of the mathematics department at Kingsborough Community College presented the results of his computer programs for Escher's case (1), listing the signatures in each equivalence class and displaying his original patterns for this case. Later he pursued the case in which the four positions of a translation block can be filled by any of the eight aspects (rotated and reflected) of a single motif. For this unrestricted case, there are $8^4 = 4096$ signatures; he found 154 equivalence classes. He also confirmed Hanson's results for the two versions of Escher's case (2). He has produced a listing of the signatures in each of the 154 equivalence classes, and also produced the pattern for each class with an original motif composed of circular arcs (see [Da97]).

After hearing my presentation at the combinatorics conference to honor Herb Wilf in June, 1996, Stan Wagon got interested in the problem of using *Mathematica* to automate the process of producing patterns according to Escher's algorithm. He has produced, along with Rick Mabry, a program that takes a motif (which can be Escher's motif of bands) and a signature, and produces the pattern determined by that signature. See [MWS97].

How well did Escher do in his attempt to find all distinct patterns for his case (2)? For his case (2A), in which two identical direct aspects and two identical reflected aspects of the motif make up the translation block, there are $6\cdot4\cdot4 = 96$ different signatures. For his case (2B), in which two different direct aspects and two different reflected aspects of the motif make up the translation block, there are 6·(4·3)(4·3) = 864 different signatures to consider. In addition to the much larger number of signatures to be considered for case (2), there is greater difficulty in recognizing when two patterns are the same—our eyes don't readily discern the coincidence of two patterns when one pattern is the rotated, shifted, and reflected version of the other! Yet Escher's careful work, in which he considered the combinatorial possibilities for signatures and drew and compared patterns, brought him very close to the correct answer. His careful inventory stops short of completion; in fact, there are indications in his summary sheet of patterns that he intended to check more cases, but these spaces remain blank. His son George has remarked that Escher simply grew bored (and no doubt tired) with the lengthy search.

For his case (2A), he was completely accurate: he found all ten distinct patterns (and numbered them 1–10). For his case (2B), he found 37 patterns (and numbered them 11–47). The correct answer for case (2B) is 39 patterns. Among the 37 patterns that he found, two are the same, but Escher did not recognize this. He sketched the patterns for his summary of case (2B) using the simple line segment motif, and his patterns numbered 27 and 37 in that inventory are not on the same page. This may have contributed to his not noticing that they were the same. In Figure 4 below we show the two different signatures for these patterns and how the sketched patterns look. This example illustrates the difficulty in deciding by visual inspection alone whether patterns are the same or different. In Figure 4 and subsequent figures in which we show patterns with a motif in both direct and reflected aspects, labels A, B, C, D represent the four rotated aspects of the motif (as before) and a, b, c, d are their respective reflections in a horizontal line.



The patterns below are the same. The one on the left can be transformed into the one on the right by performing a 90° counter-clockwise rotation about O, then a reflection in the horizontal line through O, and finally a translation that takes O to X.

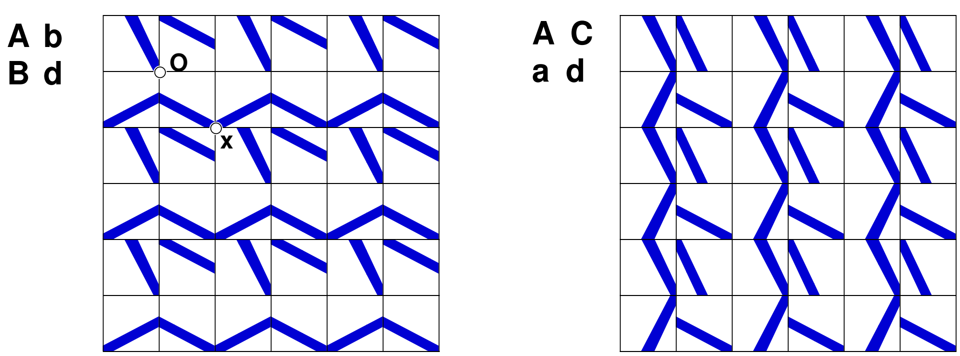
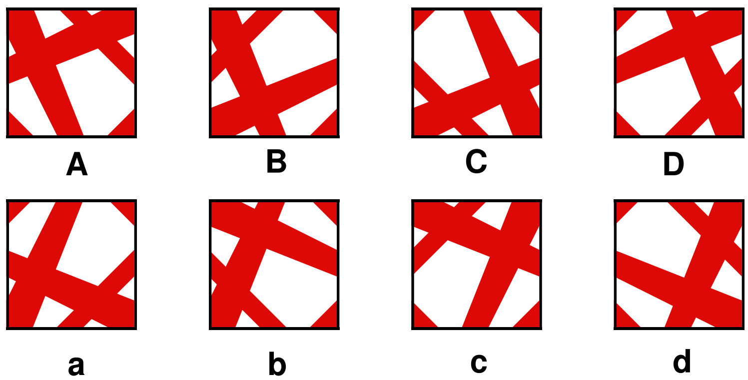


FIGURE 4. The patterns for signatures AbdB and ACda that Escher did not realize were equivalent.

In Figure 5, we display all 49 patterns for Escher's cases (2A) and (2B), using his 1938 motif of crossing bands. Note that he rotated this motif counterclockwise to obtain the four rotated aspects A, B, C, D. Patterns 1–10 are those for case (2A) and are displayed as Figure 5(a) on the next two pages. Patterns 11–49 are those for case (2B), and are displayed as Figure 5(b) at the end of this article. The three patterns that Escher missed entirely are numbers 42, 48, and 49 in this display. In Figure 5, we have listed only one signature for each pattern, and that signature always begins with aspect A. (In Escher's own inventory of patterns for case (2), he always began his signatures with aspect 1. The order in which patterns appear in our display is not exactly the same as Escher's.) In Figure 5(c), we provide a table that gives the number of signatures in each equivalence class of signatures associated to a pattern, as well as the symmetry group of each pattern. The notation for the symmetry groups in the table is that used by the International Union of Crystallography; see [Sch78].

The table makes clear the relationship between the richness of the symmetry group of a pattern and the size of its equivalence class of signatures. Those patterns generated only by translations (type p1) have the largest equivalence classes, while those generated by translations and one other symmetry (p2, pg, pm, cm) have equivalence classes half that size, and those generated by translations and two other symmetries (pgg, pmg, pmm) have equivalence classes one-fourth that size. The number of elements in the equivalence classes for patterns 1-10 (Escher's case (1)) is half the number in equivalence classes with the same symmetry group for patterns 11-49 because patterns 1-10, with two pairs of repeated aspects of the motif, have the property that the translation block is invariant under a permutation in the group K_4 that does not add to the overall symmetry of the pattern. This invariance only affects the period of the pattern. For example, the signature AAbb is invariant under the permutation that interchanges columns of the translation block, but the periodic pattern has only translation symmetry (group p1). The period of the pattern in the horizontal direction is half the length of the translation block, while its period in the vertical direction is the length of the translation block.

FIGURE 5(a). Escher's case (2A). His case (2B) is in FIGURE 5(b) at the end of this article.



Escher's 1938 motif for stamped patterns.

Motif A is rotated 90° counterclockwise to obtain the four direct aspects A, B, C, D; reflecting each of these in its bottom edge gives the four reflected aspects a, b, c, d.

Case (2A). The ten patterns whose translation block has two direct aspects and two reflected aspects of the above motif and in addition, the two direct aspects are the same and the two reflected aspects are the same.

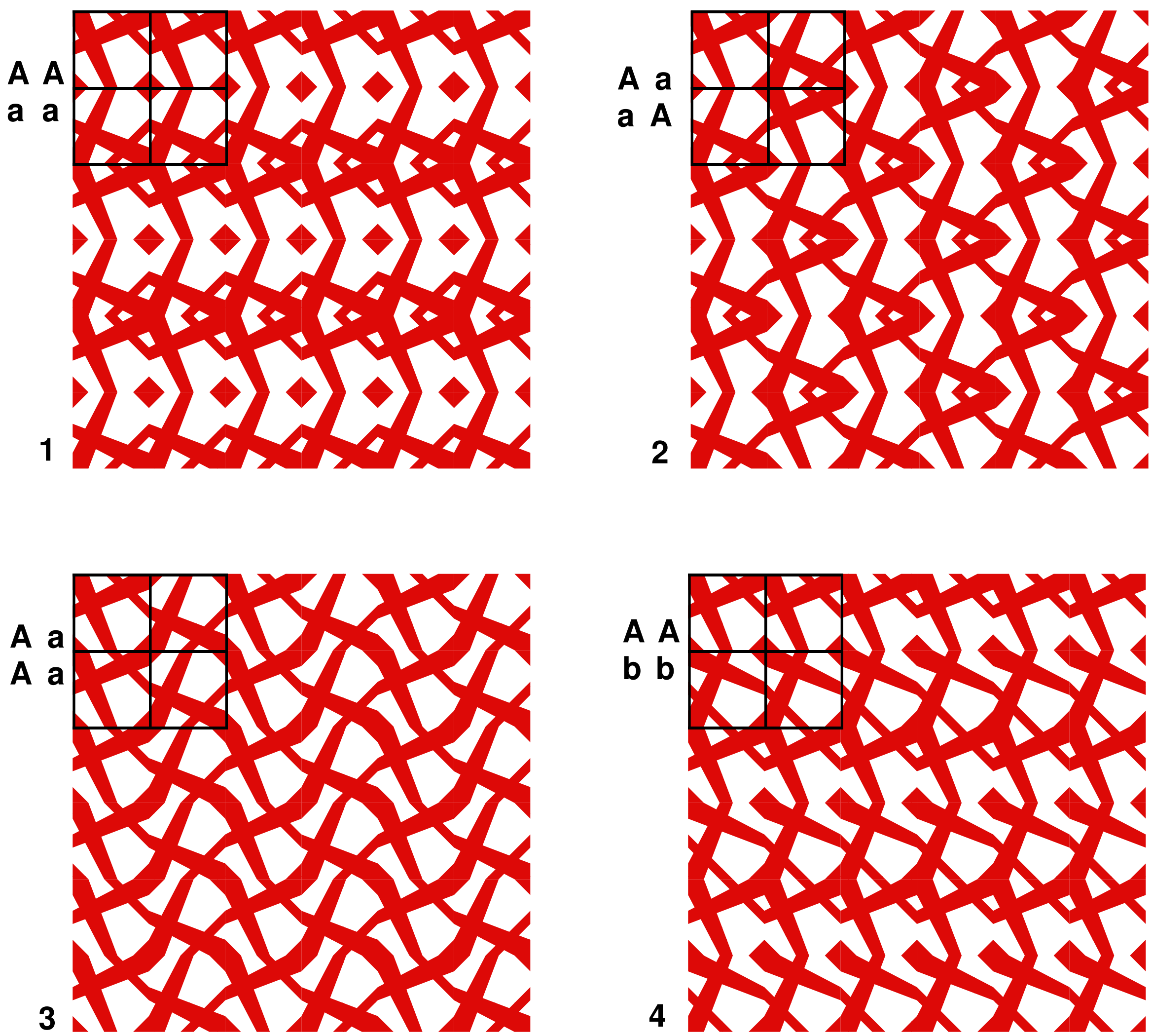
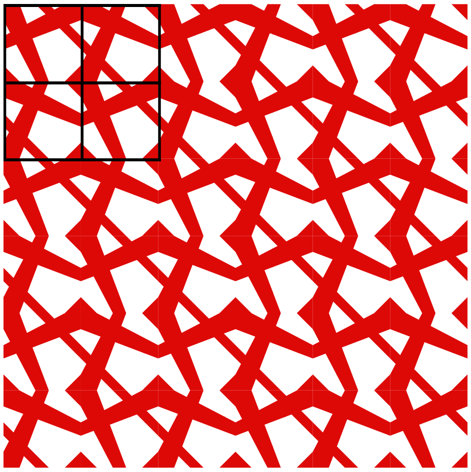


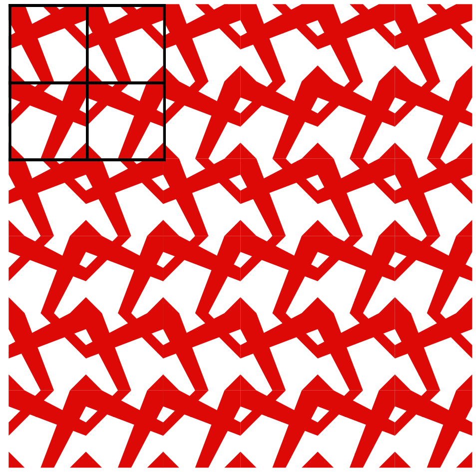
FIGURE 5(a) continued.

A b
b A



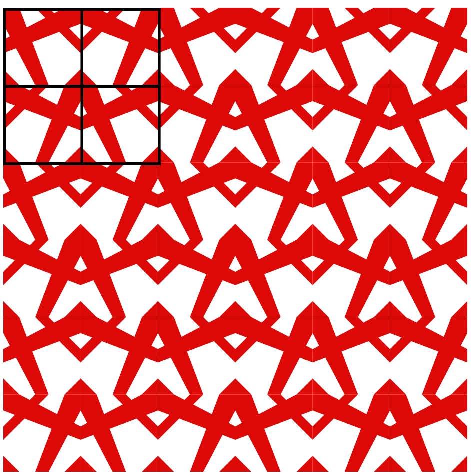
5

A A
c c



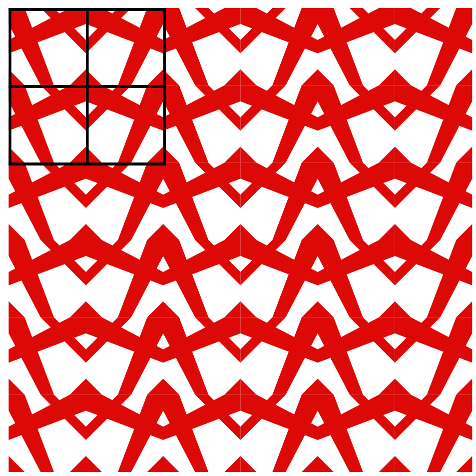
6

A c
c A



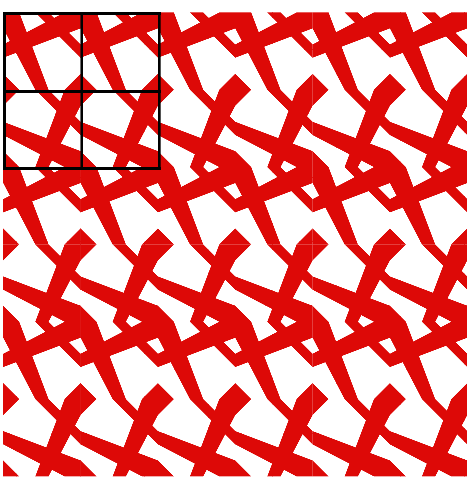
7

A c
A c



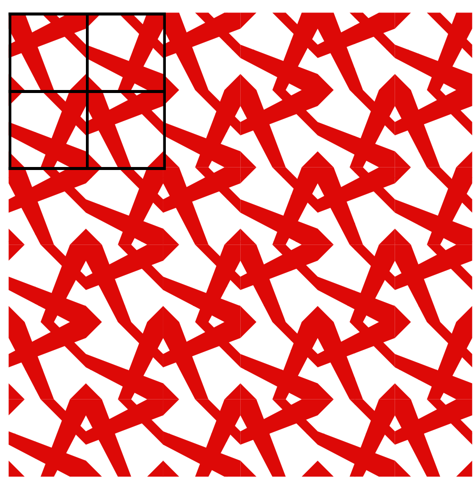
8

A A
d d



9

A d
d A



10

Escher's Other Experiments

Escher carried out several other experiments in producing patterns by his algorithm and variants of it. He sought to make patterns in which ribbons were continuous strands, weaving in and out, or in which they formed closed loops, tying together other strands. For this he used his original 1938 motif, but now made the bars into ribbons that wove over and under each other. He carved the original motif in two direct aspects: the 'under-over' relations between the strands on the second square were the reverse of those on the first. These he labeled 1 and 1a. He also carved wooden squares that gave the reflected aspects of these two, labeling the reflection of 1 and 1a in a horizontal line as $\underline{1}$ and $\underline{1a}$. As with the 1938 motif (Figure 5), he rotated the motifs in aspects 1 and 1a counterclockwise to produce the rotated aspects 2, 3, 4, and 2a, 3a, 4a. Reflections of these in a horizontal line were labeled with underlines.

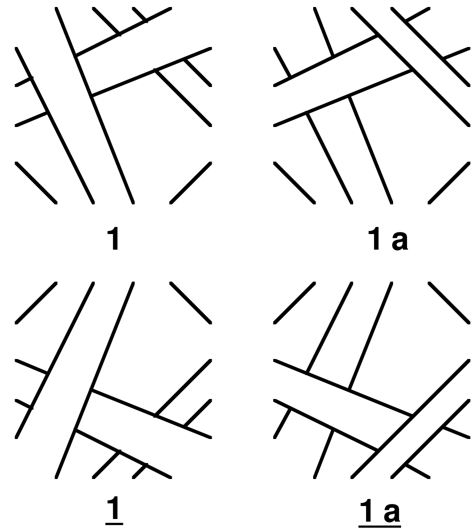
Using the four carved stamps, Escher hand-printed at least 21 different patterns, coloring in the ribbons. One of his designs has been recreated in Figure 6. Escher's son George informed me that his father made up these colored woven patterns with the intention of having the tiles produced by tile-makers. Although he showed them to tile manufacturers, he was unsuccessful in having any of the tiles produced. Several of these sample patterns were displayed in an exhibit in 1942, for which Escher made a poster explaining (and illustrating) the 16 different aspects in which motifs can appear. There is no evidence that Escher attempted to enumerate the possible patterns for this more complex case. For this case, not only is the number of choices for each tile in the 2x2 translation block increased from eight to sixteen, but in addition to rotation, reflection, and translation symmetries of the pattern, there is an under-over symmetry to consider.

The *Mathematica* program written by Wagon and Mabry (mentioned earlier) has an option in which the user can ask for the Escher motif of ribbons to be colored in the manner Escher required: ribbons are to be colored as continuous strands, crossing ribbons of different strands are to have different colors, and a minimum number of colors are to be used [MWS97].

Escher experimented with other algorithms to produce patterns. He used translation blocks that were rectangular and, as with the 2x2 squares, translated them in directions parallel to the sides of the block. He also used a translation block that was a row of four or more squares and translated it in a diagonal direction (a composition of one vertical and one horizontal move). Several of his woven ribbon patterns were produced this way. Two of these patterns can be found in [Ern76] and six more in [Sch90]. In the display of his patterns on page 49 of [Sch90], all of the signatures are wrong! (The signatures there belong to a different poster with a display of six of Escher's ribbon patterns.) This will be corrected in new printings of my book, but I would also like to correct them here. The correct signatures, displayed in the same order in which the patterns appear, can be found in the reference for [Sch90], on page 23 of this article.

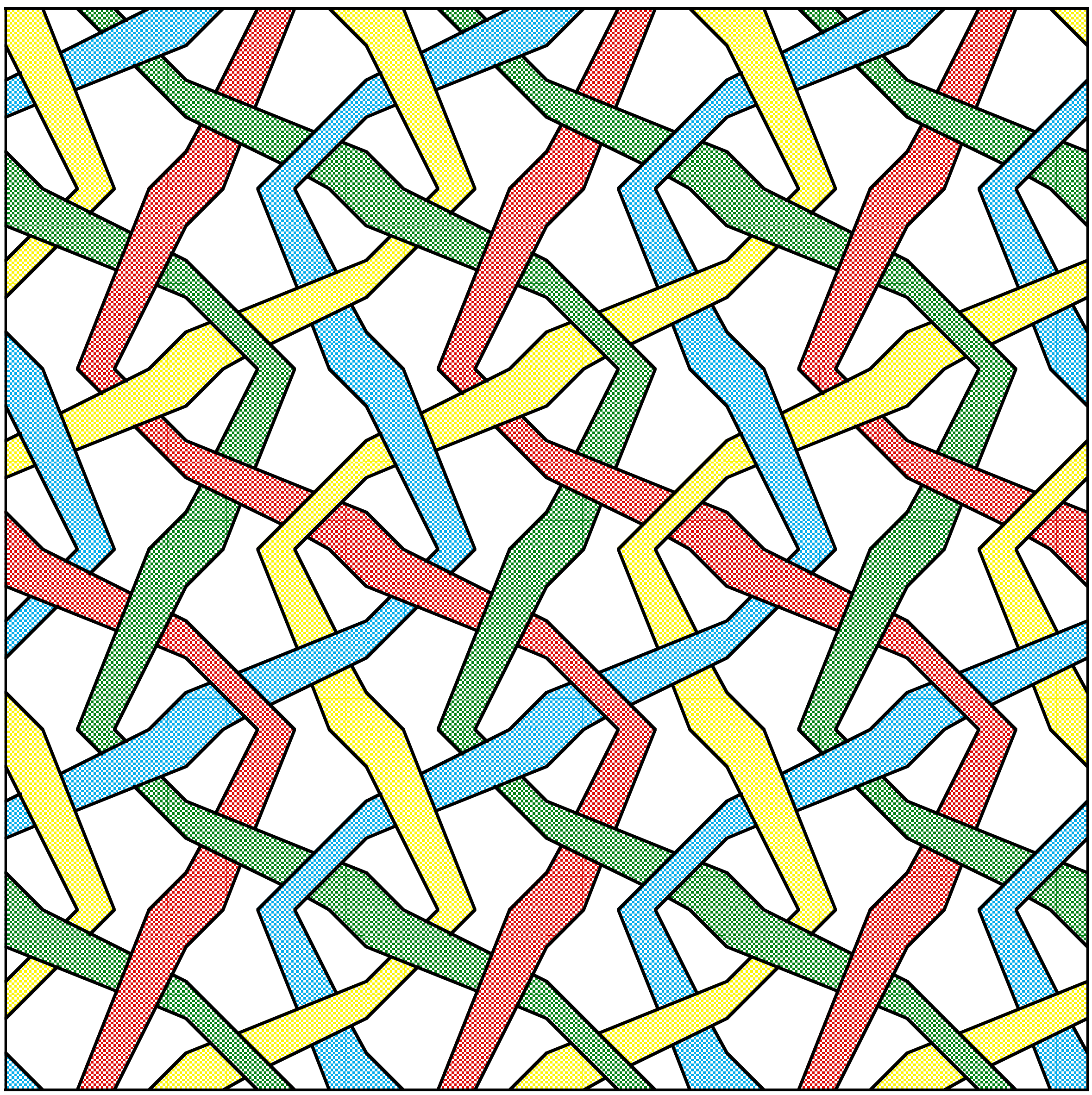
FIGURE 6. Escher's woven ribbon motifs from a 1942 exhibit. The 1938 motif now has strands that weave over and under. Copies 1 and 1a have reversed over-under relations; copies 1 and 1a are reflections of 1 and 1a in a horizontal line. Each of these 4 motifs (shown actual size) can be rotated 90 degrees for a total of 16 different aspects.

Escher carved four wooden stamps to outline the ribbons. After he stamped out a pattern, he hand-colored the woven strands.



1 1 a

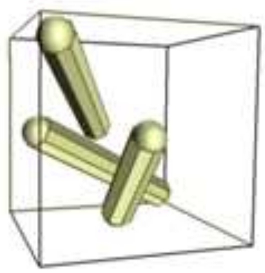
3 3a



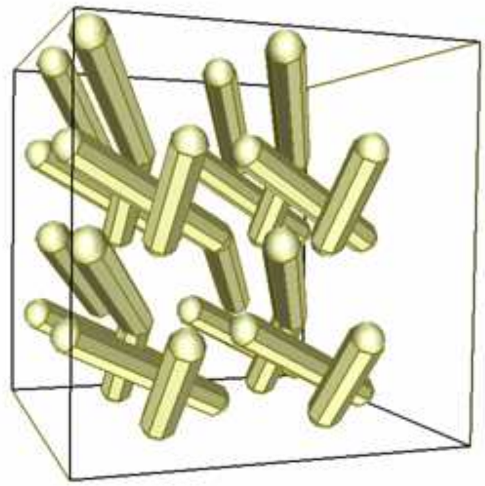
In 1943, Escher made a variant of the woven ribbon motif, this time making the strands look like yarn or cord, curving as they wove. As with the ribbon motifs, four different aspects were carved, and many patterns were stamped out, creating designs that strongly resemble knitting or crocheting. Several were hand-colored. They were collected in a binder dated IV '43. Escher at least considered making periodic plane patterns based on the honeycomb tiling by a regular hexagon. He carved a motif of crossing bands on a hexagonal tile, but it is not known if he produced patterns with it. (Illustrations of the 'knitting' patterns and the hexagonal motif can be found in [Sch90].) Of course it is possible also to create algorithms for periodic plane patterns that begin with an equilateral triangle containing a motif. Escher did investigate some tilings of the plane by triangles and create patterns by decorating the triangles, but these were not equilateral triangles. His focus of that investigation was not patterns, but rather the number of different ways in which a triangle could tile the plane in a "regular" way, that is, in which every triangle was surrounded in the same way.

A Challenge

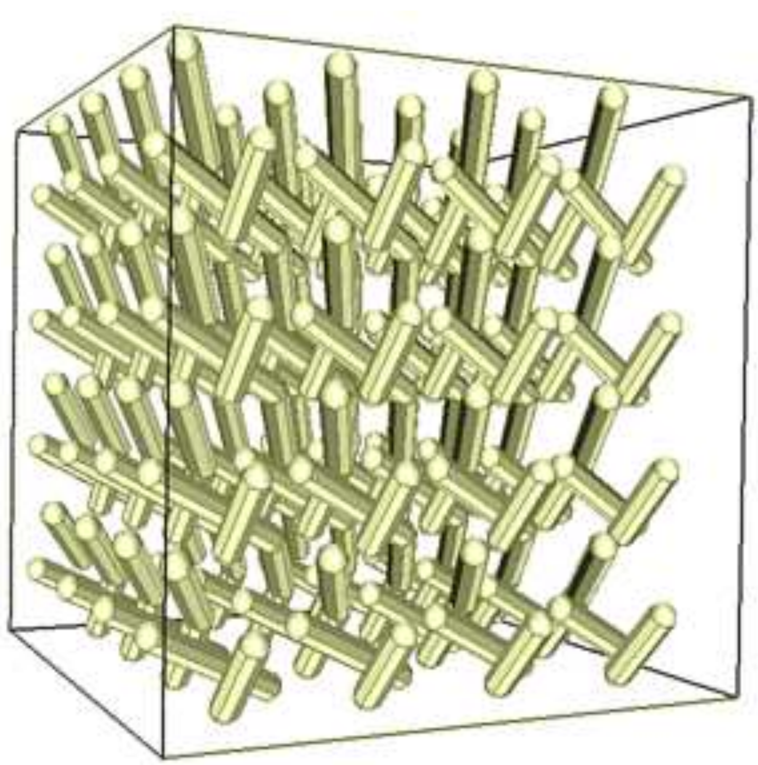
Escher's algorithm for plane patterns based on translating a 2x2 block of squares naturally suggests an analogous approach to generating three-dimensional patterns. Begin with a unit cube and put an asymmetric motif inside it-- call this a caged motif. Now build a 2x2x2 cube with eight copies of the caged motif, where the copies can appear in any direct or reflected aspect; call this a supercube. Now translate the supercube repeatedly in the three directions given by its edges to produce a periodic three-dimensional pattern. In Figure 7 we show two examples of such patterns. Here, the caged motif is a cube with three bars inside it, a bit reminiscent of Escher's 1938 motif of bars in a square. In the first pattern, all eight copies of the caged motif in the supercube are in the same aspect; this is the simplest of all such 3-D patterns, and is the analog of the plane pattern with signature AAAA. In the second pattern, the supercube is made up of four direct copies of the caged motif, all in the same aspect, and four inverted copies of the motif, all in the same aspect, arranged so that adjacent motifs in the supercube are always in opposite aspects. (An inverted copy of the motif is gotten by performing a central inversion on the motif.) Thus this 3-D pattern is the analog of the plane pattern whose signature is ACAC.



Motif of three bars in cube

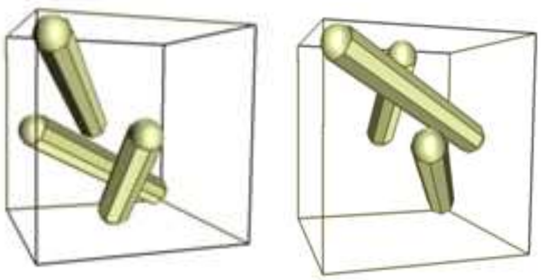


Supercube of eight copies of motif
(here all in the same aspect)

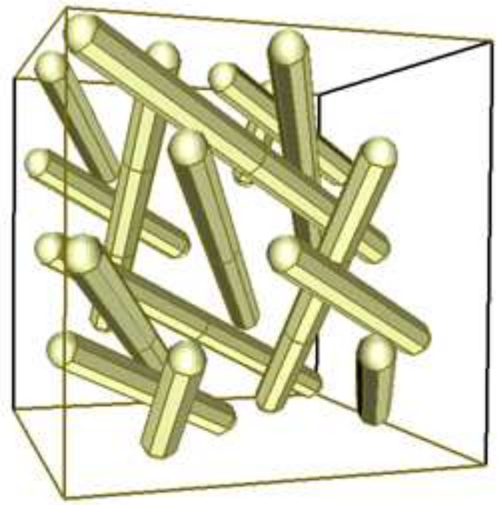


Pattern with eight translated copies of the supercube

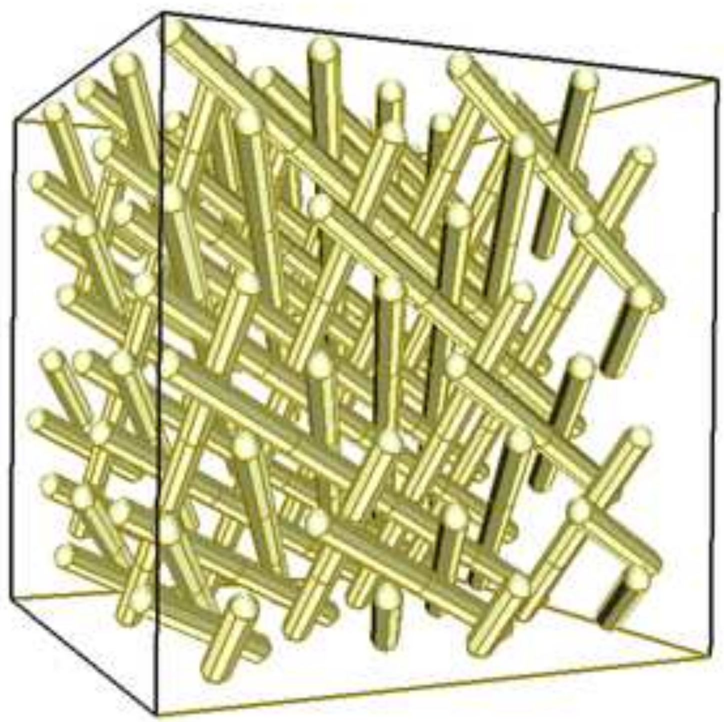
FIGURE 7(a)



Motif of three bars in cube (left) and its image under central inversion (right)



Supercubewith four copies of motif alternating with four copies of inverted motif (adjacent motifs have opposite aspects)



Pattern with eight translated copies of the supercube

FIGURE 7(b)

Of course, we ask Escher's question: *How many different 3-D patterns are possible, using this algorithm?* Even if we add restrictions, as Escher did for the planar case, the combinatorial stakes have just escalated astronomically. Since the rotation group of the cube has order 24, there are 24 different direct aspects of the caged motif, and another 24 reflected aspects. If each of the 24 direct aspects of the motif is assigned a label (such as A, B, . . . , W, X) and the image of each of these under a central inversion is assigned a companion label (a, b, . . . , w, x), then a signature can be assigned to each translation block in a manner similar to the two-dimensional case. There are 24^8 signatures for translation blocks having only direct aspects of the motif, and 48^8 signatures if both direct and reflected aspects are allowed. Each translation block has 24 rotated positions (and 48 images if reflections are also allowed); each of these produces a signature for the same pattern. (As in the two-dimensional case, we say that two patterns are the same if there is an isometry that maps the one onto the other.) Additional signatures for a pattern are produced by the eight different translation blocks that correspond to having one of the eight caged motifs in a specified corner of the block.

Thank you Escher, for a tantalizing problem. We leave its solution to the reader.

Acknowledgements. All planar patterns in this article were computer-drawn using *The Geometer's Sketchpad*. The three-dimensional patterns in Figure 6 were computer-generated by the program *Geomview*. I wish to thank the Geometry Center for their help and support during my visit in May 1996. I wish also to thank the owners of original Escher materials (Michael Sachs, Norwalk, Connecticut and the Haags Gemeentemuseum, The Hague, The Netherlands) for the opportunity to examine Escher's work.

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[Da97] Dan Davis, "On a Tiling Scheme from M. C. Escher," *The Electronic Journal of Combinatorics*, v. 4, no. 2, (1997) #R23.

[Er76] Bruno Ernst, *The Magic Mirror of M. C. Escher*. New York: Harry Abrams, 1976, pp. 40–41.

[Es86] George A. Escher, "Potato Printing, A Game for Winter Evenings", in *M. C. Escher: Art and Science*, H. S. M. Coxeter, M. Emmer, R. Penrose, M. Teuber, eds. Amsterdam: North Holland, 1986, pp. 9–11.

[MWS97] Rick Mabry, Stan Wagon, and Doris Schattschneider, "Automating Escher's Combinatorial Patterns," *Mathematica in Education and Research*, v.5, no 4 (1997).

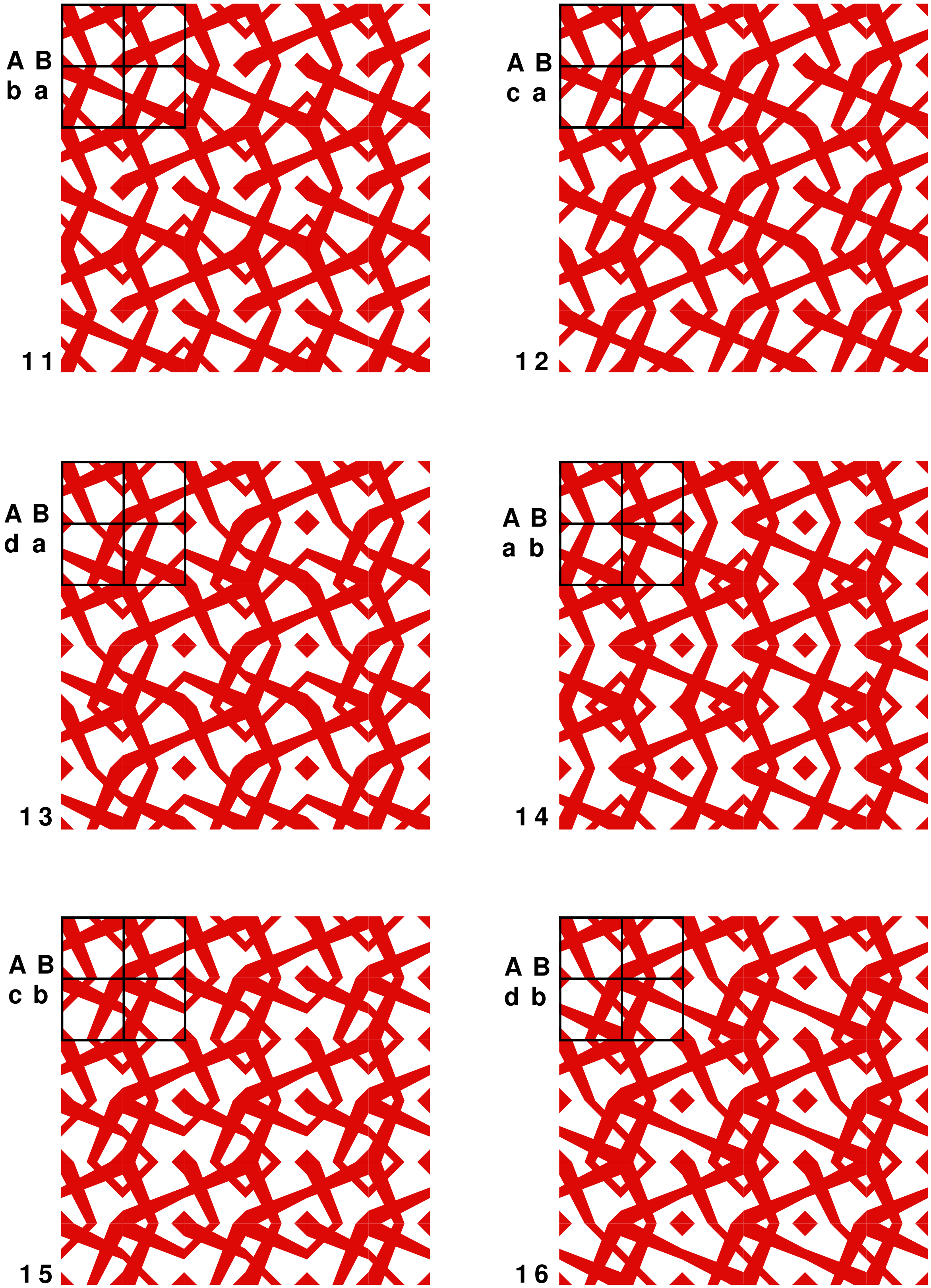
[Sch78] Doris Schattschneider, "The plane symmetry groups: their recognition and notation," *American Mathematical Monthly*, 85 (1978) 439-450.

[Sch90] Doris Schattschneider, *Visions of Symmetry: Notebooks, Periodic Drawings and Related Work of M. C. Escher*. New York: W. H. Freeman & Co., 1990, pp. 44–52. Please note the following correction to this work:

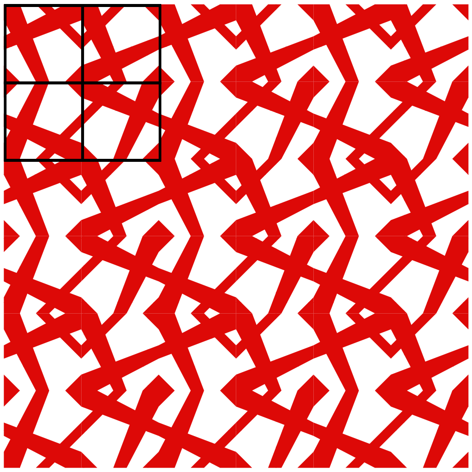
Replacement labeling for illustration on page 49, *Visions of Symmetry*

1a 4a 1 4	1 4 1a 4a
3a 2 3 2a	1 4 1a 4a
1a 1	1 2a 1a 2
3a 3	1 2a 1a 2
4a 1 3a 2	1a 2 1 2a
4a 1 3a 2	3a 4a 3 4

FIGURE 5(b). The 39 patterns for Escher's case (2B), numbered 11–49.

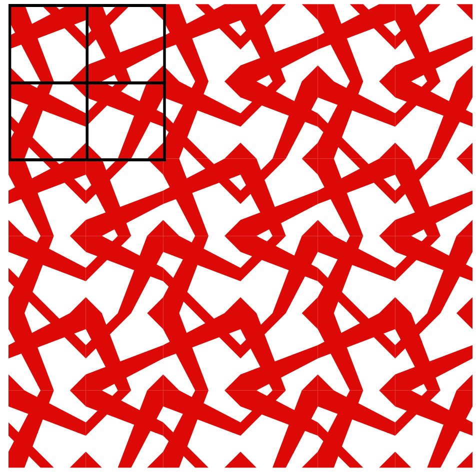


A B
a c



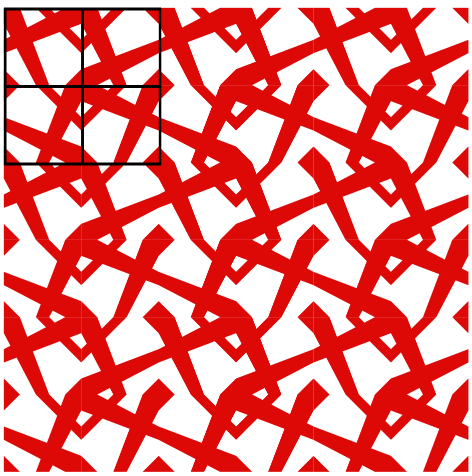
17

A B
b c



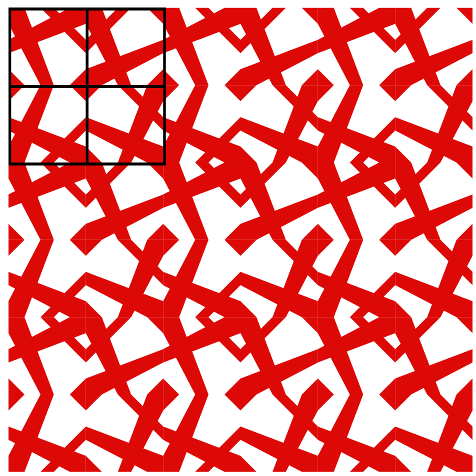
18

A B
d c



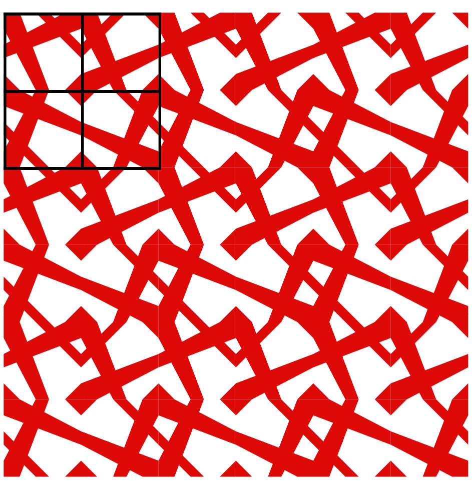
19

A B
a d



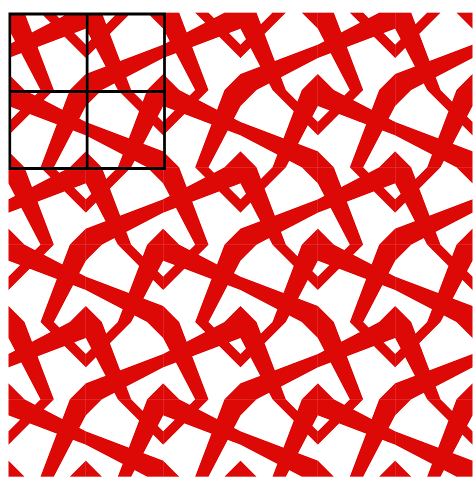
20

A B
b d



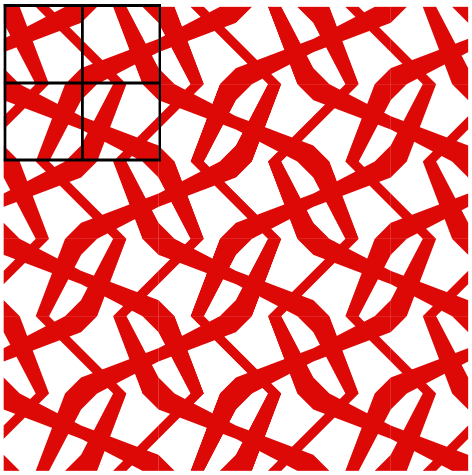
21

A B
c d



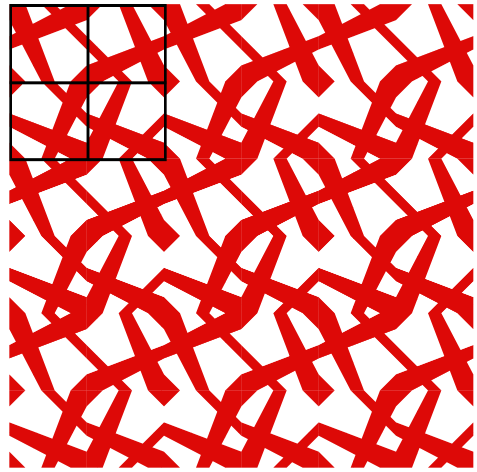
22

A C
c a



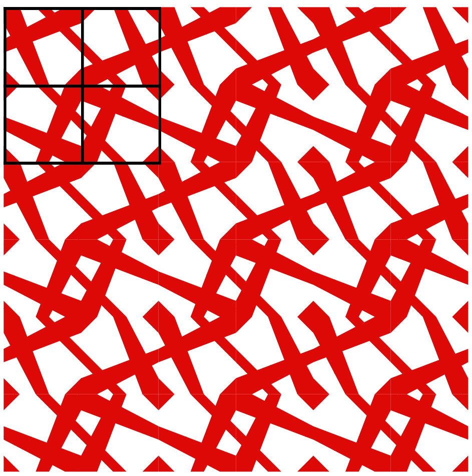
23

A C
d a



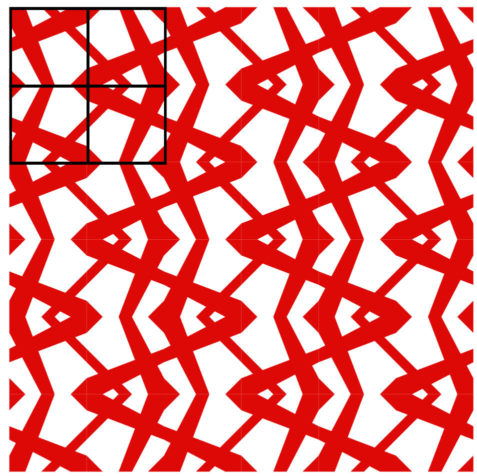
24

A C
d b



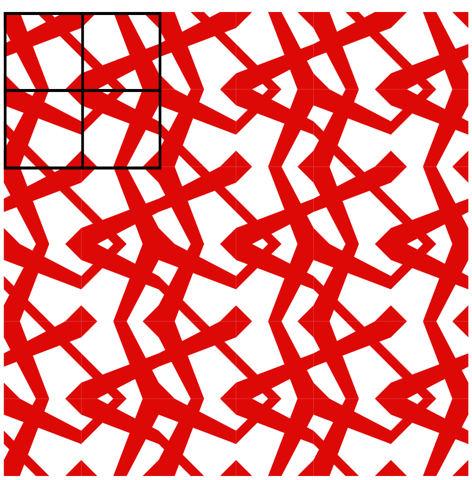
25

A C
a c



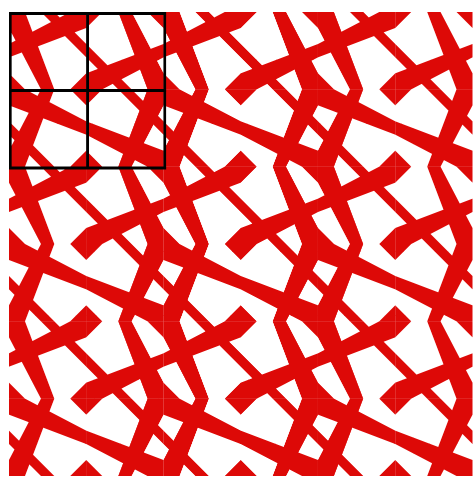
26

A C
b c



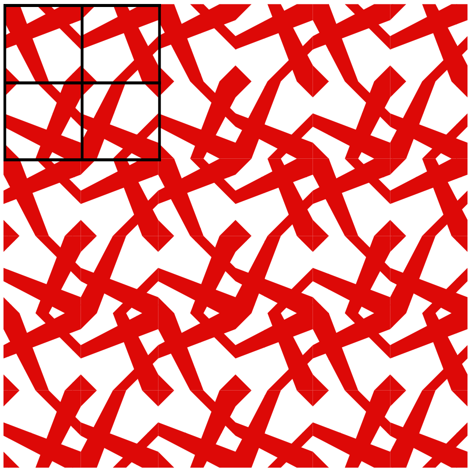
27

A C
b d



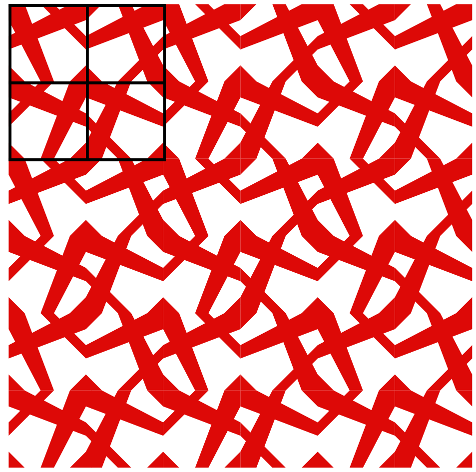
28

A D
d a



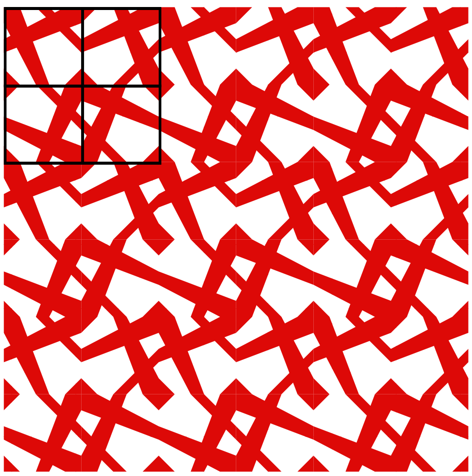
29

A D
c b



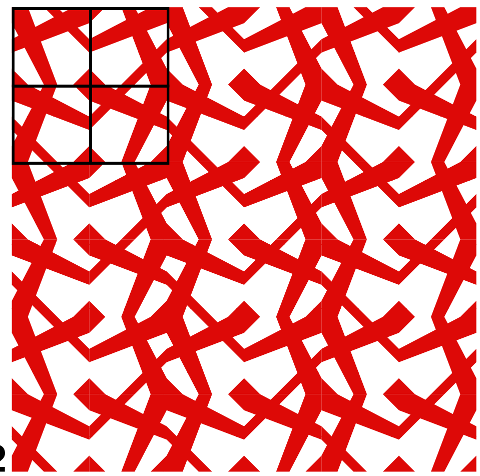
30

A D
d b



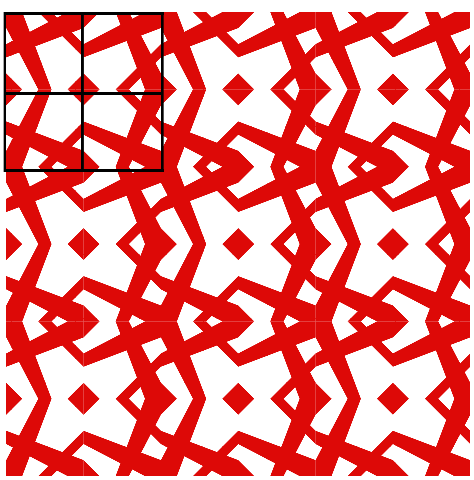
31

A D
b c



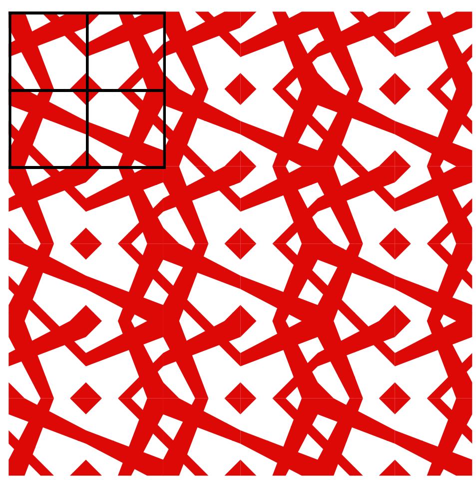
32

A D
a d



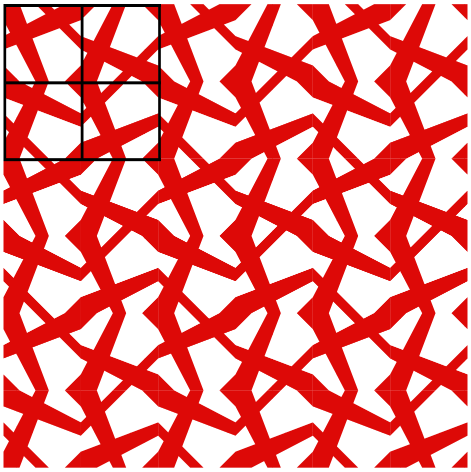
33

A D
b d



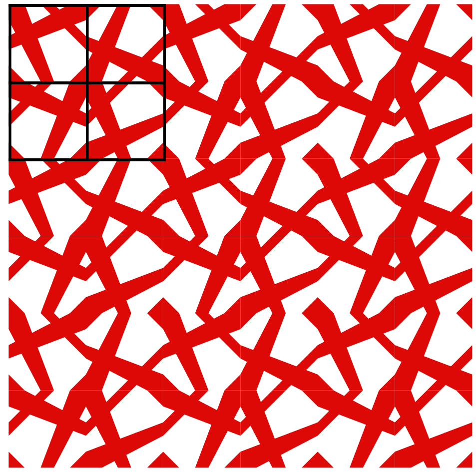
34

A a
b B



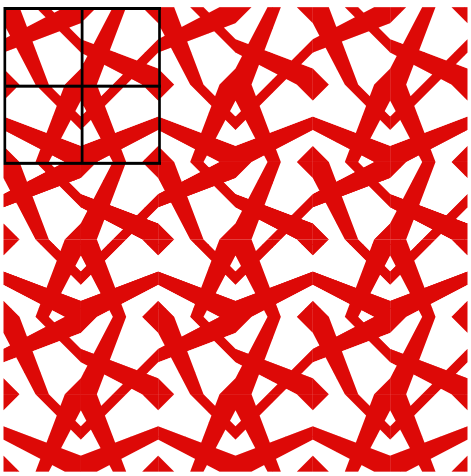
35

A a
c B



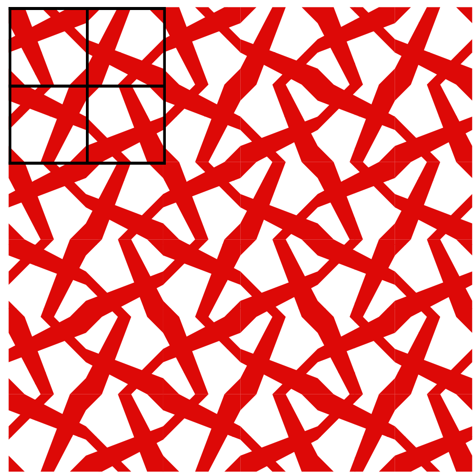
36

A a
d B



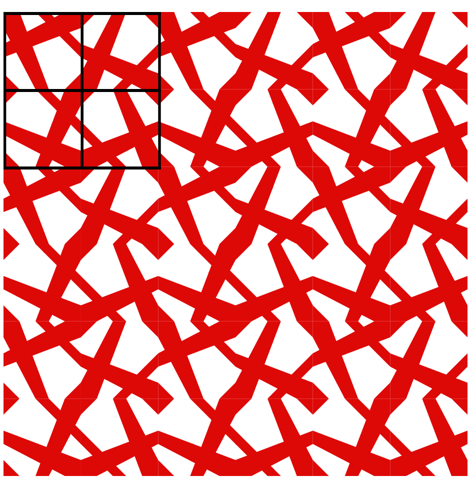
37

A a
c C



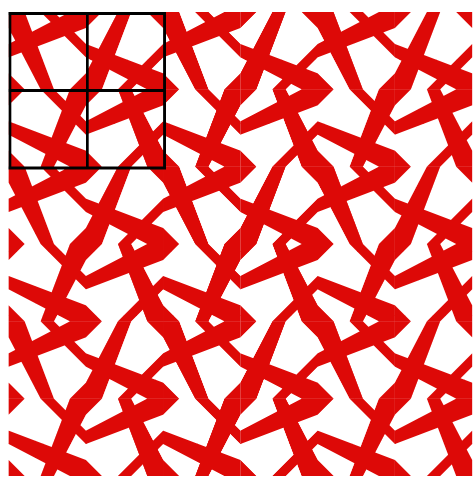
38

A a
d C



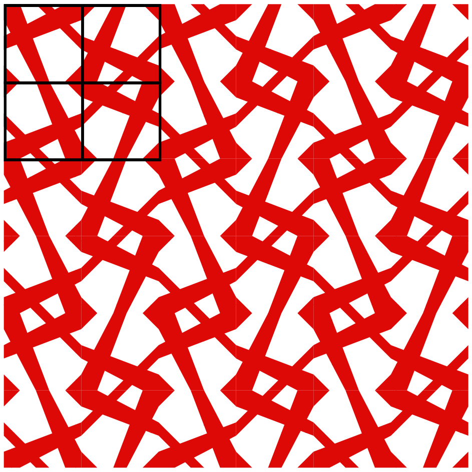
39

A a
d D



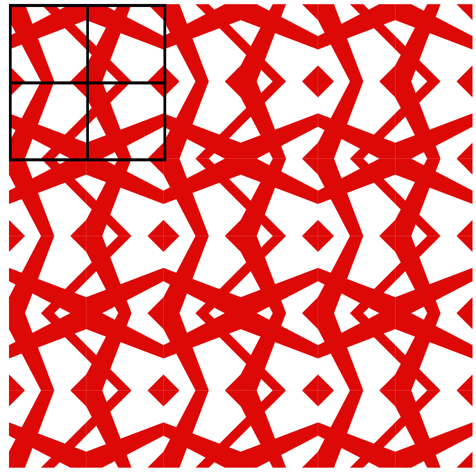
40

A a
C c



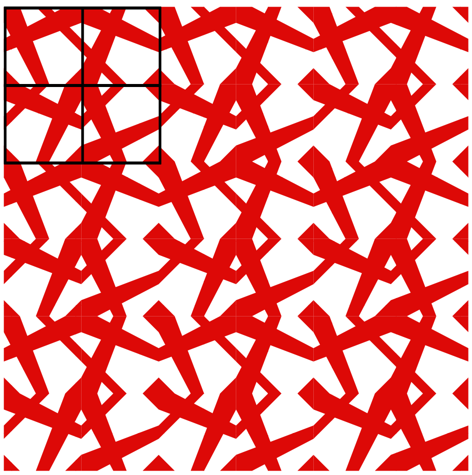
41

A b
a B



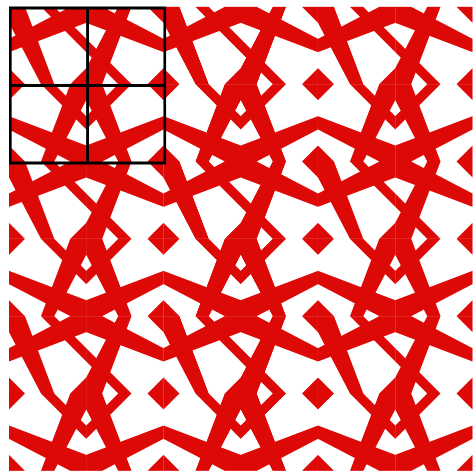
42

A b
c B



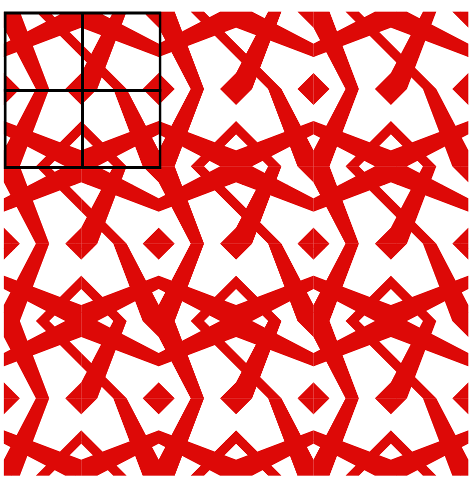
43

A b
d B



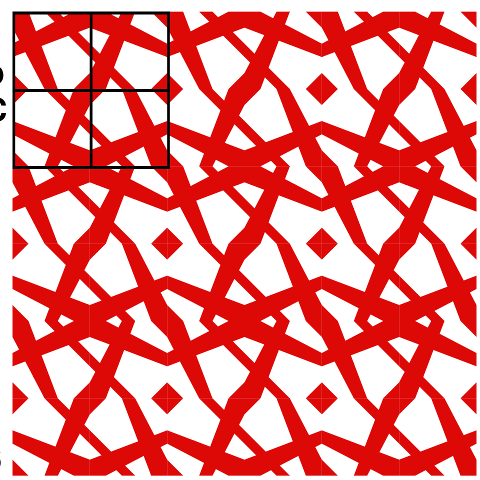
44

A b
a C



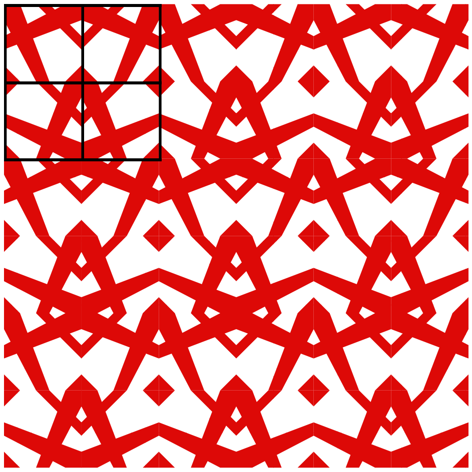
45

A b
d C



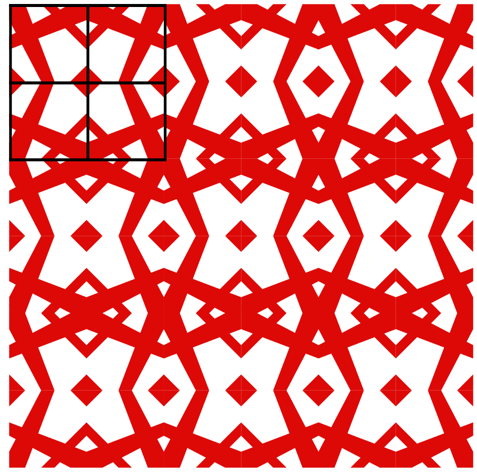
46

A c
d B



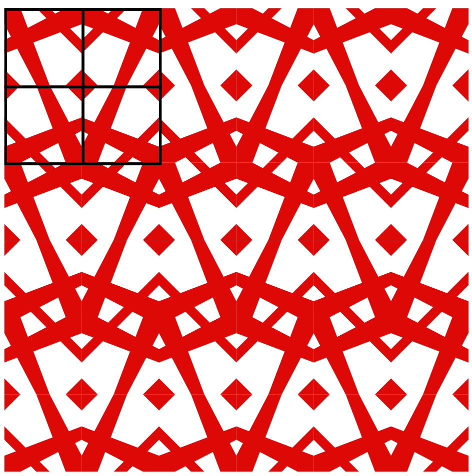
47

A c
a C



48

A c
C a



49

FIGURE 5(c). For each of the 49 patterns for Escher's case (2), the table below lists one signature for the pattern that begins with aspect A, the number of equivalent signatures for that pattern, and the symmetry group of the pattern. The notation for the symmetry groups in the table is that used by the International Union of Crystallography; see [Sch78].

PATTERN	SIGNATURE	EQUIVALENT SIGNATURES	SYMMETRY GROUP	PATTERN	SIGNATURE	EQUIVALENT SIGNATURES	SYMMETRY GROUP
1	AAaa	8	pm	26	ACca	8	pmg
2	AaAa	8	cm	27	ACcb	32	p1
3	AaaA	8	pg	28	ACdb	16	p2
4	AAbb	16	p1	29	ADad	16	pg
5	AbAb	8	pg	30	ADbc	16	pg
6	AAcc	8	pg	31	ADbd	32	p1
7	AcAc	8	cm	32	ADcb	16	pg
8	AccA	8	pm	33	ADda	16	pm
9	AAdd	16	p1	34	ADdb	32	p1
10	AdAd	8	pg	35	AaBb	16	pg
11	ABab	16	pg	36	AaBc	32	p1
12	ABac	32	p1	37	AaBd	32	p1
13	ABad	32	p1	38	AaCc	8	pgg
14	ABba	16	pm	39	AaCd	32	p1
15	ABbc	32	p1	40	AaDd	16	pg
16	ABbd	32	p1	41	AacC	8	pgg
17	ABca	32	p1	42	AbBa	16	pm
18	ABcb	32	p1	43	AbBc	32	p1
19	ABcd	16	pg	44	AbBd	32	p1
20	ABda	32	p1	45	AbCa	32	p1
21	ABdb	32	p1	46	AbCd	16	p2
22	ABdc	16	pg	47	AcBd	16	pm
23	ACac	8	pgg	48	AcCa	8	pmm
24	ACad	32	p1	49	AcaC	8	pmg
25	ACbd	16	p2				