ON A TILING SCHEME FROM M.C. ESCHER

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ABSTRACT: Escher studied certain tilings by decorated square tiles arranged into 2x2 supertiles which are copied and translated to fill the plane. It is found that there are exactly 154 such tilings, if reflections of the tiles are allowed. Illustrations of all of them are presented.

Consider a square tile marked with a motif asymmetric under rotation or reflection; call it the “prototile”. It can be rotated through successive quarter-turns, and these can be reflected, to give eight different states. Choose four copies of the prototile in any combination of states and assemble them into a larger square; call it the “supertile”. Tile the plane with identical copies of the supertile (i.e. with no further rotations or reflections), matching edges and vertices of the squares. In how many ways can this be done? The Dutch artist M.C. Escher addressed this question [1, 2] and achieved correct results, with certain limitations. He also used the idea as the basis for an “amusement for rainy afternoons” for his children [3].

Since a supertile is composed of four tiles chosen from eight possible states, there are 4096 different supertiles. But they do not all produce different patterns. For example, rotating a supertile through 90 degrees will produce another supertile that generates the same pattern, but rotated through 90 degrees. Also, the mirror image of a supertile is formed from a different choice of prototile states, but we will consider a tiling and its mirror image to be the same. And for any particular tiling, the frame containing the four prototile states can be translated by the length of one prototile either horizontally, vertically, or diagonally; giving four supertiles (including the original) for the same pattern. We will refer to such a translation as a “shift”.

A computer program has been written which segregates the supertiles into classes in which all members produce the same pattern up to rotation, reflection, or translation. The prototile in its original position is represented by the letter A, and its clockwise quarter-turns by B, C, and D. The letter a represents A reflected across a vertical line, and b, c, and d stand for successive clockwise quarter-turns of a. A supertile is represented by a “word” of four letters taken from the set {A, B, C, D, a, b, c, d}. The four letters in the word represent the four states of the motif in the supertile, beginning with the state of the motif in the upper left position and traveling clockwise around the supertile. The program applies appropriate transformations to these words and places them by a sieving process into the classes. It is found that there are 154 classes. In other words, it is possible to create 154 different patterns from a single prototile and its reflection, using Escher’s scheme.

If one uses a motif which touches the edges of the prototile in such a way that there is always some contact no matter now the copies of the prototile abut each other in the supertile, and if the borders of the tile are suppressed, then the prototile is not obvious to the eye. Indeed, one sometimes has to look hard to see that two different patterns are made from the same decorated tile. This lends a nice esthetic element to the subject, which is not surprising since it comes from the most mathematical of artists, Escher.

As noted above, there are 4096 words which can be formed by choosing four letters at a time from the eight letters which represent the prototile in its various rotated and reflected states. (In the choice of the four letters, repetitions are allowed.) The program uses lexicographical ordering to search and find the first word not previously assigned to a class and making it the first member of a new class. This word is referred to as the "class captain"; it serves to represent the class. In the illustrations which accompany this article, the patterns are labelled by their class captains. (In our lexicographical ordering, all the capital letters come first, then the lower-case letters. Thus, AAAA is the captain of the first class.)

The remaining words in the class are found by transforming the letters of the class captain to produce equivalent words, that is, words that produce the same pattern. These transformations form a permutation group G of order 32. G is the product of a group of order 8 (generated by a permutation induced by a 90-degree rotation of the supertile and a permutation induced by a reflection of the supertile in its vertical center line) and a group of order 4 (generated by permutations induced by the vertical, horizontal, and diagonal shifts described earlier). Most classes have a distinct word for each transformation and hence have 32 members. But if the pattern associated with a class has some symmetry, some transformations will repeat words already assigned to the class instead of generating new ones, and the class membership will fall short of the usual 32.

We will use the following notation for the transformations, which we identify with the transformations of the supertiles that induce them: *M* represents reflection of a supertile about a center vertical line; *R0* and *K0* represent the identity; *R1*, *R2*, and *R3* represent clockwise rotations of a supertile about its center by 90, 180, and 270 degrees respectively; and *K1*, *K2*, and *K3* represent horizontal, vertical, and diagonal shifts, respectively, of the frame of a supertile in the pattern. These transformations act on a word as follows:

ROTATION (*R1*). Advance each letter cyclically one space to the right in the word (with the last letter moving to the first position). Then advance each letter to its successor thus: (A -> B -> C -> D -> A and a -> b -> c -> d -> a).

EXAMPLE: *R1*(ACdB) = CBDa

Then $R2 = (R1)^2$ and $R3 = (R1)^3$.

REFLECTION (*M*). Replace each letter in the word by this rule: A <-> a, B <-> d, C <-> c, D <-> b; then permute the letters by (12)(34).

EXAMPLE: *M*(ACdB) = cadB

SHIFTS

Horizontal (*K1*). Permute letters by (12)(34).

EXAMPLE: *K1*(ACdB) = CABd

Vertical (K2). Permute letters by (14)(23).

EXAMPLE: *K2*(ACdB) = BdCA

Diagonal (K3). Permute letters by (13)(24).

EXAMPLE: *K3*(ACdB) = dBAC

Let us analyze a specific case to see how this works. Consider Pattern # 133, whose class captain is AaBd. It is is symmetric about vertical mirror lines at the center and at the edges of each supertile, and so it has crystallographic symmetry group pm (for a rigorous discussion of the one- and two-dimensional crystallographic symmetry groups, see Ref. 4). Applying M to the class captain, we have

$$M(\text{AaBd}) = \text{AaBd}$$

so the class captain is invariant under *M*. Since the pattern has vertical mirror lines, the three words that are transformations of the class captain by shifts are invariant under *M*. It can be shown that the four words generated by the product of *R2* and a shift are invariant under *M*. Finally, all eight words generated by *R3* and a shift and/or *M* are repetitions of words produced by *R1* and a shift and/or *M*. Thus, 16 of the words produced by the 32 transformations are repetitions, so the class has only 16 members. The class list is:

| | | | |
|---|---|---|---|
| AaBd(*R0K0* ) | aAdB(*R0K1* ) | dBaA(*R0K2* ) | BdAa(*R0K3* ) |
| aBbC(*R1K0* ) | dAcD(*R1K0M*) | BaCb(*R1K1* ) | AdDc(*R1K1M*) |
| CbBa(*R1K2* ) | DcAd(*R1K2M*) | bCaB(*R1K3* ) | cDdA(*R1K3M*) |
| DbCc(*R2K0* ) | bDcC(*R2K1* ) | cCbD(*R2K2* ) | CcDb(*R2K3* ) |

The above example shows that if a pattern has pm symmetry, its class will have 16 members. Similarly, patterns with symmetry group p2 (generated by half-turns) or symmetry group pg (generated by two parallel glide reflections) have 16 members.

There is another category of classes with 16 members: those invariant under one of the three shifts. The shifts do not induce additional symmetry into the patterns, but their unit translation blocks consist of just two unit squares rather than four. Their unit translation blocks are 1x2 rectangles (for *K1* and *K2*) or two unit squares that share a vertex (for *K3*).

Additionally there is a set of classes with only eight members. This includes two patterns invariant under a quarter-turn (p4 symmetry); those patterns with a symmetry group generated by three elements (pgg, pmg, pmm); those invariant under a single shift and having a symmetry group generated by two elements; and the single pattern whose class captain is AAAA, which is invariant under all three shifts.

There are 22 classes with 16 members and 20 classes with 8 members, accounting for 512 of the 4096 words. The remaining 3584 words belong to classes with 32 members. These describe patterns with p1 symmetry; that is, no symmetry other than translation. There are 112 such classes. In all, the number of classes is 112 + 22 + 20 = 154.

The table on the next page lists the classes with fewer than 32 members, along with their symmetry groups and the shifts that leave the patterns invariant. There follows an illustration of how the eight tile states A, B, C, D, a, b, c, d are derived from a single prototile, and how a typical pattern is produced from this prototile. We conclude with an illustrated catalog of all 154 patterns that can be made from this supertile by Escher's scheme.

It should be noted, for those who wish to compare the classes in this catalog with those in [2], that Escher's transformation convention was different from that used here. He turned his tiles counter-clockwise, used a horizontal reflection line, and reflected after rotating; in each instance the reverse of what we have done. The classes generated by the two conventions will almost always have different names, and will occur in different order, but there is a simple one-to-one correspondence between the two sets of classes.

A catalog of all 4096 words sorted into the 154 classes, with their class captains and the transformation for each word, is available in the form of an ASCII file. Anyone wishing a copy should contact the author by e-mail at the address given in the title of this paper.

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REFERENCES.

1. Doris Schattschneider, *Visions of Symmetry: Notebooks, Periodic Drawings, and Related Work of M.C. Escher*, W.H. Freeman, New York, 1990.
2. __________________, "M.C. Escher's Combinatorial Tilings", *The Electronic Journal of Combinatorics*, V.4, no.2 (1997) #R17
3. George Escher, "M.C. Escher at Work" in *"M.C. Escher: Art and Science"*, edited by H.S.M. Coxeter et al. North-Holland, Amsterdam, 1986.
4. George Martin, *Transformation Geometry: An Introduction to Symmetry*. Springer-Verlag, New York, 1982.

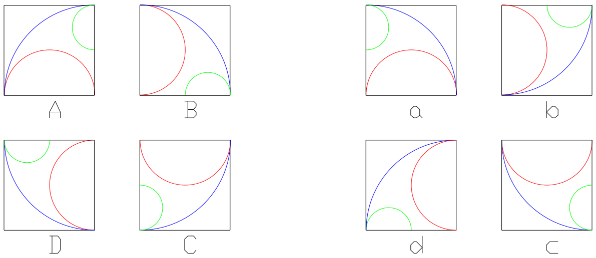
### THE CLASSES WITH FEWER THAN 32 MEMBERS

**I. The 20 classes with 8 members** **II. The 22 classes with 16 members**

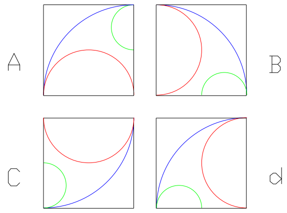
| CLASS NUMBERS AND CAPTAINS | SYMMETRY | CLASS NUMBERS AND CAPTAINS | SYMMETRY |
|---|---|---|---|
| 1. AAAA | All shifts, p1 | 9. AABB; 22. AADD; 31. AAbb; 36. AAdd; 37. ABAB | 1 shift, p1 |
| 45. ABCD; 106. ADCB | p4 | 52. ABDC; 94. ACDB; 103. ACbd; 105. ACdb; 144. AbCd | p2 |
| 16. AACC; 83. ACAC; 88. ACCA | 1 shift, p2 | 62. ABad; 74. ABcb; 79. ABda; 113. ADab; 116. ADba; 121. ADcb; 145. AbDa; 151. AcDd | pg |
| 27. AAaa; 140. AbAb; 153. AccA; 154. AdAd | 1 shift, pg | 68. ABbc; 125. ADdc; 133. AaBd; 136. AaDb | pm |
| 34. AAcc; 127. AaAa; 137. AaaA | 1 shift, pm | | |
| 100. ACac; 139. AacC | pmg | | |
| 104. ACca; 150. AcCa; 152. AcaC | pgg | | |
| 135. AaCc | pmm | | |
| 148. AcAc | 1 shift, cm | | |

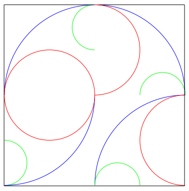
# THE 154 PATTERNS GENERATED FROM A SINGLE PROTOTILE, USING ESCHER'S SCHEME

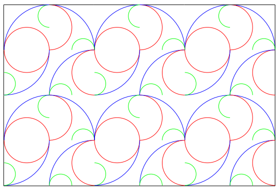
The rotated and reflected states of the prototile are labelled thus:



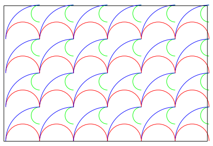

Four of these are used to make a supertile which is translated horizontally and vertically to generate a pattern. The word for the supertile is the label of the pattern.



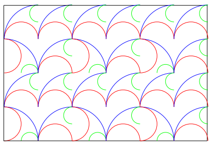





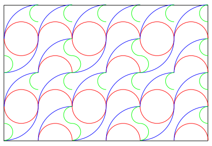
78. ABdC



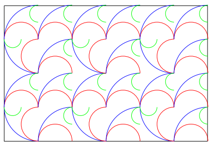
1. AAAA



2. AAAB



3. AAAC



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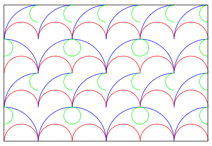
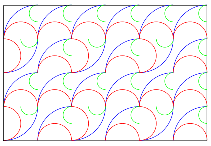
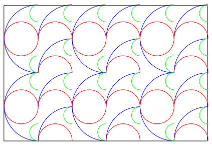


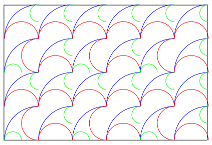
5. AAAa



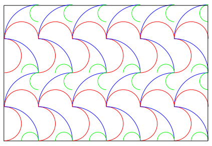
6. AAAb



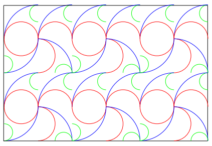
7. AAAc



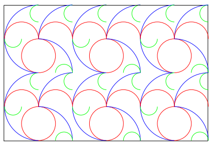
8. AAAd



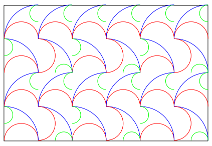
9. AABB



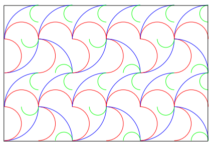
10. AABC



11. AABD



12. AABa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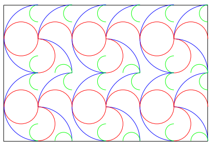
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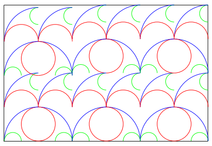
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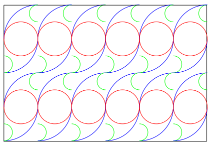
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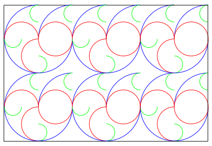
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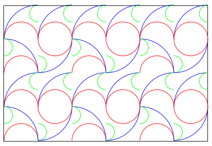
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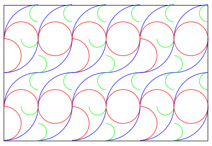
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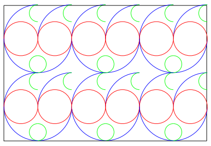
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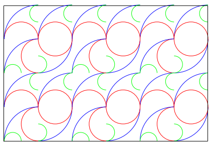
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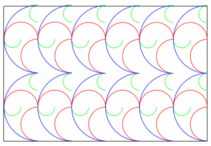
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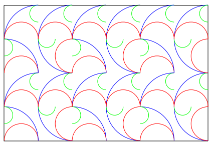
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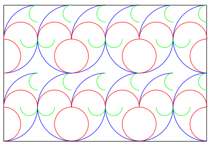
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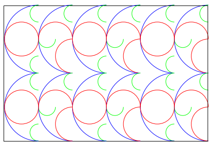
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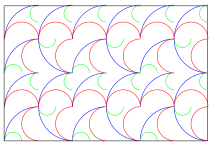
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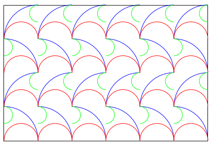
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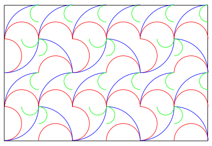
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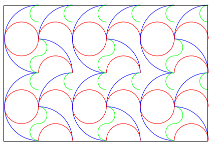
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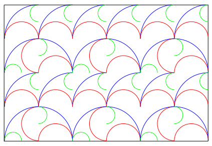
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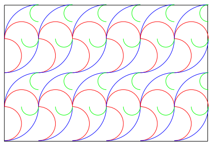
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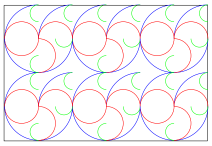
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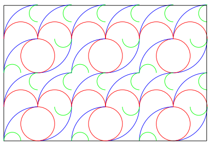
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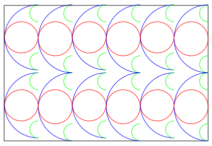
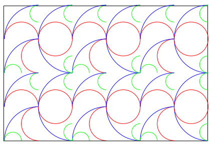

34 AAcc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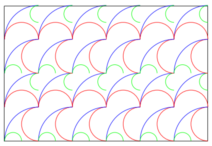
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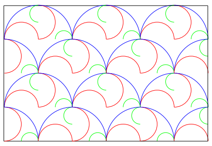
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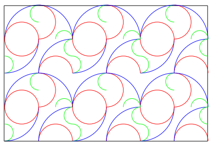
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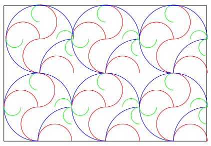
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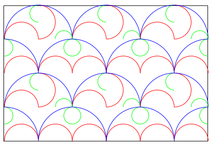
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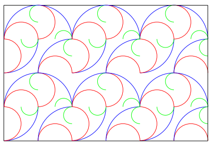
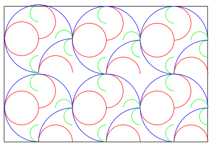
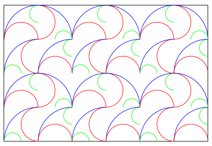
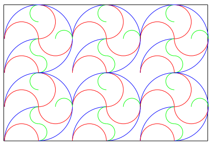
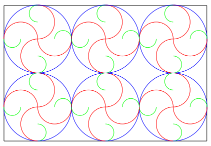
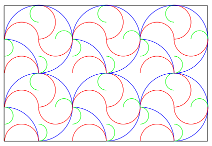

41. ABAb

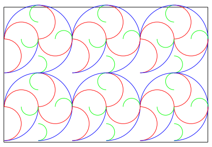

42. ABAc

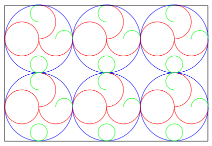

43. ABAd

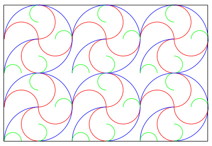

44. ABCA

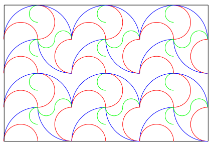

45. ABCD

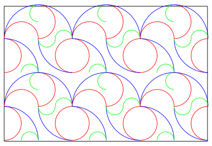

46. ABCa

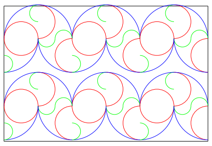

47. ABCb

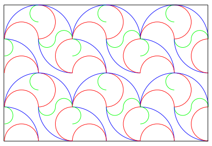

48. ABCc

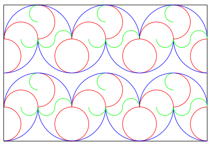

49. ABCd

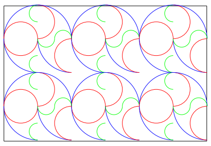

50. ABDA

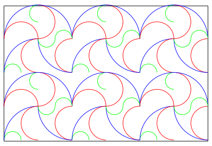

51. ABDB

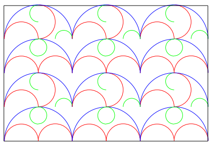

52. ABDC

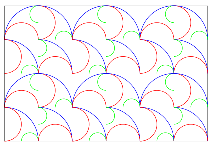

53. ABDa

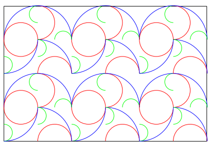

54. ABDb

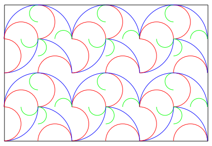

55. ABDc

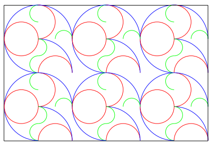

56. ABDd

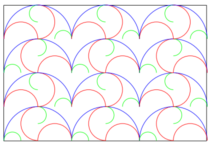

57. ABaA

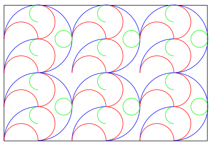

58. ABaB

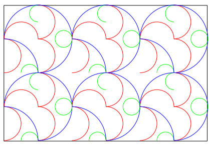

59. ABaC

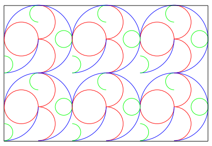

60. ABab

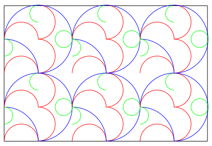

61. ABac

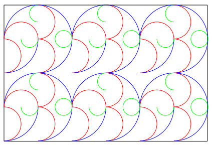

62. ABad

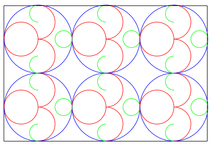

63. ABbA

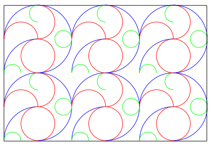

64. ABbB

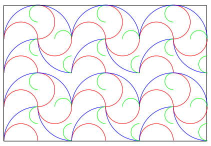

65. ABbC

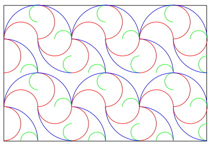

66. ABba

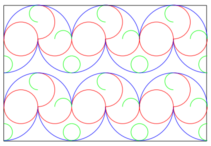

67. ABbb

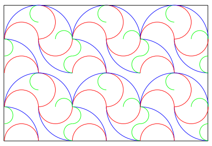

68. ABbc

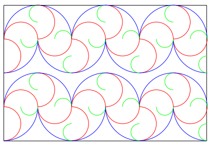

69. ABbd

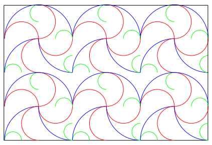

70. ABcA

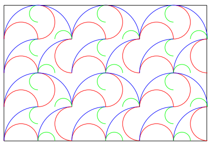

71. ABcB

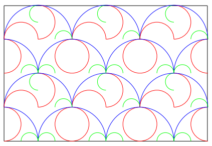

72. ABcC

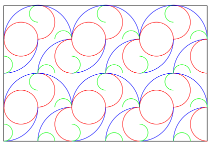

73. ABca

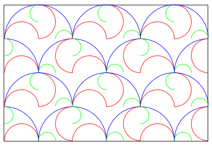

74. ABcb

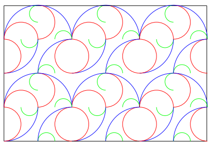

75. ABcd

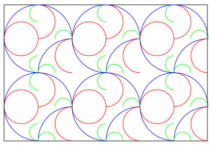

76.ABdA

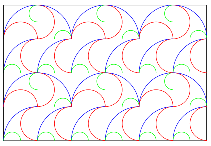

77. ABdB

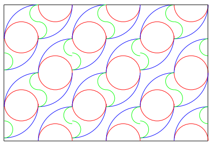

78. ABdC

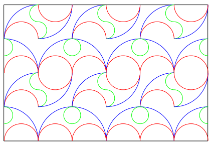

79. ABda

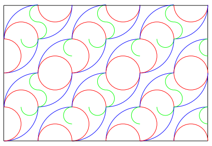

80. ABdb

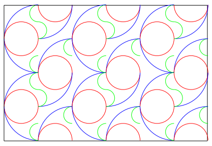

81. ABdc

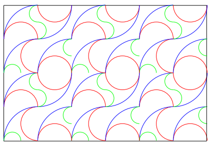

82. ABdd

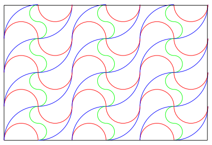

83. ACAC

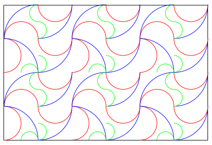

84. ACAa

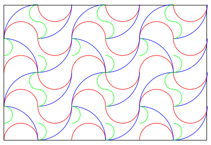

85. ACAb

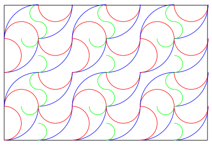

86. ACAc

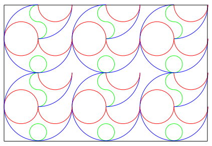

87. ACAd

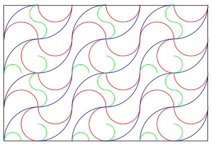

88. ACCA

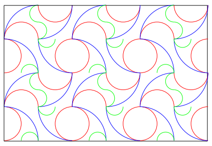

89. ACCB

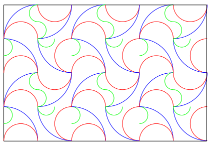

90. ACCa


91. ACCb


92. ACCc


93. ACCd


94. ACDB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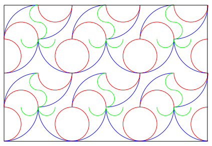
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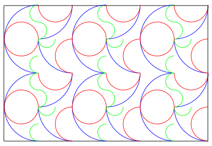
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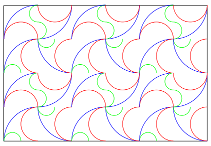
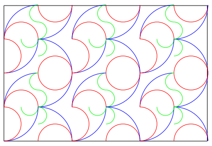
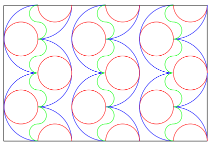
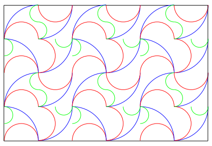
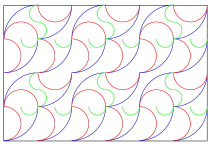
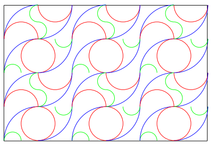

98. ACDd


99. ACab


100. ACac


101. ACba


102. ACbb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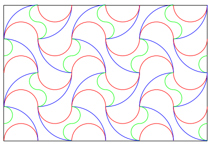
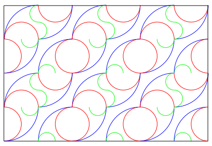
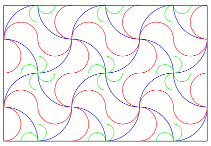
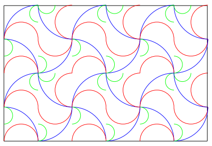
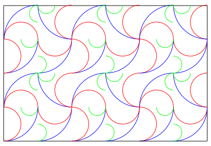
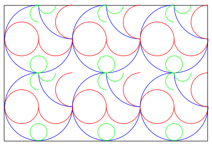

104. ACca


105. ACdb


106. ADCB


107. ADCa


108. ADCb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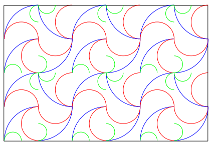
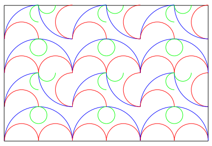
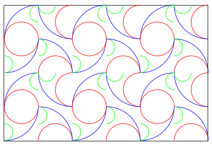
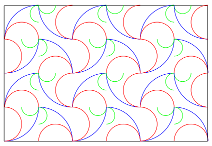
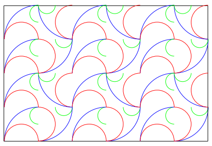
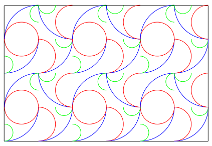

110. ADCd

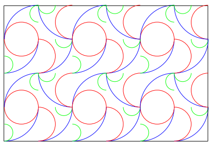

111. ADaA

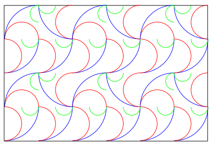

112. ADaC

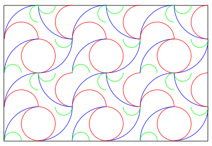

113. ADab

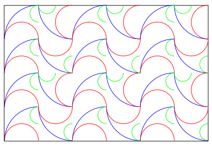

114. ADbA

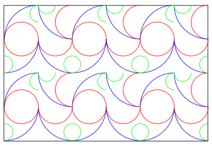

115. ADbC

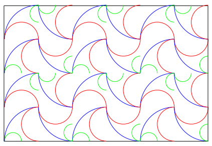

116. ADba

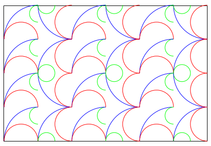

117. ADbb

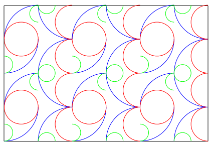

118. ADbd

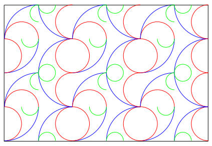

119. ADcA

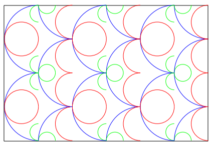

120. ADcC

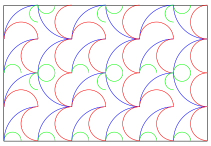

121. ADcd

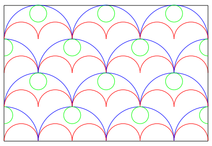

122. ADdA

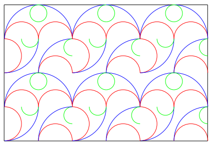

123. ADdC

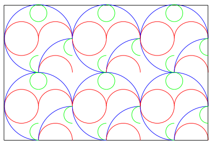

124. ADdb

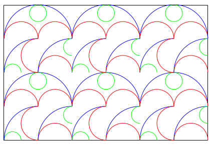

125. ADdc

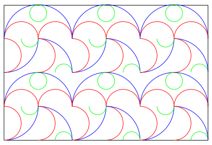

126. ADdd

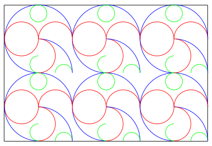

127. AaAa

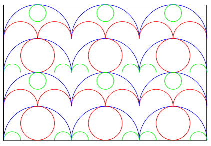

128. AaAb

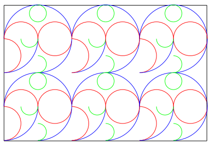

129. AaAc

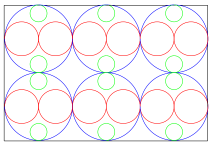

130. AaAd

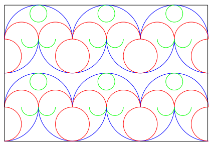

131. AaBb

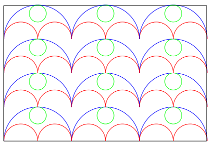

132. AaBc

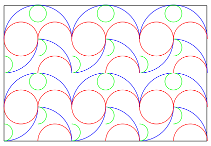

133. AaBd

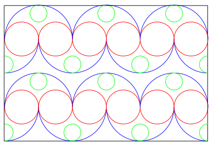

134. AaCb

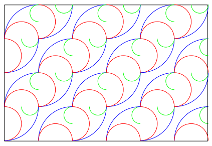

135. AaCc

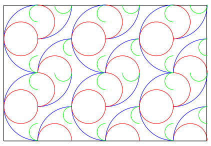

136. AaDb

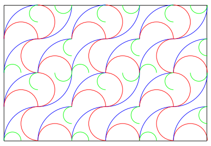

137. AaaA

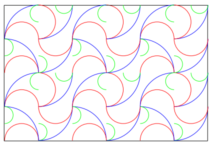

138. AaaC

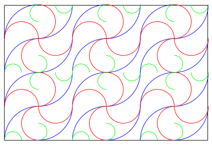

139. AacC

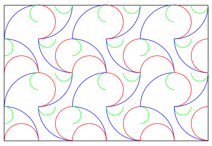

140. AbAb

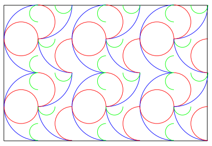

141. AbAc

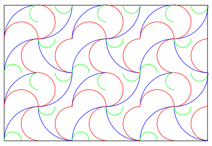

142. AbAd

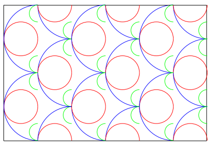

143. AbCa

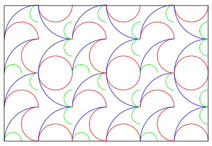

144. AbCd


145. AbDa


146. AbDc


147. AbDd


148. AcAc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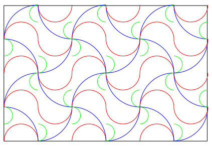
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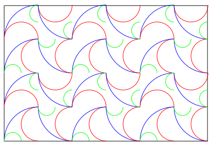
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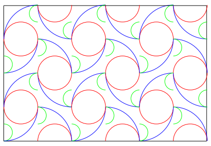
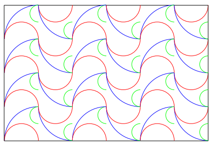
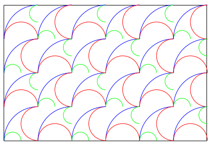

152. AcaC


153. AccA


154. AdAd