Parking Functions and Noncrossing Partitions

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Dedicated to Herb Wilf on the occasion of his sixty-fifth birthday

Abstract

A *parking function* is a sequence $(a_1, \ldots, a_n)$ of positive integers such that if $b_1 \le b_2 \le \cdots \le b_n$ is the increasing rearrangement of $a_1, \ldots, a_n$, then $b_i \le i$. A *noncrossing partition* of the set $[n] = \{1, 2, \ldots, n\}$ is a partition π of the set $[n]$ with the property that if $a < b < c < d$ and some block B of π contains both a and c, while some block B' of π contains both b and d, then $B = B'$. We establish some connections between parking functions and noncrossing partitions. A generating function for the flag f-vector of the lattice NC_{n+1} of noncrossing partitions of $[n+1]$ is shown to coincide (up to the involution ω on symmetric function) with Haiman's parking function symmetric function. We construct an edge labeling of NC_{n+1} whose chain labels are the set of all parking functions of length n. This leads to a local action of the symmetric group $\mathfrak{S}_n$ on NC_{n+1}.

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1. Introduction. A *parking function* is a sequence $(a_1, \ldots, a_n)$ of positive integers such that if $b_1 \le b_2 \le \cdots \le b_n$ is the increasing rearrangement of $a_1, \ldots, a_n$, then $b_i \le i$.[1] Parking functions were introduced by Konheim and Weiss [14] in connection with a hashing problem (though the term "hashing"

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[1] Minor variations of this definition appear in the literature, but they are equivalent to the definition given here. For instance, in [31] parking functions are obtained from the definition given here by subtracting one from each coordinate.

was not used). See this reference for the reason (formulated in a way which now would be considered politically incorrect) for the terminology "parking function." Parking functions were subsequently related to labelled trees and to hyperplane arrangements. For further information on these connections see [31] and the references given there. In this paper we will develop a connection between parking functions and another topic, viz., noncrossing partitions.

A *noncrossing partition* of the set $[n] = \{1, 2, \ldots, n\}$ is a partition π of the set $[n]$ (as defined e.g. in [29, p. 33]) with the property that if $a < b < c < d$ and some block B of π contains both a and c, while some block B' of π contains both b and d, then $B = B'$. The study of noncrossing partitions goes back at least to H. W. Becker [1], where they are called "planar rhyme schemes." The systematic study of noncrossing partitions began with Kreweras [15] and Poupard [22]. For some further work on noncrossing partitions, see [5][21][25][28] and the references given there.

A fundamental property of the set of noncrossing partitions of $[n]$ is that it can be given a natural partial ordering. Namely, we define $\pi \leq \sigma$ if every block of π is contained in a block of σ. In other words, π is a *refinement* of σ. Thus the poset NC_n of all noncrossing partitions of $[n]$ is an induced subposet of the lattice Π_n of all partitions of $[n]$ [29, Example 3.10.4]. In fact, NC_n is a lattice with a number of remarkable properties. We will develop additional properties of the lattice NC_n which connect it directly with parking functions.

2. The parking function symmetric function. Let P be a finite graded poset of rank n with $\hat{0}$ and $\hat{1}$ and with rank function ρ. (See [29, Ch. 3] for poset terminology and notation used here.) Let S be a subset of $[n-1] = \{1, 2, \ldots, n-1\}$, and define $\alpha_P(S)$ to be the number of chains $\hat{0} = t_0 < t_1 < \cdots < t_s = \hat{1}$ of P such that $S = \{\rho(t_1), \rho(t_2), \ldots, \rho(t_{s-1})\}$. The function α_P is called the *flag f-vector* of P. For $S \subseteq [n-1]$ further define

$$\beta_P(S) = \sum_{T \subseteq S} (-1)^{|S-T|} \alpha_P(T).$$

The function β_P is called the *flag h-vector* of P. Knowing α_P is the same as knowing β_P since

$$\alpha_P(S) = \sum_{T \subseteq S} \beta_P(T).$$

For further information on flag f-vectors and h-vectors (using a different terminology), see [29, Ch. 3.12].

There is a kind of generating function for the flag h-vector which is often useful in understanding the combinatorics of P. Regarding n as fixed, let $S \subseteq [n-1]$ and define a formal power series $Q_S = Q_S(x) = Q_S(x_1, x_2, \ldots)$ in the (commuting) indeterminates $x_1, x_2, \ldots$ by

$$Q_S = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n \\ i_j < i_{j+1} \text{ if } j \in S}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

Q_S is known as *Gessel's quasisymmetric function* [10] (see also [16, §5.4][18][24, Ch. 9.4]). The functions Q_S, where S ranges over all subsets of $[n-1]$, are linearly independent over any field. For our ranked poset P we then define

$$F_P = \sum_{S \subseteq [n-1]} \beta_P(S) Q_S.$$

This definition (in a different but equivalent form) was first suggested by R. Ehrenborg [6, Def. 4.1] and is further investigated in [30]. One of the results of [30] (Thm. 1.4) is the following proposition (which is equivalent to a simple generalization of [29, Exercise 3.65]).

2.1 Proposition. *Let P be as above. If every interval $[u, v]$ of P is rank-symmetric (i.e., $[u, v]$ has as many elements of rank i as of corank i), then F_P is a symmetric function of $x_1, x_2, \ldots$.*

We now consider the case $P = \mathrm{NC}_{n+1}$. (We take NC_{n+1} rather than NC_n because NC_{n+1} has rank n.) It is well-known that every interval in NC_{n+1} is self-dual and hence rank-symmetric. (This follows from the fact that NC_{n+1} is itself self-dual [15, §3][27, Thm. 1.1] and that every interval of NC_{n+1} is a product of NC_i's [21, §1.3].) Hence $F_{\mathrm{NC}_{n+1}}$ is a symmetric function, and we can ask whether it is already known. In fact, $F_{\mathrm{NC}_{n+1}}$ has previously appeared in connection with parking functions, as stated below in Theorem 2.3. First we provide some background information related to parking functions.

Let $\mathcal{P}_n$ denote the set of all parking functions of length n. The symmetric group $\mathfrak{S}_n$ acts on $\mathcal{P}_n$ by permuting coordinates. Let $\mathrm{PF}_n = \mathrm{PF}_n(x)$ denote the Frobenius characteristic of the character of this action [17, Ch. 1.7]. Thus if

$$\mathrm{PF}_n = \sum_{\lambda \vdash n} \tau_{\lambda,n} s_\lambda$$

is the expansion of PF_n in terms of Schur functions, then $\tau_{\lambda,n}$ is the multiplicity of the irreducible character of $\mathfrak{S}_n$ indexed by λ in the action of $\mathfrak{S}_n$ on $\mathcal{P}_n$. The symmetric function PF_n was first considered in the context of parking functions by Haiman [13, §§2.6 and 4.1]. Following Haiman, we will give a formula for PF_n from which its expansion in terms of various symmetric function bases is immediate. The key observation (due to Pollack [8, p. 13] and repeated in [13, p. 28]) is the following (which we state in a slightly different form than Pollack). Let $\mathbb{Z}_{n+1}$ denote the set $\{1, 2, \ldots, n+1\}$, with addition modulo $n+1$. Then every coset of the subgroup H of $\mathbb{Z}_{n+1}^n$ generated by $(1, 1, \ldots, 1)$ contains exactly one parking function. From this it follows easily that

$$\mathrm{PF}_n = \frac{1}{n+1} [t^n] H(t)^{n+1}, \tag{1}$$

where $[t^n]G(t)$ denotes the coefficient of t^n in the power series $G(t)$, and where

$$H(t) = 1 + h_1 t + h_2 t^2 + \cdots = \frac{1}{(1 - x_1 t)(1 - x_2 t) \cdots},$$

the generating function for the complete symmetric functions h_i. (Throughout this paper we adhere to symmetric function terminology and notation as in Macdonald [17].)

The following proposition summarizes some of the properties of PF_n which follow easily from equation (1).

2.2 Proposition. (a) *We have the following expansions.*

$$\begin{aligned}
\mathrm{PF}_n &= \sum_{\lambda\vdash n}(n+1)^{\ell(\lambda)-1}z_\lambda^{-1}p_\lambda && (2)\\
&= \sum_{\lambda\vdash n}\frac{1}{n+1}s_\lambda(1^{n+1})s_\lambda && (3)\\
&= \sum_{\lambda\vdash n}\frac{1}{n+1}\left[\prod_i\binom{\lambda_i+n}{n}\right]m_\lambda && (4)\\
&= \sum_{\lambda\vdash n}\frac{n(n-1)\cdots(n-\ell(\lambda)+2)}{m_1(\lambda)!\cdots m_n(\lambda)!}h_\lambda. && (5)\\
\omega\mathrm{PF}_n &= \sum_{\lambda\vdash n}\frac{1}{n+1}\left[\prod_i\binom{n+1}{\lambda_i}\right]m_\lambda. && (6)
\end{aligned}$$

Here $s_\lambda(1^{n+1})$ denotes s_λ with $n+1$ variables set equal to 1 and the others to 0, and is evaluated explicitly e.g. in [17, Example 4, p. 45]. Moreover, $\ell(\lambda)$ is the number of parts of λ; z_λ is as in [17, p. 24]; $m_i(\lambda)$ denotes the number of parts of λ equal to i; and ω is the standard involution [17, pp. 21–22] on symmetric functions.

(b) *We also have that*

$$\sum_{n\geq 0}\mathrm{PF}_n\, t^{n+1} = (tE(-t))^{\langle -1\rangle}, \tag{7}$$

where $E(t)=\sum_{n\geq 0}e_nt^n$, e_n denotes the nth elementary symmetric function, and ${}^{\langle -1\rangle}$ denotes compositional inverse.

Proof. (a) Let $C(x,y)=\prod_{i,j}(1-x_iy_j)^{-1}$, the well-known "Cauchy product." Then $H(t)^{n+1}$ is obtained by setting $n+1$ of the y_i's equal to t and the others equal to 0. From this all the expansions in (a) follow from (1) and well-known expansions for $C(x,y)$ and $\omega_xC(x,y)$ (where ω_x denotes ω acting on the x variables only). To give just one example (needed in the first proof of Theorem 2.3), we have

$$\omega_xC(x,y)=\sum_\lambda m_\lambda(x)e_\lambda(y).$$

Hence

$$\omega\mathrm{PF}_n=\sum_{\lambda\vdash n}\frac{1}{n+1}e_\lambda(1^{n+1})m_\lambda.$$

Equation (6) now follows from the simple fact that $e_k(1^{n+1})=\binom{n+1}{k}$. We should point out that (2) appears (in a dual form) in [11, (9)], (3) appears in [13, (28)], and (5) appears (again in dual form) in [13, (82)][17, Example 24(a), p. 35]. A q-analogue of PF_n and of much of our Proposition 2.2 appears in [9].

(b) This is an immediate consequence of (1), the fact that

$$\frac{1}{H(t)}=E(-t), \tag{8}$$

and the Lagrange inversion formula, as in [13, §4.1]. See also [17, Examples 2.24–2.25, pp. 35–36]. □

Let P be a Cohen-Macaulay poset with $\hat{0}$ and $\hat{1}$ such that every interval is rank-symmetric. Thus F_P is a symmetric function. In [30, Conj. 2.3] it was conjectured that F_P is *Schur positive*, i.e., a nonnegative linear combination of Schur functions. Equation (3) confirms this conjecture in the case $P = \mathrm{NC}_{n+1}$. However, it turns out that the conjecture is in fact *false*. A counterexample is provided by the following poset P. The elements of P consist of all integer vectors $(a_1, a_2, b_1, b_2, b_3, b_4)$ such that $0 \le a_1 \le 5$, $0 \le a_2 \le 1$, $0 \le b_1 \le 3$, $0 \le b_2, b_3, b_4 \le 1$, and $a_1 + a_2 = b_1 + b_2 + b_3 + b_4$, ordered componentwise. It can be shown that P is lexicographically shellable and hence Cohen-Macaulay, and it is easy to see that P is locally rank-symmetric (even locally self-dual). Moreover,

$$F_P = s_6 + 7s_{51} + 6s_{42} + 2s_{33} + 18s_{411} + 10s_{321} - s_{222} + 20s_{3111} + 5s_{2211} + 8s_{21111}.$$

The symmetric functions PF_n also have an unexpected connection with the multiplication of conjugacy classes in the symmetric group (the work of Farahat-Higman [7]). For further details see [11][17, Ch. I, Example 7.25, pp. 132–134]. This connection was exploited by Goulden and Jackson [11] to compute some connection coefficients for the symmetric group.

The expansion (5) of PF_n in terms of the h_λ's has a simple interpretation in terms of parking functions. Suppose that $a = (a_1, \ldots, a_n) \in \mathcal{P}_n$. Let $r_1, \ldots, r_k$ be the positive multiplicities of the elements of the multiset $\{a_1, \ldots, a_n\}$ (so $r_1 + \cdots + r_k = n$). Then the action of $\mathfrak{S}_n$ on the orbit $\mathfrak{S}_n a$ has characteristic $h_{r_1} \cdots h_{r_k}$. For instance, a set of orbit representatives in the case $n = 3$ is $(1,1,1), (2,1,1), (3,1,1), (2,2,1)$, and $(3,2,1)$. Hence $\mathrm{PF}_3 = h_3 + h_2h_1 + h_2h_1 + h_2h_1 + h_1^3 = h_3 + 3h_{21} + h_{111}$. In general it follows that the coefficient q_λ of h_λ in PF_n is equal to the number of orbits of parking functions of length n such that the terms of their elements have multiplicities $\lambda_1, \lambda_2, \ldots$ (in some order). Equation (5) then gives an explicit formula for this number. The *total* number of parking functions whose terms have multiplicities $\lambda_1, \lambda_2, \ldots$ is q_λ times the size of the orbit, i.e., $q_\lambda \binom{n}{\lambda_1, \lambda_2, \ldots}$.

We are now ready to discuss the connection between PF_n and noncrossing partitions. The basic result is the following.

2.3 Theorem. *For any $n \ge 0$ we have*

$$F_{\mathrm{NC}_{n+1}} = \omega \mathrm{PF}_n.$$

Proof. Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be a partition of n with $\lambda_\ell > 0$. It is immediate from the definition of F_P in Section 1 (see [30, Prop. 1.1]) that if F_P is symmetric and $F_P = \sum_\lambda c_\lambda m_\lambda$, then

$$c_\lambda = \alpha_P(\lambda_1, \lambda_1 + \lambda_2, \ldots, \lambda_1 + \lambda_2 + \cdots + \lambda_{\ell-1}). \tag{9}$$

The proof now follows by comparing equation (6) with the evaluation of $\alpha_{\mathrm{NC}_{n+1}}(S)$ due to Edelman [4, Thm. 3.2]. □

It follows from the above discussion that PF_n encodes in a simple way the flag f-vector and flag h-vector of NC_{n+1}, viz., (1) the coefficient of Q_S in the expansion of PF_n in terms of Gessel's quasisymmetric function is equal to $\beta_{\mathrm{NC}_{n+1}}(S)$, and (2) if the elements of $S \subseteq [n-1]$ are $j_1 < \cdots < j_r$ and if λ is the partition whose parts are the numbers $j_1, j_2 - j_1, j_3 - j_2, \ldots, n - j_r$, then the coefficient of m_λ in the expansion of PF_n in terms of monomial symmetric functions is equal to $\alpha_{\mathrm{NC}_{n+1}}(S)$. There is a further statistic on NC_{n+1} closely related to PF_n, namely, the number of noncrossing partitions of $[n+1]$ of *type* λ, i.e., with block sizes $\lambda_1, \lambda_2, \ldots$.

2.4 Proposition. *Let λ be a partition of n. The coefficient of h_λ in the expansion of PF_n in terms of complete symmetric functions is equal to the number u_λ of noncrossing partitions of type λ.*

First proof. Compare equation (5) with the explicit value of the number of noncrossing partitions of type λ found by Kreweras [15, Thm. 4]. □

Second proof. Our second proof is based on the following noncrossing analogue of the exponential formula due to Speicher [28, p. 616]. (For a more general result, see [21].) Given a function $f : \mathbb{N} \to R$ (where R is a commutative ring with identity) with $f(0) = 1$, define a function $g : \mathbb{N} \to R$ by $g(0) = 1$ and

$$g(n) = \sum_{\pi=\{B_1,\ldots,B_k\}\in\mathrm{NC}_n} f(\#B_1)\cdots f(\#B_k). \tag{10}$$

Let $F(t) = \sum_{n\geq 0} f(n)t^n$. Then

$$\sum_{n\geq 0} g(n)t^{n+1} = \left(\frac{t}{F(t)}\right)^{\langle -1\rangle}. \tag{11}$$

In equation (10) take $f(n) = h_n$, the complete symmetric function. Then $g(n)$ becomes $\sum_{\lambda\vdash n} u_\lambda h_\lambda$. But Proposition 2.2(b), together with equations (11) and (8), shows that $g(n) = \mathrm{PF}_n$, and the proof follows. □

Note the curious fact that Theorem 2.3 refers to NC_{n+1}, while Proposition 2.4 refers to NC_n. Proposition 2.4, together with the definition of PF_n, show that the number of noncrossing partitions of type $\lambda \vdash n$ is equal to the number of $\mathfrak{S}_n$-orbits of parking functions of length n and part multiplicities λ. It is easy to give a bijective proof of this fact (shown to me by R. Simion), which we omit.

3. An edge labeling of the noncrossing partition lattice. If P is a locally finite poset, then an *edge* of P is a pair $(u, v) \in P \times P$ such that v covers u (i.e., $u < v$ and no element t satisfies $u < t < v$). An *edge labeling* of P is a map $\Lambda : \mathcal{E}(P) \to \mathbb{Z}$, where $\mathcal{E}(P)$ is the set of edges of P. Edge labelings of posets have many applications; in particular, if P has what is known as an *EL-labeling*, then P is lexicographically shellable and hence Cohen-Macaulay [2][3]. An EL-labeling of NC_{n+1} was defined by Björner [2, Example 2.9] and further exploited by Edelman and Simion [5]. Here we define a new labeling, which up to an unimportant reindexing is EL and is intimately related to parking functions.

Let (π, σ) be an edge of NC_{n+1}. Thus σ is obtained from π by merging together two blocks B and B'. Suppose that $\min B < \min B'$, where $\min S$ denotes the minimum element of a finite set S of integers. Define

$$\Lambda(\pi, \sigma) = \max\{i \in B : i < B'\}, \tag{12}$$

where $i < B'$ denotes that i is less than every element of B'. For instance, if $B = \{2, 4, 5, 15, 17\}$ and $B' = \{7, 10, 12, 13\}$, then $\Lambda(\pi, \sigma) = 5$. Note that $\Lambda(\pi, \sigma)$ always exists since $\min B < B'$.

The labeling Λ of the edges of NC_{n+1} extends in a natural (and well-known) way to a labeling of the maximal chains. Namely, if $\mathfrak{m} : \hat{0} = \pi_0 < \pi_1 < \cdots < \pi_n = \hat{1}$ is a maximal chain of NC_{n+1}, then set

$$\Lambda(\mathfrak{m}) = (\Lambda(\pi_0, \pi_1), \Lambda(\pi_1, \pi_2), \ldots, \Lambda(\pi_{n-1}, \pi_n)).$$

3.1 Theorem. *The labels $\Lambda(\mathfrak{m})$ of the maximal chains of NC_{n+1} consist of the parking functions of length n, each occuring once.*

Proof. If $\Lambda(\pi_j, \pi_{j+1}) = i$, then the block of π_{j+1} containing i also contains an element $k > i$. Hence the number of j for which $\Lambda(\pi_j, \pi_{j+1}) = i$ cannot exceed $n + 1 - i$, from which it follows that $\Lambda(\mathfrak{m})$ is a parking function.

Suppose that $\mathfrak{m}$ and $\mathfrak{m}'$ are maximal chains of NC_{n+1} for which $\Lambda(\mathfrak{m}) = \Lambda(\mathfrak{m}')$. We will prove by induction on n that $\mathfrak{m} = \mathfrak{m}'$. The assertion is clear for $n = 0$. Assume true for $n - 1$. Let the elements of $\mathfrak{m}$ be $\hat{0} = \pi_0 < \pi_1 < \cdots < \pi_n = \hat{1}$. Suppose that $\Lambda(\mathfrak{m}) = (a_1, \ldots, a_n)$. Let $r = \max\{a_i : 1 \le i \le n\}$, and let $s = \max\{i : a_i = r\}$. We claim that one of the blocks of π_{s-1} is just the singleton set $\{r + 1\}$. If r and $r + 1$ are in the same block of π_{s-1}, then we can't have $\Lambda(\pi_{s-1}, \pi_s) = r$, contradicting $a_s = r$. Hence r and $r + 1$ are in different blocks of π_{s-1}. If the block B of π_{s-1} containing $r+1$ contained some element $t < r$, then by the noncrossing property and the fact that $a_s = r$ we have that B is merged with the block B_1 of π_{s-1} containing r to get π_s. But $\min B \le t < r \in B_1$, contradicting $a_s = r$. Hence every element of B is greater than r. If B contained some element $t > r + 1$, then (since $r + 1 = \min B$) we would have $a_k = r + 1$ for some $k < r$, contradicting maximality of r. This proves the claim.

We next claim that π_s is obtained from π_{s-1} by merging the block B_1 containing r with the block $\{r+1\}$. Otherwise (since $a_s = r$) π_s is obtained by merging B_1 with some block B_2 all of whose elements are greater than $r + 1$. For some $t > s$ we must obtain π_t from π_{t-1} by merging the block B_3 containing $r + 1$ with the block B_4 containing r. Now B_3 can't contain an element less than $r + 1$ by the noncrossing property of π_{s-1} (since B_4 contains both r and an element greater than $r + 1$). It follows that $\Lambda(\pi_{t-1}, \pi_t) = r$, contradicting the maximality of s and proving the claim.

It is now clear by induction that the chain $\mathfrak{m}$ can be uniquely recovered from the parking function $\Lambda(\mathfrak{m}) = (a_1, \ldots, a_n)$. Namely, let a' be the sequence obtained from $\Lambda(\mathfrak{m})$ by removing a_s. Then a' is a parking function of length $n - 1$. By induction there is a unique maximal chain $\mathfrak{m}^* : \hat{0} = \pi_0^* < \pi_1^* < \cdots < \pi_{n-1}^* = \hat{1}$ of NC_n such that $\Lambda(\mathfrak{m}^*) = a'$. By the discussion above we can then obtain $\mathfrak{m}$ uniquely from $\mathfrak{m}^*$ by (1) replacing each element $i > r$ of the ambient set $[n]$ with $i + 1$, (2) adjoining a singleton block $\{r + 1\}$ to each π_i^* for $i \le s - 1$, (3) inserting between π_{s-1}^* and π_s^* a new element obtained from π_{s-1}^* by merging the block containing r with the singleton block $\{r + 1\}$, and (4) for $i > s$ adjoining the element $r + 1$ to the block of π_i^* containing r. Hence we have shown that if $\Lambda(\mathfrak{m}) = \Lambda(\mathfrak{m}')$, then $\mathfrak{m} = \mathfrak{m}'$. But it is known [15, Cor. 5.2][4, Cor. 3.3] that NC_{n+1} has $(n + 1)^{n-1}$ maximal chains, which is just the number of parking functions of length n [14, Lemma 1 and §6][8]. Thus every parking function of length n occurs exactly once among the sequences $\Lambda(\mathfrak{m})$, and the proof is complete. □

The above proof of the injectivity of the map Λ from maximal chains to parking functions is reminiscent of the proof [20, p. 5] that the Prüfer code of a labelled tree determines the tree. Our proof "cheated" by using the fact that the number of maximal chains is the number of parking functions. We only gave a direct proof of the injectivity of Λ. However, our proof actually suffices to show also surjectivity since the argument of the above paragraph is valid for any parking function, the key point being that removing an occurrence of the largest element of a parking function preserves the property of being a parking function.

If we define a new labeling Λ^* of NC_{n+1} by

$$\Lambda^*(\pi,\sigma) = |\pi| - \Lambda(\pi,\sigma),$$

where $|\pi|$ is the number of blocks of π, then it is easy to check (using the fact that every interval of NC_{n+1} is a product of NC_i's) that every interval $[\pi,\tau]$ has a unique maximal chain $\mathfrak{m} : \pi = \pi_0 < \pi_1 < \cdots < \pi_j = \tau$ such that

$$\Lambda^*(\pi_0,\pi_1) \leq \Lambda^*(\pi_1,\pi_2) \leq \cdots \leq \Lambda^*(\pi_{k-1},\pi_k).$$

In other words, Λ^* is an *R-labeling* in the sense of [29, Def. 3.13.1]. Moreover, this maximal chain $\mathfrak{m}$ has the lexicographically least label $\Lambda^*(\mathfrak{m})$ of any maximal chain of the interval $[\pi,\tau]$. Thus Λ^* is in fact an *EL-labeling*, as defined in [2, Def. 2.2] (though there it is called just an "L-labeling."). For the significance of the EL-labeling property, see the first paragraph of this section. Here we will just be concerned with the weaker R-labeling property.

Define the *descent set* $D(a)$ of a parking function $a = (a_1,\ldots,a_n)$ by

$$D(a) = \{i : a_i > a_{i+1}\}.$$

From the fact that Λ^* is an R-labeling and [29, Thm. 3.13.2], we obtain the following proposition.

3.2 Proposition. (a) *Let $S \subseteq [n-1]$. The number of parking functions a of length n satisfying $D(a) = S$ is equal to $\beta_{\mathrm{NC}_{n+1}}([n-1]-S)$.*

(b) *Let $S \subseteq [n-1]$. The number of parking functions a of length n satisfying $D(a) \supseteq S$ is equal to $\alpha_{\mathrm{NC}_{n+1}}([n-1]-S)$. This number is given explicitly by [4, Thm. 3.2] or by equations (4) and (9).*

The labeling Λ is closely related to a bijection between the maximal chains of NC_{n+1} and labelled trees, different from the earlier bijection of Edelman [4, Cor. 3.3]. Let $\mathfrak{m} : \hat{0} = \pi_0 < \pi_1 < \cdots < \pi_n = \hat{1}$ be a maximal chain of NC_{n+1}. Define a graph $\Gamma_\mathfrak{m}$ on the vertex set $[n+1]$ as follows. There will be an edge e_i for each $1 \leq i \leq n$. Suppose that π_i is obtained from π_{i-1} by merging blocks B and B' with $\min B < \min B'$. Then the vertices of e_i are defined to be $\Lambda(\pi_{i-1},\pi_i)$ and $\min B'$. It is easy to see that $\Gamma_\mathfrak{m}$ is a tree. Root $\Gamma_\mathfrak{m}$ at the vertex 1 and erase the vertex labels. If v_i is the vertex of e_i farthest from the root, then move the label i of the edge e_i from e_i to the vertex v_i. Label the root with 0 and unroot the tree. We obtain a labelled tree $T_\mathfrak{m}$ on $n+1$ vertices, and one can easily check that the map $\mathfrak{m} \mapsto T_\mathfrak{m}$ is a bijection between maximal chains of NC_{n+1} and labelled trees on $n+1$ vertices.

4. A local action of the symmetric group. Suppose that P is a graded poset of rank n with $\hat{0}$ and $\hat{1}$ such that F_P is a symmetric function. If F_P is Schur positive, then it is the Frobenius characteristic of a representation of $\mathfrak{S}_n$ whose dimension is the number of maximal chains of P. Thus we can ask whether there is some "nice" representation of $\mathfrak{S}_n$ on the vector space V_P (over a field of characteristic zero) whose basis is the set of maximal chains of P. This question was discussed in [30, §5]. A "nice" representation should somehow reflect the poset structure. With this motivation, an action of $\mathfrak{S}_n$ on V_P is defined to be *local* [30, §5] if for every adjacent transposition $\sigma_i = (i, i+1)$ and every maximal chain

$$\mathfrak{m} : \hat{0} = t_0 < t_1 < \cdots < t_n = \hat{1}, \tag{13}$$

we have that $\sigma_i(\mathfrak{m})$ is a linear combination of maximal chains of the form $t_0 < t_1 < \cdots < t_{i-1} < t'_i < t_{i+1} < \cdots < t_n$, i.e., of maximal chains which agree with $\mathfrak{m}$ except possibly at t_i.

Now let $P = \mathrm{NC}_{n+1}$. Every interval $[\pi, \tau]$ of NC_{n+1} of length two contains either two or three elements in its middle level. In the latter case, there are three blocks B_1, B_2, B_3 of π such that τ is obtained from π by merging B_1, B_2, B_3 into a single block. Moreover, any two of these blocks can be merged to form a noncrossing partition. Let π_{ij} be the noncrossing partition obtained by merging B_i and B_j, so that the middle elements of the interval $[\pi, \tau]$ are $\pi_{12}, \pi_{13}, \pi_{23}$. Exactly one of these partitions π_{ij} will have the property that $\Lambda(\pi, \pi_{ij}) = \Lambda(\pi_{ij}, \tau)$, where Λ is defined by (12). Let us call this partition π_{ij} the *special* element of the interval $[\pi, \tau]$. Now define linear transformations $\sigma'_i : V_{\mathrm{NC}_{n+1}} \to V_{\mathrm{NC}_{n+1}}$, $1 \le i \le n-1$ as follows. Let $\mathfrak{m}$ be a maximal chain of NC_{n+1} with elements $\hat{0} = \pi_0 < \pi_1 < \cdots < \pi_n = \hat{1}$.

Case 1. The interval $[\pi_{i-1}, \pi_{i+1}]$ contains exactly two middle elements π_i and π'_i. Then set $\sigma'_i(\mathfrak{m}) = \mathfrak{m}'$, where $\mathfrak{m}'$ is given by $\pi_0 < \pi_1 < \cdots < \pi_{i-1} < \pi'_i < \pi_{i+1} < \cdots < \pi_n$.

Case 2. The interval $[\pi_{i-1}, \pi_{i+1}]$ contains exactly three middle elements, of which π_i is special. Then set $\sigma'_i(\mathfrak{m}) = \mathfrak{m}$.

Case 3. The interval $[\pi_{i-1}, \pi_{i+1}]$ contains exactly three middle elements π_i, π'_i, and π''_i, of which π''_i is special. Then set $\sigma'_i(\mathfrak{m}) = \mathfrak{m}'$, where $\mathfrak{m}'$ is given by $\pi_0 < \pi_1 < \cdots < \pi_{i-1} < \pi'_i < \pi_{i+1} < \cdots < \pi_n$.

4.1 Proposition. *The action of each σ'_i on $V_{\mathrm{NC}_{n+1}}$ defined above yields a local action of $\mathfrak{S}_n$ on $V_{\mathrm{NC}_{n+1}}$. Equivalently, there is a homomorphism $\varphi : \mathfrak{S}_n \to \mathrm{GL}(V_{\mathrm{NC}_{n+1}})$ satisfying $\varphi(\sigma_i) = \sigma'_i$. The Frobenius characteristic of this action is given by* PF_n.

Proof. Each maximal chain $\mathfrak{m}$ corresponds to a parking function $\Lambda(\mathfrak{m})$ *via* Theorem 3.1. Thus the natural action of $\mathfrak{S}_n$ on $\mathcal{P}_n$ defined in Section 2 may be "transferred" to an action ψ of $\mathfrak{S}_n$ on the set of maximal chains of NC_{n+1}. It is easy to check that ψ and φ agree on the σ_i's, and the proof follows. □

The action φ does not quite have the property mentioned at the beginning of this section that its characteristic is $F_{\mathrm{NC}_{n+1}}$. By Theorem 2.3, the characteristic is actually $\omega F_{\mathrm{NC}_{n+1}}$. However, we only have to multiply φ by the sign character (equivalently, define a new action φ' by $\varphi'(\sigma_i) = -\varphi(\sigma_i)$) to get the desired property.

It is rather surprising that the simple "local" definition we have given of φ defines an action of $\mathfrak{S}_n$. Perhaps it would be interesting to look for some more examples. (We need to exclude trivial examples such as $w(\mathfrak{m}) = \mathfrak{m}$ for all $w \in \mathfrak{S}_n$ and all maximal chains $\mathfrak{m}$.) A few other examples appear in the next section and in [30, §5]. A further example (the posets of shuffles of C. Greene [12]) is discussed in [26] together with the rudiments of a systematic theory of such actions, but much work needs to be done for a satisfactory understanding of local $\mathfrak{S}_n$-actions.

5. Generalizations. In this section we will briefly discuss two generalizations of what appears above. All proofs are entirely analogous and will be omitted. Fix an integer $k \in \mathbb{P}$. A *k-divisible noncrossing partition* is a noncrossing partition π for which every block size is divisible by k. Thus π is a noncrossing partition of a set $[kn]$ for some $n \geq 0$. Let $\mathrm{NC}_n^{(k)}$ be the poset of all k-divisible noncrossing partitions of $[kn]$. ($\mathrm{NC}_n^{(k)}$ is actually a join-semilattice of NC_{kn}. It has $\hat{1}$ but not a $\hat{0}$ when $k > 1$.) The combinatorial properties of the poset $\mathrm{NC}_n^{(k)}$ were first considered by Edelman [4, §4]. If a pair (π, σ) is an edge of $\mathrm{NC}_n^{(k)}$ (i.e., σ covers π in $\mathrm{NC}_n^{(k)}$), then (π, σ) is an edge of NC_n. Hence the edge-labeling Λ of NC_n restricts to an edge-labeling of $\mathrm{NC}_n^{(k)}$.

Define a *k-parking function* of length n to be a sequence $(a_1, \ldots, a_n)$ of positive integers such that if $b_1 \leq b_2 \leq \cdots \leq b_n$ is the increasing rearrangement of $a_1, \ldots, a_n$, then $b_i \leq ki$. Let $\mathcal{P}_n^{(k)}$ denote the set of all k-parking functions of length n. The argument of Pollack mentioned in Section 2 that $\#\mathcal{P}_n = (n+1)^{n-1}$ easily extends to $\mathcal{P}_n^{(k)}$. Namely, let $\mathbb{Z}_{k(n+1)}$ denote the set $\{1, 2, \ldots, k(n+1)\}$ with addition modulo $k(n+1)$. Then every coset of the subgroup H of $\mathbb{Z}_{k(n+1)}^n$ generated by $(1, 1, \ldots, 1)$ contains exactly k k-parking functions. Hence $\#\mathcal{P}_n^{(k)} = k(k(n+1))^{n-1}$. The symmetric group $\mathfrak{S}_n$ acts on $\mathcal{P}_n^{(k)}$ by permuting coordinates, and we can consider its Frobenius characteristic $\mathrm{PF}_n^{(k)}$ just as we did for $\mathcal{P}_n$. The above generalization of Pollack's argument shows that

$$\mathrm{PF}_n^{(k)} = \frac{1}{n+1}[t^n]H(t)^{k(n+1)}.$$

Proposition 2.2 generalizes straightforwardly to the case of $\mathrm{PF}_n^{(k)}$. In particular, Proposition 2.2(b) takes the form

$$\sum_{n \geq 0} \mathrm{PF}_n^{(k)} t^{n+1} = \left(tE(-t)^k\right)^{\langle -1 \rangle}.$$

Theorem 3.1 generalizes as follows.

5.1 Theorem. *The labels $\Lambda(\mathfrak{m})$ of the maximal chains of $\mathrm{NC}_{n+1}^{(k)}$ consist of the k-parking functions of length n, each occuring once.*

Proposition 3.2 requires some modification when extended to k-parking functions because the posets $\mathrm{NC}_{n+1}^{(k)}$ do not have a $\hat{0}$ when $k > 1$. For these posets we regard the minimal elements as having rank 0, and we define $\alpha_{\mathrm{NC}_{n+1}^{(k)}}(S)$ and $\beta_{\mathrm{NC}_{n+1}^{(k)}}(S)$ for $S \subseteq \{0, 1, \ldots, n-1\}$. Thus for instance $\alpha_{\mathrm{NC}_{n+1}^{(k)}}(\emptyset) =$

$\beta_{\mathrm{NC}^{(k)}_{n+1}}(\emptyset) = 1$, and $\alpha_{\mathrm{NC}^{(k)}_{n+1}}(0)$ is the number of minimal elements of $\mathrm{NC}^{(k)}_{n+1}$. Write $[0, n-1] = \{0, 1, \ldots, n-1\}$.

5.2 Proposition. (a) *Let $S \subseteq [n-1]$. The number of k-parking functions a of length n satisfying $D(a) = S$ is equal to*

$$\beta_{\mathrm{NC}^{(k)}_{n+1}}([n-1] - S) + \beta_{\mathrm{NC}^{(k)}_{n+1}}([0, n-1] - S).$$

(b) *Let $S \subseteq [n-1]$. The number of k-parking functions a of length n satisfying $D(a) \supseteq S$ is equal to $\alpha_{\mathrm{NC}^{(k)}_{n+1}}([0, n-1] - S)$. This number is given explicitly by [4, Thm. 4.2].*

Note, however, that there does not seem to be a nice generalization of Theorem 2.3. The quasisymmetric function $F_{\mathrm{NC}^{(k)}_{n+1}}$ is not a symmetric function when $k > 1$, and we know of no simple connection between the flag f-vector of $\mathrm{NC}^{(k)}_{n+1}$ and the symmetric function $\mathrm{PF}^{(k)}_n$, nor between the number of k-divisible noncrossing partitions of a given type and $\mathrm{PF}^{(k)}_n$.

Proposition 4.1 extends straightforwardly to $\mathrm{NC}^{(k)}_{n+1}$. The natural action of $\mathfrak{S}_n$ on $\mathcal{P}^{(k)}_n$ is transferred *via* Theorem 5.1 to an action on $V_{\mathrm{NC}^{(k)}_{n+1}}$. This action is a permutation representation on the maximal chains, and is readily seen to be local. Its characteristic is $\mathrm{PF}^{(k)}_n$.

There is a different generalization of noncrossing partitions due to Reiner [23] (a special case had earlier appeared in a different guise in [19], as explained in [23]) that we have not looked at very closely. Reiner regards ordinary noncrossing partitions as corresponding to the root system A_n and constructs analogues for the root systems B_n and D_n. Actually, for every subset $S \subseteq [n]$ he constructs a lattice $\mathrm{NC}^{BD}_n(S)$ interpolating between the B_n analogue (the case $S = \emptyset$) and the D_n analogue (the case $S = [n]$). The lattices $\mathrm{NC}^{BD}_n(S)$ do not always have self-dual intervals [23, Remark on p. 13], but at least every interval is rank-symmetric. Thus by Proposition 2.1, this implies that $\mathrm{NC}^{BD}_n(S)$ is a symmetric function. This symmetric function only depends on the cardinality s of S, so we write $F^{BD}_n(s)$ for $F_{\mathrm{NC}^{BD}_n(S)}$. We also write F^B_n for $F^{BD}_n(0)$.

Let $[n]^n$ denote the set of all sequences $(a_1, \ldots, a_n)$ of positive integers with $a_i \leq n$. We call such a sequence a *B_n-parking function*, for the following reason. Let PF^B_n be the Frobenius characteristic of the action of $\mathfrak{S}_n$ on $[n]^n$ obtained by permuting coordinates. It follows from [23, Prop. 7] that

$$F^B_n = \omega \mathrm{PF}^B_n.$$

Thus in analogy with Theorem 2.3 it makes sense to think of the elements of $[n]^n$ as B_n-parking functions. Reiner's result [23, Prop. 7] makes it easy to give an analogue of Proposition 2.2. Let us simply mention the formula

$$\mathrm{PF}^B_n = [t^n] H(t)^n,$$

from which the analogues of all parts of Proposition 2.2 follow easily. In particular, the analogue of Proposition 2.2(b) takes the form

$$\sum_{n \geq 1} \mathrm{PF}^B_n \frac{t^n}{n} = \log \frac{(tE(-t))^{\langle -1 \rangle}}{t}. \tag{14}$$

Comparing Proposition 2.2 with equation (14) yields the curious result that

$$\exp \sum_{n\geq 1} \mathrm{PF}_n^B \frac{t^n}{n} = \sum_{n\geq 0} \mathrm{PF}_n \, t^n.$$

What is missing from the analogy between A_n and B_n noncrossing partitions is the analogue of Theorem 3.1, i.e., a labeling of NC_{n+1}^B such that the labels of the maximal chains are the B_n-parking functions. We have not looked at this question and recommend it as an interesting open problem.

For the general case of $\mathrm{NC}_n^{BD}(S)$, it follows from [23, Thm. 11] that

$$F_n^{BD}(s) = F_n^B - s \cdot \mathrm{PF}'_n,$$

where PF'_n is the Frobenius characteristic of the action of $\mathfrak{S}_n$ on all sequences $(a_1, \ldots, a_n)$ of positive integers whose increasing rearrangement $b_1 \leq \cdots \leq b_n$ satisfies $b_1 = 1$ and $b_i \leq i-1$ for $2 \leq i \leq n$. We have not considered further properties of the symmetric function $F_n^{BD}(s)$.

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