A Multivariate Lagrange Inversion Formula for Asymptotic Calculations

Edward A. Bender
Department of Mathematics
University of California, San Diego
La Jolla, CA 92093-0112, USA
ebender@ucsd.edu

L. Bruce Richmond
Department of Combinatorics and Optimization
University of Waterloo
Waterloo, Ontario N2L 3G1, Canada
lbrichmond@watdragon.uwaterloo.ca

Submitted: March 3, 1998
Accepted: June 30, 1998

Abstract

The determinant that is present in traditional formulations of multivariate Lagrange inversion causes difficulties when one attempts to obtain asymptotic information. We obtain an alternate formulation as a sum of terms, thereby avoiding this difficulty.

1991 AMS Classification Number. Primary: 05A15 Secondary: 05C05, 40E99

1. Introduction

Many researchers have studied the Lagrange inversion formula, obtaining a variety of proofs and extensions. Gessel [4] has collected an extensive set of references. For more recent results see Haiman and Schmitt [6], Goulden and Kulkarni [5], and Section 3.1 of Bergeron, Labelle, and Leroux [3].

Let boldface letters denote vectors and let a vector to a vector power be the product of componentwise exponentiation as in $\mathbf{x}^{\mathbf{n}} = x_1^{n_1} \cdots x_d^{n_d}$. Let $[\mathbf{x}^{\mathbf{n}}]\, h(\mathbf{x})$ denote the coefficient of $\mathbf{x}^{\mathbf{n}}$ in $h(\mathbf{x})$. Let $\|a_{i,j}\|$ denote the determinant of the $d \times d$ matrix with entries $a_{i,j}$. A traditional formulation of multivariate Lagrange inversion is

Theorem 1. *Suppose that $g(\mathbf{x}),\ f_1(\mathbf{x}), \cdots, f_d(\mathbf{x})$ are formal power series in $\mathbf{x}$ such that $f_i(\mathbf{0}) \neq 0$ for $1 \le i \le d$. Then the set of equations $w_i = t_i f_i(\mathbf{w})$ for $1 \le i \le d$ uniquely determine the w_i as formal power series in $\mathbf{t}$ and*

$$[\mathbf{t}^{\mathbf{n}}]\, g(\mathbf{w}(\mathbf{t})) = [\mathbf{x}^{\mathbf{n}}] \left\{ g(\mathbf{x})\ \mathbf{f}(\mathbf{x})^{\mathbf{n}} \left\| \delta_{i,j} - \frac{x_i}{f_j(\mathbf{x})} \frac{\partial f_j(\mathbf{x})}{\partial x_i} \right\| \right\}, \tag{1}$$

where $\delta_{i,j}$ is the Kronecker delta.

If one attempts to use this formula to estimate $[\mathbf{t}^{\mathbf{n}}]\, g(\mathbf{w}(\mathbf{t}))$ by steepest descent or stationary phase, one finds that the determinant vanishes near the point where the integrand is maximized, and this can lead to difficulties as $\min(n_i) \to \infty$. We derive an alternate formulation of (1) which avoids this problem. In [2], we apply the result to asymptotic problems.

Let $\mathcal{D}$ be a directed graph with vertex set V and edge set E. Let the vectors $\mathbf{x}$ and $\mathbf{f}(\mathbf{x})$ be indexed by V. Define

$$\frac{\partial \mathbf{f}}{\partial \mathcal{D}} = \prod_{j \in V} \left\{ \left(\prod_{(i,j) \in E} \frac{\partial}{\partial x_i} \right) f_j(\mathbf{x}) \right\}.$$

We prove

Theorem 2. *Suppose that $g(\mathbf{x}),\ f_1(\mathbf{x}), \cdots, f_d(\mathbf{x})$ are formal power series in $\mathbf{x}$ such that $f_i(\mathbf{0}) \neq 0$ for $1 \le i \le d$. Then the set of equations $w_i = t_i f_i(\mathbf{w})$ for $1 \le i \le d$ uniquely determine the w_i as formal power series in $\mathbf{t}$ and*

$$[\mathbf{t}^{\mathbf{n}}]\, g(\mathbf{w}(\mathbf{t})) = \frac{1}{\prod n_i}\, [\mathbf{x}^{\mathbf{n}-\mathbf{1}}] \sum_{\mathcal{T}} \frac{\partial(g, f_1^{n_1}, \ldots, f_d^{n_d})}{\partial \mathcal{T}}, \tag{2}$$

where $\mathbf{1} = (1, \ldots, 1)$, the sum is over all trees $\mathcal{T}$ with $V = \{0, 1, \ldots, d\}$ and edges directed toward 0, and the vector in $\partial/\partial\mathcal{T}$ is indexed from 0 to d.

When $d = 1$, this reduces to the classical formula

$$[t^n]\, g(w(t)) = \frac{[x^{n-1}]\, g'(t) f(t)^n}{n}.$$

Derivatives with respect to trees have also appeared in Bass, Connell, and Wright [1].

2. Proof of Theorem 2

Expand the determinant $\|\delta_{i,j} - a_{i,j}\|$. For each subset S of $\{1,\ldots,d\}$ and each permutation π on S, select the entries $-a_{i,\pi(i)}$ for $i \in S$ and $\delta_{i,i}$ for $i \notin S$. The sign of the resulting term will be $(-1)^{|S|}$ times the sign of π. Since (i) the sign of π is -1 to the number of even cycles in π and (ii) $|S|$ has the same parity as the number of odd cycles in π, it follows that

$$\|\delta_{i,j} - a_{i,j}\| = \sum_{S,\pi} (-1)^{c(\pi)} \prod_{i\in S} a_{i,\pi(i)}, \tag{3}$$

where $c(\pi)$ is the number of cycles of π and the sum is over all S and π as described above. (When $S=\emptyset$, the product is 1 and $c(\pi)=0$.)

Applying (3) to (1) with $h_0 = g$, $h_1 = f_1^{n_1},\ldots,h_d = f_d^{n_d}$ and understanding that $S \subseteq \{1,\ldots,d\}$, we obtain

$$\begin{aligned}
&\left(\textstyle\prod n_i\right)[\mathbf{x}^{\mathbf{n}}]\, g(\mathbf{w}(\mathbf{t}))\\
&\quad= [\mathbf{x}^{\mathbf{n}}] \sum_{S,\pi} (-1)^{c(\pi)} \left\{ \prod_{\substack{i\notin S\\ i\neq 0}} n_i \times \prod_{i\notin S} h_i(\mathbf{x}) \times \prod_{i\in S} x_i n_i\, f_{\pi(i)}(\mathbf{x})^{n_i-1} \frac{\partial f_{\pi(i)}(\mathbf{x})}{\partial x_i} \right\}\\
&\quad= [\mathbf{x}^{\mathbf{n-1}}] \sum_{S,\pi} (-1)^{c(\pi)} \left\{ \prod_{\substack{i\notin S\\ i\neq 0}} \frac{n_i}{x_i} \times \prod_{i\notin S} h_i(\mathbf{x}) \times \prod_{i\in S} \frac{\partial h_{\pi(i)}(\mathbf{x})}{\partial x_i} \right\}\\
&\quad= [\mathbf{x}^{\mathbf{n-1}}] \sum_{S,\pi} (-1)^{c(\pi)} \left\{ \left(\prod_{\substack{i\notin S\\ i\neq 0}} \frac{\partial}{\partial x_i} \right) \left(\prod_{i\notin S} h_i(\mathbf{x}) \times \prod_{i\in S} \frac{\partial h_{\pi(i)}(\mathbf{x})}{\partial x_i} \right) \right\},
\end{aligned} \tag{4}$$

where, in the last line, the $\partial/\partial x_i$ operators replaced n_i/x_i because we are extracting the coefficient of $x_i^{n_i-1}$.

If we expand a particular S,π term in (4) by distributing the partial derivative operators, we obtain a sum of terms of the form

$$\prod_{j\in V} \left\{ \left(\prod_{(i,j)\in E} \frac{\partial}{\partial x_i} \right) h_j(\mathbf{x}) \right\},$$

where $V = \{0,1,\ldots,d\}$ and $E \subset V \times V$. Since each $\partial/\partial x_i$ appears exactly once per term, all vertices in the directed graph $\mathcal{D} = (V,E)$ have outdegree one, except for vertex 0 which has outdegree zero. Thus adding the edge $(0,0)$ to $\mathcal{D}$ gives a functional digraph. The cycles of π are among the cycles of $\mathcal{D}$, and, since the $\partial/\partial x_i$ for $i \notin S$ can be applied to any factor, the remaining edges are arbitrary. Hence

$$\left(\prod_{\substack{i\notin S\\ i\neq 0}} \frac{\partial}{\partial x_i} \right) \left(\prod_{i\notin S} h_i(\mathbf{x}) \times \prod_{i\in S} \frac{\partial h_{\pi(i)}(\mathbf{x})}{\partial x_i} \right) = \sum_{\mathcal{D}} \frac{\partial \mathbf{h}}{\partial \mathcal{D}},$$

where the sum ranges over all directed graphs $\mathcal{D}$ on $V = \{0, 1, \ldots, d\}$ such that (i) adjoining $(0,0)$ produces a functional digraph and (ii) the cycles of $\mathcal{D}$ include π. Denote condition (ii) by $\pi \subseteq \mathcal{D}$. We have shown that

$$\begin{aligned}(\textstyle\prod n_i)\ [\mathbf{x}^{\mathbf{n}}]\, g(\mathbf{w}(\mathbf{t})) &= [\mathbf{x}^{\mathbf{n}-\mathbf{1}}] \sum_{S,\pi} (-1)^{c(\pi)} \sum_{\mathcal{D}:\pi\subseteq\mathcal{D}} \frac{\partial \mathbf{h}}{\partial \mathcal{D}} \\ &= [\mathbf{x}^{\mathbf{n}-\mathbf{1}}] \sum_{\mathcal{D}} \sum_{\pi:\pi\subseteq\mathcal{D}} (-1)^{c(\pi)} \frac{\partial \mathbf{h}}{\partial \mathcal{D}}.\end{aligned}$$

Since $\sum_{\pi\subseteq\mathcal{D}} (-1)^{c(\pi)} = 0$ when $\mathcal{D}$ has cyclic points and is 1 otherwise, the sum reduces to a sum over acyclic directed graphs $\mathcal{D}$ such that adjoining $(0,0)$ gives a functional digraph. Since these are precisely the trees with edges directed toward 0, the proof is complete.

References

[1] H. Bass, E. H. Connell, and D. Wright, The Jacobian conjecture: Reduction of degree and formal expansion of the inverse, *Bull. Amer. Math. Soc. (N.S.)* **7** (1982) 287–330.

[2] E. A. Bender and L. B. Richmond, Asymptotics for multivariate Lagrange inversion, in preparation.

[3] F. Bergeron, G. Labelle, and P. Leroux (trans. by M. Readdy), *Combinatorial Species and Tree-Like Structures*, Encylopedia of Math. and Its Appl. Vol 67, Cambridge Univ. Press, 1998.

[4] I. M. Gessel, A combinatorial proof of the multivariate Lagrange inversion formula, *J. Combin. Theory Ser. A* **45** (1987) 178–195.

[5] I. P. Goulden and D. M. Kulkarni, Multivariable Lagrange invers, Gessel-Viennot cancellation and the Matrix Tree Theorem, *J. Combin. Theory Ser. A* **80** (1997) 295–308.

[6] M. Haiman and W. Schmitt, Incidence algebra antipodes and Lagrange inversion in one and several variables, *J. Combin. Theory Ser. A* **50** (1989) 172–185.