ASYMPTOTICS OF THE NUMBER OF k-WORDS WITH AN ℓ-DESCENT

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Submitted: December , 1997; Accepted: February 25, 1998

Abstract. The number of words $w = w_1 \cdots w_n$, $1 \le w_i \le k$, for which there are $1 \le i_1 < \cdots < i_\ell \le n$ and $w_{i_1} > \cdots > w_{i_\ell}$, is given, by the Schensted-Knuth correspondence, in terms of standard and semi-standard Young tableaux. When $n \to \infty$, the asymptotics of the number of such words is calculated.

*Work partially supported by N.S.F. Grant No. DMS-94-01197.

The Main Results.

Let $k, n > 0$ be integers and let $W(k;n) = \{w_1 \cdots w_n \mid 1 \le w_i \le k \text{ for all } 1 \le i \le n\}$ denote the set of words of length n on the alphabet $\{1, \cdots, k\}$. A word $w = w_1 \cdots w_n \in W(k,n)$ is said to have a descent of length ℓ if there exist indices $1 \le i_1 < \cdots < i_\ell \le n$ such that $w_{i_1} > \cdots > w_{i_\ell}$ (trivially, such words exist if and only if $\ell \le k$).

Let $W(k,\ell;n)$ denote the set of words in $W(k;n)$ having descent $\le \ell$, and denote $w(k,\ell;n) = |W(k,\ell;n)|$. Thus $W(k;n) = W(k,k;n)$, and $w(k,k;n) = k^n$.

Recall: given two sequences $\{a_n\}$ and $\{b_n\}$ of real numbers, we denote $a_n \underset{n\to\infty}{\simeq} b_n$ (or simply $a_n \simeq b_n$) if $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 1$.

The main result here is

Theorem 1. *Let $1 \le \ell \le k$, then*

$$w(k,\ell;n) \underset{n\to\infty}{\simeq} \frac{1!2!\cdots(\ell-1)!}{(k-\ell)!\cdots(k-1)!} \cdot \left(\frac{1}{\ell}\right)^{\ell(k-\ell)} \cdot n^{\ell(k-\ell)} \cdot \ell^n$$

Remark. $\frac{1!\cdots(\ell-1)!}{(k-\ell)!\cdots(k-1)!} = \left[\frac{k!!}{\ell!!(k-\ell)!!}\right]^{-1}$, where $m!! \overset{\text{def}}{=} 1!2!\cdots(m-1)!$

Standard and Semistandard Tableaux.

Let $\lambda \vdash n$ (i.e. λ is a partition of n). A tableau of shape λ, filled with $1, \cdots, n$, is standard if the numbers in it are increasing both in rows and in columns. Let d_λ denote the number of such tableaux. It is well known that $d_\lambda = \deg(\chi_\lambda)$, where χ_λ is the corresponding irreducible character of the symmetric group S_n.

A k-tableau of shape λ is a tableau filled with $1, \cdots, k$ possibly with repetitions; it is semi-standard if the numbers are weakly increasing in rows and strictly increasing in columns. Let $s_k(\lambda)$ denote the number of such k-tableaux. It is well known that $s_k(\lambda)$ is the degree of a corresponding irreducible character of $GL(k,\mathbb{C})$ (or of $SL(k,\mathbb{C})$).

The numbers $w(k,\ell;n)$ are given by

Theorem 2. *Let $\wedge_\ell(n) = \{(\lambda_1, \lambda_2, \cdots) \vdash n \mid \lambda_{\ell+1} = 0\}$. Then*

$$w(k,\ell;n) = \sum_{\lambda \in \wedge_\ell(n)} s_k(\lambda) \cdot d_\lambda\,.$$

Formulas for calculating d_λ's and $s_k(\lambda)$'s are well known. Here we shall need the following formula:

Let $\lambda = (\lambda_1, \lambda_2, \cdots)$. If $\lambda_{k+1} > 0$ then $s_k(\lambda) = 0$. Assume $\lambda_{k+1} = 0$. Then

$$s_k(\lambda) = [1!2!\cdots(k-1)!]^{-1} \cdot \prod_{1\le i<j\le k} (\lambda_i - \lambda_j + j - i) \qquad (*)$$

We turn now to the proofs of Theorems 1 and 2, starting with

The proof of Theorem 2:

Apply the Schensted-Knuth correspondence [K] to $w \in W(k;n) : w \to (P_\lambda, Q_\lambda)$, where P_λ and Q_λ are tableaux of same shape λ, Q_λ is standard and P_λ is k-semistandard. This gives a bijection

$$W(k;n) \leftrightarrow \{(P_\lambda, Q_\lambda) \mid \lambda \in \wedge_k(n),\ P_\lambda \text{ is } k\text{-semistandard},\ Q_\lambda \text{ is standard}\}.$$

Moreover, let $w \leftrightarrow (P_\lambda, Q_\lambda)$ under this correspondence, then w has a descent of length $\geq r$ if and only if $\lambda_r \gneqq 0$. It clearly follows that the Schensted-Knuth correspondence gives a bijection

$$W(k,\ell;n) \leftrightarrow \{(P_\lambda, Q_\lambda) \mid \lambda \in \wedge_\ell(n),\ P_\lambda \text{ is } k\text{-semistandard},\ Q_\lambda \text{ is standard}\}.$$

Hence

$$w(k,\ell;n) = \sum_{\lambda \in \Lambda_\ell(n)} s_k(\lambda) d_\lambda$$

Q.E.D.

Remark. Let $1 \leq \ell \leq k$ and let $\lambda \in \wedge_\ell(n)$, then it is easy to verify that $(*)$ implies that

$$s_k(\lambda) = a \cdot b \cdot c \qquad (**)$$

where $a = [(k-\ell)! \cdots (k-1)!]^{-1}$, $b = \prod_{1 \leq i \leq \ell} \left[\prod_{\ell+1 \leq j \leq k} (\lambda_i + j - i) \right]$ and $c = \prod_{1 \leq i < j \leq \ell} (\lambda_i - \lambda_j + j - i)$.

The Proof of Theorem 1.

Here the results of [C.R] are applied. Let $\lambda \in \wedge_\ell(n)$, $1 \leq \ell \leq k$, and write:

$$\lambda = (\lambda_1, \cdots, \lambda_\ell) = (\lambda_1, \cdots, \lambda_k), \text{ where } \lambda_{\ell+1} = \cdots = \lambda_k = 0.$$

Also write $\lambda_j = \frac{n}{\ell} + c_j\sqrt{n}$. By the notations of [C.R], the factors b and c of $(**)$ satisfy

$$b \approx \prod_{1 \leq i \leq \ell} \left(\frac{n}{\ell}\right)^{k-\ell} = \left(\frac{n}{\ell}\right)^{\ell(k-\ell)}$$

and

$$c \approx \left[\prod_{1 \leq i < j \leq \ell} (c_i - c_j) \right] (\sqrt{n})^{\frac{\ell(\ell-1)}{2}}.$$

Thus

$$s_k(\lambda) \approx [(k-\ell)! \cdots (k-1)! \ell^{\ell(k-\ell)}]^{-1} \cdot \left[\prod_{1 \le i < j \le \ell} (c_i - c_j) \right] \cdot n^{\ell(k-\ell) + \frac{\ell(\ell-1)}{4}} .$$

Apply now [C.R. Theorem 2] with $\beta = 1$ (also ℓ replacing k and $s_k(\lambda)$ replacing $f(\lambda)$):

$$\begin{aligned} w(k,\ell;n) &\underset{\text{Thm } 1}{=} \sum_{\lambda \in \Lambda_\ell(n)} s_k(\lambda) d_\lambda \simeq \\ &\simeq [(k-\ell)! \cdots (k-1)! \cdot \ell^{\ell(k-\ell)}]^{-1} \cdot \left(\frac{1}{\sqrt{2\pi}} \right)^{\ell-1} \cdot \ell^{\frac{1}{2}\ell^2} \cdot n^{\ell(k-\ell)} \cdot \ell^n \cdot I_\ell, \end{aligned} \tag{$***$}$$

where

$$I_\ell = \underset{\substack{x_1+\cdots+x_\ell=0 \\ x_1 \ge \cdots \ge x_\ell}}{\int \cdots \int} \left[\prod_{1 \le i < j \le \ell} (x_i - x_j) \right]^2 \exp\left(-\frac{\ell}{2} \sum_{j=1}^{\ell} x_j^2 \right) d^{(\ell-1)} x .$$

Special Case: Let $\ell = k$. Then $w(k,k;n) = k^n$. Cancelling k^n from both sides of $(***)$ implies that

$$I_k = [1!2! \cdots (k-1)!] \sqrt{2\pi}^{k-1} \cdot \left(\frac{1}{k} \right)^{\frac{1}{2}k^2}$$

(Note: by [R, §4] I_k can also be calculated by the Mehta-Selberg integral).

In particular,

$$I_\ell = [1!2! \cdots (\ell-1)!] \sqrt{2\pi}^{\ell-1} \cdot \left(\frac{1}{\ell} \right)^{\frac{1}{2}\ell^2} .$$

Substituting for I_ℓ in $(***)$ implies that

$$w(k,\ell;n) \simeq \frac{1!2! \cdots (\ell-1)!}{(k-\ell)! \cdots (k-1)!} \cdot \left(\frac{1}{\ell} \right)^{\ell(k-\ell)} \cdot n^{\ell(k-\ell)} \cdot \ell^n$$

which completes the proof of Theorem 1. Q.E.D.

Acknowledgement. I am thankful to Dr. H. Wilf for suggesting this problem.

References

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