A Macdonald Vertex Operator and Standard Tableaux Statistics for the Two-Column (q,t)-Kostka Coefficients

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Abstract

The two parameter family of coefficients $K_{\lambda\mu}(q,t)$ introduced by Macdonald are conjectured to (q,t) count the standard tableaux of shape λ. If this conjecture is correct, then there exist statistics $a_\mu(T)$ and $b_\mu(T)$ such that the family of symmetric functions $H_\mu[X;q,t] = \sum_\lambda K_{\lambda\mu}(q,t)s_\lambda[X]$ are generating functions for the standard tableaux of size $|\mu|$ in the sense that $H_\mu[X;q,t] = \sum_T q^{a_\mu(T)}t^{b_\mu(T)}s_{\lambda(T)}[X]$ where the sum is over standard tableau of of size $|\mu|$. We give a formula for a symmetric function operator H_2^{qt} with the property that $H_2^{qt}H_{(2^a1^b)}[X;q,t] = H_{(2^{a+1}1^b)}[X;q,t]$. This operator has a combinatorial action on the Schur function basis. We use this Schur function action to show by induction that $H_{(2^a1^b)}[X;q,t]$ is the generating function for standard tableaux of size $2a+b$ (and hence that $K_{\lambda(2^a1^b)}(q,t)$ is a polynomial with non-negative integer coefficients). The inductive proof gives an algorithm for 'building' the standard tableaux of size $n+2$ from the standard tableaux of size n and divides the standard tableaux into classes that are generalizations of the catabolism type. We show that reversing this construction gives the statistics $a_\mu(T)$ and $b_\mu(T)$ when μ is of the form (2^a1^b) and that these statistics prove conjectures about the relationship between adjacent rows of the (q,t)-Kostka matrix that were suggested by Lynne Butler.

1 Introduction

The Macdonald basis for the symmetric functions generalizes many other bases by specializing the values of t and q. The symmetric function basis $\{P_\mu[X;q,t]\}_\mu$ is defined ([14] p. 321) as being self-orthogonal and having an upper triangularity condition with the monomial symmetric functions and the integral form of the basis is defined by setting $J_\mu[X;q,t] = P_\mu[X;q,t]h_\mu(q,t)$ for some q,t-polynomial coefficients $h_\mu(q,t)$. The $\{J_\mu[X;q,t]\}_\mu$ have the expansion

$$J_\mu[X;q,t] = \sum_\lambda K_{\lambda\mu}(q,t) S_\lambda[X;t]$$

where $S_\lambda[X;t]$ is the dual Schur basis. The coefficients $K_{\lambda\mu}(q,t)$ are referred to as the Macdonald (q,t)-Kostka coefficients. These coefficients are known to be polynomials and conjectured to have non-negative integer coefficients. It is known that $K_{\lambda\mu}(1,1) = K_\lambda$ and so it is conjectured that these coefficients (q,t) count the standard tableau of shape λ.

We are interested here in the basis

$$H_\mu[X;q,t] = \sum_\lambda K_{\lambda\mu}(q,t) s_\lambda[X]$$

It has the specializations that $H_\mu[X;0,t] = H_\mu[X;t]$ (the Hall-Littlewood basis of symmetric functions), $H_\mu[X;0,0] = s_\mu[X]$, $H_\mu[X;0,1] = h_\mu[X]$, and the property that $H_\mu[X;q,t] = q^{n(\mu')}t^{n(\mu)}\omega H_\mu[X;1/q,1/t]$ and $H_\mu[X;q,t] = \omega H_{\mu'}[X;t,q]$.

For each of the homogeneous, Schur, and Hall-Littlewood symmetric functions there are vertex operators with the property that for $m \geq \mu_1$ $h_m h_\mu[X] = h_{(m,\mu)}[X]$, $S_m s_\mu[X] = s_{(m,\mu)}[X]$, and $H_m^t H_\mu[X;t] = H_{(m,\mu)}[X;t]$ where (m,μ) represents the partition $(m, \mu_1, \mu_2, \ldots, \mu_k)$. These are each given by the following formulas:

$$i) \quad h_m = h_m[X] \tag{1.1}$$

$$ii) \quad S_m = \sum_{i \geq 0} (-1)^i h_{m+i}[X] e_i^{\perp} \tag{1.2}$$

$$iii) \quad H_m^t = \sum_{j \geq 0} t^j S_{m+j} h_j^{\perp} \tag{1.3}$$

The action of each of these operators on the Schur basis is known ([15]). It is hopeful that a similar vertex operator can be found for the $H_m[X;q,t]$ symmetric functions and the action on the Schur basis can be expressed easily.

Define H_m^{qt} to be "the" operator that has the property that $H_m^{qt} H_\mu[X;q,t] = H_{(m,\mu)}[X;q,t]$. This condition alone is not sufficient to define this operator uniquely, but it is sufficient to calculate the action on the Schur basis for certain partitions. Since the $\{H_\mu[X;q,t]\}_\mu$ is a basis for the symmetric functions, $s_\lambda[X] = \sum_\mu d_{\lambda\mu}(q,t) H_\mu[X;q,t]$, and for $m \geq |\lambda|$, H_m^{qt} may be calculated by the expression

$$H_m^{qt} s_\lambda[X] = \sum_\mu d_{\lambda\mu}(q,t) H_{(m,\mu)}[X;q,t]$$

These calculations are enough to inspire the following conjecture

Conjecture 1.1

$$H_m^{qt} = \sum_{T \in ST^m} q^{co(T)} H_m^T(t)$$

for some polynomial symmetric functions operators $H_m^T(t)$ that are only dependent on t with the following properties:

$$i) \quad H_m^T(1) = s_{\lambda(T)}[X]$$

$$ii) \quad H_m^{\omega T}(t) = \omega H_m^T(1/t) \omega R^t$$

$$iii) \quad H_m^{\boxed{1 \mid 2 \mid \;\; m}} = H_m^t$$

where T is a standard tableau of size m, $co(T)$ is the cocharge statistic on the tableau, $\lambda(T)$ is the shape of the tableau, H_m^t is the Hall-Littlewood vertex operator, ωT is the tableau flipped about the diagonal and R^t is a linear operator that acts on homogeneous symmetric functions $P[X]$ of degree n with the action $R^t P[X] = t^n P[X]$.

These vertex operators do not seem to be transformed versions of the vertex operators known for the $\{P_\mu[X;q,t]\}_\mu$ ([12], [7]).

In the case that $m = 2$, this conjecture completely determines the operator H_2^{qt} and the main result presented in the first section of this paper will be

Theorem 1.2 *The operator*

$$H_2^{qt} = H_2^t + q\omega H_2^{\frac{1}{t}} \omega R^t$$

has the property that $H_2^{qt} H_{(2^a 1^b)}[X;q,t] = H_{(2^{a+1} 1^b)}[X;q,t]$.

This theorem will follow from a formula by John Stembridge [13] that gives an expression for the Macdonald polynomial indexed by a shape with two columns in terms of Hall-Littlewood polynomials. Susanna Fischel [2] has already used this result to find statistics on rigged configurations that are known to be isomorphic to standard tableaux. It would be better to have these statistics directly for standard tableau since the bijection between standard tableau and rigged configurations is not trivial ([8], [9], [5]).

Our main purpose for finding the vertex operator H_m^{qt} and its action on the Schur function basis is to use it to discover statistics $a_\mu(T)$ and $b_\mu(T)$ on standard tableau so that $K_{\lambda\mu}(q,t) = \sum_{T \in ST^\lambda} q^{a_\mu(T)} t^{b_\mu(T)}$. If these statistics exist, then the family of symmetric functions $\{H_\mu[X;q,t]\}_\mu$ can be thought of as generating functions for the standard tableaux in the sense that $H_\mu[X;q,t] = \sum_{T \in ST^{|\mu|}} q^{a_\mu(T)} t^{b_\mu(T)} s_{\lambda(T)}[X]$.

The vertex operator property has the interpretation that H_m^{qt} changes the generating function for the standard tableaux of size n to the generating function for the standard tableaux of size $n+m$. Knowing the action of H_m^{qt} on the Schur function basis gives a description of how the shape of the tableau changes when a block of size m is added.

In the case of $m = 2$, the action of H_2^t (and $\omega H_2^{\frac{1}{t}} \omega R^t$ and hence H_2^{qt}) on the Schur function basis is well understood. The operator H_2^{qt} can be interpreted as instructions for building the standard tableaux of size $n+2$ from the standard tableaux of size n. The second section of this paper will define a tableaux operator and show how it can be used to build tableaux of larger content from smaller and state explicitly how cancellation of any negative terms in the expression $H_2^{qt} H_{(2^a 1^b)}[X; q, t] = H_{(2^{a+1} 1^b)}[X; q, t]$ occurs. This operator suggests that the standard tableaux are divided into subclasses of tableaux and that each subclass is represented by a piece of the expression for $H_{(2^a 1^b)}[X; q, t]$. The last section will be exposition of the statistics $a_\mu(T)$ and $b_\mu(T)$ and on the subclasses of tableaux.

1.1 Notation

A partition λ is a weakly decreasing sequence of non-negative integers with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq 0$. The length $l(\lambda)$ of the partition is the largest i such that $\lambda_i > 0$. The partition λ is a partition of n if $\lambda_1 + \lambda_2 + \cdots + \lambda_{l(\lambda)} = n$. We associate a partition with its diagram and often use the two interchangeably. We use the French convention and draw the largest part on the bottom of the diagram. One partition is contained in another, $\lambda \subseteq \mu$ if $\lambda_i \leq \mu_i$ for all i (the notation is to suggest that if the diagram for λ were placed over the diagram for μ that one would be contained in the other).

For every partition λ there is a corresponding conjugate partition denoted by λ' where $\lambda'_i =$ the number of cells in the i^{th} column of λ.

A skew partition is denoted by λ/μ, where it is assumed that $\mu \subseteq \lambda$, and represents the cells that are in λ but are not in μ. A skew partition λ/μ is said to be a horizontal strip if there is at most one cell in each column. Denote the class of horizontal strips of size k by $\mathcal{H}_k$ so that the notation $\lambda/\mu \in \mathcal{H}_k$ means that λ/μ is a horizontal strip with k cells. Similarly, the class of vertical strips (skew partitions with only one cell in each row) will be denoted by $\mathcal{V}_k$.

A useful statistic defined on compositions, μ, is $n(\mu) = \sum_i \mu_i(i-1)$.

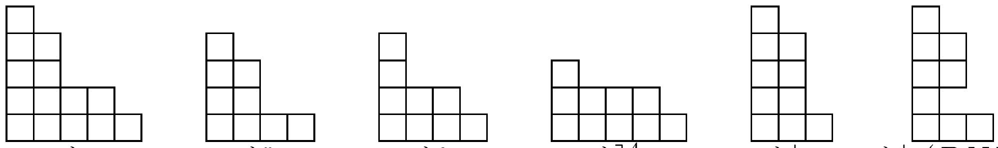
If λ is a partition, then let λ^r denote the partition with the first row removed, that is $\lambda^r = (\lambda_2, \lambda_3, \ldots, \lambda_{l(\lambda)})$. Let λ^c denote the partition with the first column removed, so that $\lambda^c = (\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_{l(\lambda)} - 1)$. This allows us to define the border of a partition μ to be the skew partition μ/μ^{rc}.

Define the k-snake of a partition μ to be the k bottom most right hand cells of the border of μ (the choice of the word "snake" is supposed to suggest the cells that slink with its belly on the ground from the bottom of the partition up along the right hand edge). We use the symbol $ht_k(\mu)$ to denote the height of the k-snake. The symbol $\mu\rfloor_k = (\mu_2 - 1, \mu_3 - 1, \ldots, \mu_h - 1, \mu_1 + h - k - 1, \mu_{h+1}, \ldots, \mu_{l(\lambda)})$ will be used to represent a partition with the k-snake removed with the understanding that if removing the k-snake does not leave a partition that this symbol is undefined.

Define the k-attic of a partition μ to be the top most left hand cells of the border of μ. The symbol $\bar{ht}_k(\mu)$ will represent the width of the k-attic ($\bar{ht}_k(\mu) = ht_k(\mu')$), and $\mu\rceil^k = \mu'\rfloor_k'$ will represent a partition with the k-attic removed with the understanding that if removing the k-attic does not leave a partition that this symbol is undefined.

Assume the convention that a Schur symmetric function indexed by a partition $\rho\rfloor_n$ or $\rho\rceil^n$ that does not exist is 0.

Example 1.3



λ λ^r λ^c $\lambda\rceil^4$ $\lambda\rfloor_4$ $\lambda\rfloor_5(DNE)$

If $\lambda = (5,4,2,2,1)$ *is the partition, then the* $\lambda^r = (4,2,2,1)$, $\lambda^c = (4,3,1,1)$, $\lambda\rceil^4 = (5,4,1)$, $\lambda\rfloor_4 = (3,2,2,2,1)$ *can all be calculated by drawing the diagram for* λ *and crossing off the appropriate cells. Note that in this example that* $\lambda\rfloor_5$ *does not exist.*

If the shape of $\rho = \lambda\rfloor_k$ is given and the height of the k-snake is specified then λ can be recovered (λ is determined from ρ by adding a k-snake of height h). This is because

$$\lambda = (\rho_h + k - h + 1, \rho_1 + 1, \rho_2 + 1, \dots, \rho_{h-1} + 1, \rho_{h+1}, \rho_{h+2}, \dots, \rho_{l(\rho)}) \tag{1.4}$$

and so λ will be a partition as long as k is sufficiently large.

A standard tableau is a diagram of a partition of n filled with the numbers 1 to n such that the labels increase moving from left to right in the rows and from bottom to top in the columns. The set of standard tableaux of size n will be denoted by ST^n.

We will consider the ring of symmetric functions in an infinite number of variables as a subring of $\mathbb{Q}[x_1, x_2, \dots]$. A more precise construction of this ring can be found in [14] section I.2.

We make use of plethystic notation for symmetric functions here. This is a notational device for expressing the substitution of the monomials of one expression, $E = E(t_1, t_2, t_3, \dots)$ for the variables of a symmetric function, P. The result will be denoted by $P[E]$ and represents the expression found by expanding P in terms of the power symmetric functions and then substituting for p_k the expression $E(t_1^k, t_2^k, t_3^k, \dots)$.

More precisely, if the power sum expansion of the symmetric function P is given by

$$P = \sum_\lambda c_\lambda p_\lambda$$

then the $P[E]$ is given by the formula

$$P[E] = \sum_{\lambda} c_\lambda p_\lambda \Big|_{p_k \to E(t_1^k, t_2^k, t_3^k, \ldots)}$$

.

To express a symmetric function in a single set of variables $x_1, x_2, \ldots, x_n$, let $X_n = x_1 + x_2 + \cdots + x_n$. The expression $P[X_n]$ represents the symmetric function P evaluated at the variables $x_1, x_2, \ldots, x_n$ since

$$P(x_1, x_2, \ldots, x_n) = \sum_{\lambda} c_\lambda p_\lambda \Big|_{p_k \to x_1^k + x_2^k + \cdots + x_n^k} = P[X_n]$$

The Cauchy kernel is a ubiquitous formula in the theory of symmetric functions (especially when working with plethystic notation).

Definition 1.4 *The Cauchy kernel*

$$\Omega[X] = \prod_i \frac{1}{1 - x_i}$$

It follows using plethystic notation that $\Omega[X]\Omega[Y] = \Omega[X+Y]$ and $\Omega[-X] = \prod_i (1 - x_i)$.

The Cauchy kernel evaluated at the product of two sets of variables has the formula ([14] p 63)

$$\Omega[XY] = \prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_{\lambda} s_\lambda[X] s_\lambda[Y] = \sum_{\lambda} h_\lambda[X] m_\lambda[Y]$$

We will use the notation that $f^\perp$ to denote the adjoint to multiplication for a symmetric function f with respect to the standard inner product. Therefore $\left\langle f^\perp g, h \right\rangle = \langle g, fh \rangle$. Note that $h_k^\perp$ and $e_k^\perp$ act on the Schur function basis with the formulas

$$e_k^\perp s_\mu = \sum_{\mu/\lambda \in \mathcal{V}_k} s_\lambda$$

$$h_k^{\perp} s_\mu = \sum_{\mu/\lambda \in \mathcal{H}_k} s_\lambda$$

The Macdonald basis [14] for the symmetric functions are defined by the following two conditions

$$a) \quad P_\lambda = s_\lambda + \sum_{\mu<\lambda} s_\mu c_{\mu\lambda}(q,t)$$

$$b)\, \langle P_\lambda, P_\mu \rangle_{qt} = 0 \ \ for \ \ \lambda \neq \mu$$

where $\langle , \rangle_{qt}$ denotes the scalar product of symmetric functions defined on the power symmetric functions by $\langle p_\lambda, p_\mu \rangle_{qt} = \delta_{\lambda\mu} z_\lambda p_\lambda \left[\frac{1-q}{1-t}\right]$ (z_λ is the size of the stablizer of the permuations of cycle structure λ and $\delta_{xy} = 1$ if $x = y$ and 0 otherwise). We will also refer to the basis $H_\mu[X;q,t] = \prod_{s\in\mu}(1 - q^{a_\mu(s)} t^{l_\mu(s)+1}) P_\mu \left[\frac{X}{1-t}; q, t\right] = \sum_\lambda K_{\lambda\mu}(q,t) s_\lambda[X]$ that is of interest in this paper as Macdonald symmetric functions ($s \in \mu$ means run over all cells s in μ and $a_\mu(s)$ and $l_\mu(s)$ are the arm and leg of s in μ respectively).

The Hall-Littlewood symmetric functions $H_\mu[X;t]$ can be defined by the following formula.

Definition 1.5 *The Hall-Littlewood symmetric function*

$$H_\mu[X;t] = \prod_{i\geq 0, 1\leq j\leq k} \frac{1}{1 - z_j x_i} \prod_{1\leq i\leq j\leq k} \frac{1 - z_j/z_i}{1 - t z_j/z_i}\Big|_{Z^\mu}$$

where μ is a partition with k parts and $\Big|_{Z^\mu}$ represents taking the coefficient of the monomial $z_1^{\mu_1} z_2^{\mu_2} \cdots z_k^{\mu_k}$.

These symmetric functions are not the same, but are related to the symmetric functions referred to as Hall-Littlewood polynomials in [14] p. 208. The Hall-Littlewood functions are related to the Schur symmetric functions by letting $t \to 0$ and to the homogeneous symmetric functions by letting $t \to 1$.

The Hall-Littlewood functions can be expanded in terms of the Schur symmetric function basis with coefficients $K_{\lambda\mu}(t)$, that is, $H_\mu[X;t] = \sum_\lambda K_{\lambda\mu}(t) s_\lambda[X]$. The $K_{\lambda\mu}(t)$ are well studied and referred to as the Kostka-Foulkes polynomials. The vertex operator, H_m^t in formula (1.3), that has $H_m^t H_\mu[X;t] = H_{(m,\mu)}[X;t]$ is due to Jing ([6], [4]). The Schur function vertex operator of equation (1.2) is due to Bernstein [16] (p. 69).

2 The Vertex Operator

Define the following symmetric function operator by the following equivalent formulas

Definition 2.1 *Let $P[X]$ be a homogeneous symmetric function of degree n.*

$$
\begin{aligned}
H_2^{qt} P[X] &= (H_2^t + q\omega H_2^{\frac{1}{t}} \omega R^t) P[X] && (2.1)\\
&= P\left[X - \frac{1-t}{z}\right] \Omega[zX] + qP\left[tX - \frac{1-t}{z}\right] \Omega[-zX]\Big|_{z^2} && (2.2)\\
&= \sum_{i \geq 0} (t^i S_{2+i} h_i^\perp + q t^{n-i} \omega S_{2+i} \omega e_i^\perp) P[X] && (2.3)\\
&= \sum_{i,j \geq 0} (t^j (-1)^i h_{2+i+j}[X] + q t^{n-i} (-1)^j e_{2+i+j}[X]) e_i^\perp h_j^\perp P[X] && (2.4)
\end{aligned}
$$

where the symbol $\Big|_{z^2}$ means take the coefficient of z^2 in the expression and R^t is an operator that has the property $R^t P[X] = t^n P[X]$.

For the remainder of this paper the symbol $H_2^{\boxed{\substack{2\\1}}}$ will represent the expression $\omega H_2^{\frac{1}{t}} \omega R^t$ and the symbol $H_2^{\boxed{1|2}}$ will represent the operator H_2^t so that $H_2^{qt} = H_2^{\boxed{1|2}} + qH_2^{\boxed{\substack{2\\1}}}$.

A formula for the (q,t) Kostka coefficients $K_{\lambda\mu}(q,t)$ when μ is a two column partition was given in [13]. That result will be used to prove that the H_2^{qt} operator has the vertex operator property. The proof first requires the following four lemmas:

Lemma 2.2

$$H_{(1^{b+2})}[X;t] = t^{b+1} H_{(21^b)}[X;t] + t^{\binom{b+1}{2}} \omega H_{(21^b)}[X;t^{-1}]$$

Proof There are combinatorial interpretations of each term of this equation and a bijective proof is easy enough to state. The left hand side of this equation is given by

$$H_{(1^{b+2})}[X;t] = \sum_{T \in ST^{b+2}} t^{c(T)} s_{\lambda(T)}[X]$$

Each term on the right hand side of the equation is given by the sums

$$t^{b+1} H_{(21^b)}[X;t] = \sum_{T \in CST^{(21^b)}} t^{c(T)+b+1} s_{\lambda(T)}[X]$$

$$t^{\binom{b+1}{2}} \omega H_{(21^b)}[X;t^{-1}] = \sum_{T \in CST^{(21^b)}} t^{\binom{b+1}{2} - c(T)} s_{\lambda(\omega T)}[X]$$

where ωT is the tableau that is flipped about the diagonal.

Each standard tableau has either the label of 2 lying to the immediate right of 1 or above it.

A tableau that has a 2 that lies immediately to the right of the 1 is isomorphic to a tableau that has content (21^b) and charge that is $b+1$ higher. The isomorphism simply decreases the label any cell with a label higher than 2 by 1 and the inverse is to increase the label of every cell except the 1 in the corner. The charge of the standard tableau is $b+1$ more than the charge of the corresponding tableau of content (21^b) because in the word definition of charge, the index of every letter (except the 1) of the word of the tableau decreases by 1 when the labels are decreased.

A tableau that has a label of 2 lying above the 1 can be transposed about the diagonal and this tableau is isomorphic to a tableau of content (21^b) by the same map. The charge of standard tableau is the cocharge of the transposed tableau so $c(T) = \binom{b+2}{2} - c(\omega T)$. The transformation that decreases the label in each cell by 1 (except the first cell) decreases the charge of the tableau by $b+1$ and so the charge of the tableau of content (21^b) is $\binom{b+2}{2} - (b+1) - c(T)$. $\square$

Lemma 2.3

$$H_2^{\substack{\boxed{2}\\\boxed{1}}} H_{(1^b)}[X;t] = H_{(1^{b+2})}[X;t] - t^{b+1} H_{(21^b)}[X;t]$$

Proof Note that for the Hall-Littlewood symmetric function indexed by the partition (1^b) we know from [14] p. 364 that $H_{(1^b)}[X;t] = (t;t)_n e_n \left[\frac{X}{1-t}\right]$. From this we derive

$$\begin{aligned}
H_{(1^b)}[X;t] &= (t;t)_b e_b \left[\frac{X}{1-t}\right] \\
&= (-1)^b t^{\binom{b+1}{2}} (t^{-1};t^{-1})_b e_b \left[-\frac{X}{(1-1/t)t}\right] \\
&= (-1)^{2b} t^{\binom{b+1}{2}-b} (t^{-1};t^{-1})_b h_b \left[\frac{X}{(1-1/t)}\right] \\
&= t^{\binom{b}{2}} \omega H_{(1^b)}[X;t^{-1}]
\end{aligned}$$

So that using the last lemma and the vertex operator property gives that

$$\begin{aligned}
\omega H_2^{\frac{1}{t}} \omega t^b H_{(1^b)}[X;t] &= t^{\binom{b}{2}+b} \omega H_2^{\frac{1}{t}} H_{(1^b)}[X;t^{-1}] \\
&= t^{\binom{b+1}{2}} \omega H_{(21^b)}[X;t^{-1}] \\
&= H_{(1^{b+2})}[X;t] - t^{b+1} H_{(21^b)}[X;t]
\end{aligned}$$

$\square$

Lemma 2.4

$$H_2^{\substack{\boxed{2}\\\boxed{1}}} H_2^{\boxed{1}\boxed{2}} = t H_2^{\boxed{1}\boxed{2}} H_2^{\substack{\boxed{2}\\\boxed{1}}}$$

Proof Let $H(z) = P\left[X - \frac{1-t}{z}\right] \Omega[zX]$ so that $H_2 = H(z)\Big|_{z^2} = t^2 H(z/t)\Big|_{z^2}$.

$$
\begin{aligned}
H_2^{\begin{smallmatrix}\boxed{2}\\ \boxed{1}\end{smallmatrix}} H_2^{\boxed{1}\boxed{2}} P[X] &= t^2 H_2^{\begin{smallmatrix}\boxed{2}\\ \boxed{1}\end{smallmatrix}} P\left[X - \frac{1-t}{(z/t)}\right] \Omega[(z/t)X]\Big|_{z^2} \\
&= t^2 P\left[tX - \frac{1-t}{u} - \frac{t-t^2}{z}\right] \Omega\left[(z/t)\left(tX - \frac{1-t}{u}\right)\right] \Omega\left[-uX\right]\Big|_{z^2u^2} \\
&= t^2 P\left[tX - \frac{1-t}{u} - \frac{t-t^2}{z}\right] \Omega[zX]\Omega[-uX]\Omega\left[\frac{z}{u} - \frac{z}{ut}\right]\Big|_{z^2u^2} \\
&= t^2 P\left[t\left(X - \frac{1-t}{z}\right) - \frac{1-t}{u}\right] \Omega[-uX]\Omega[zX] \frac{1-\frac{z}{ut}}{1-\frac{z}{u}}\Big|_{z^2u^2} \\
&= tP\left[t\left(X - \frac{1-t}{z}\right) - \frac{1-t}{u}\right] \Omega[-uX]\Omega[zX] \frac{1-\frac{tu}{z}}{1-\frac{u}{z}}\Big|_{z^2u^2} \\
&= tP\left[t\left(X - \frac{1-t}{z}\right) - \frac{1-t}{u}\right] \Omega[-uX]\Omega[zX]\Omega\left[u\frac{1-t}{z}\right]\Big|_{z^2u^2} \\
&= tH_2^{\boxed{1}\boxed{2}} P\left[tX - \frac{1-t}{u}\right] \Omega[-uX]\Big|_{u^2} \\
&= tH_2^{\boxed{1}\boxed{2}} H_2^{\begin{smallmatrix}\boxed{2}\\ \boxed{1}\end{smallmatrix}} P[X]
\end{aligned}
$$

□

Lemma 2.5

$$
H_2^{\begin{smallmatrix}\boxed{2}\\ \boxed{1}\end{smallmatrix}} H_{(2^a1^b)}[X;t] = t^a H_{(2^a1^{b+2})}[X;t] - t^{a+b+1} H_{(2^{a+1}1^b)}[X;t]
$$

Proof We show by induction on a that this is true. By the Lemma 2.3, the statement holds for $a = 0$ and by using the previous lemma we have that

$$
\begin{aligned}
H_2^{\begin{smallmatrix}\boxed{2}\\ \boxed{1}\end{smallmatrix}} H_{(2^a1^b)}[X;t] &= tH_2^{\boxed{1}\boxed{2}} H_2^{\begin{smallmatrix}\boxed{2}\\ \boxed{1}\end{smallmatrix}} H_{(2^{a-1}1^b)}[X;t] \\
&= tH_2^{\boxed{1}\boxed{2}} (t^{a-1} H_{(2^{a-1}1^{b+2})}[X;t] - t^{a+b} H_{(2^a1^b)}[X;t]) \\
&= t^a H_{(2^a1^{b+2})}[X;t] - t^{a+b+1} H_{(2^{a+1},1^b)}[X;t]
\end{aligned}
$$

$\square$

Note that there is a bijective proof of this identity that follows by rewriting the equation as

$$H_2^{\begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix}} H_{(2^a1^b)}[X;t] + t^{a+b+1} H_{(2^{a+1}1^b)}[X;t] = t^a H_{(2^a1^{b+2})}[X;t]$$

and realizing the combinatorial interpretation of each piece of this equation as the sum over tableaux. The combinatorial interpretation of the operator $H_2^{\begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix}}$ will be explained later and so an algebraic proof (except for the first lemma) is provided here instead.

Theorem 2.6

$$H_2^{qt} H_{(2^a1^b)}[X;q,t] = H_{(2^{a+1}1^b)}[X;q,t]$$

Proof For integers $n \geq 0$, define

$$(a;t)_n = (1-a)(1-at)\cdots(1-at^{n-1})$$

In Theorem 1.1 of [13], an expansion of the 2-column Macdonald polynomials in terms of the Hall-Littlewood polynomials is given as

$$H_{(2^a1^b)}[X;q,t] = \sum_{i=0}^{a} q^{a-i} (qt^{a+b};t^{-1})_i \frac{(t^a;t^{-1})_i}{(t^i;t^{-1})_i} H_{(2^i1^{b+2a-2i})}[X;t]$$

By Lemma 2.3, we have that

$$\begin{aligned} H_2^{qt} H_{(2^i1^{b+2a-2i})}[X;t] = (1-qt^{b+2a+1-i}) &H_{(2^{i+1}1^{b+2a-2i})}[X;t] \\ &+ qt^i H_{(2^i1^{b+2a-2i+2})}[X;t] \end{aligned}$$

So then using these two expressions we have that

$$
\begin{aligned}
H_2^{qt} H_{(2^a 1^b)}[X;q,t] &= \sum_{i=0}^{a} q^{a-i} (qt^{a+b};t^{-1})_i \frac{(t^a;t^{-1})_i}{(t^i;t^{-1})_i} H_2^{qt} H_{(2^i 1^{b+2a-2i})}[X;t] \\
&= \sum_{i=0}^{a} q^{a-i} (qt^{a+b};t^{-1})_i \frac{(t^a;t^{-1})_i}{(t^i;t^{-1})_i} \\
&\qquad ((1-qt^{b+2a+1-i}) H_{(2^{i+1} 1^{b+2a-2i})}[X;t] \\
&\qquad + qt^i H_{(2^i 1^{b+2a-2i+2})}[X;t])
\end{aligned}
$$

Algebraic manipulation and changing the index of the sums reduces this expression to one for the symmetric function $H_{(2^{a+1}1^b)}[X;q,t]$.

$$
= \sum_{i=0}^{a+1} q^{a-i+1} (qt^{a+b+1};t^{-1})_i \frac{(t^{a+1};t^{-1})_i}{(t^i;t^{-1})_i} H_{(2^i 1^{b+2a-2i+2})}[X;t] = H_{(2^{a+1}1^b)}[X;q,t]
$$

□

One result that follows from this theorem is that the $H_\mu[X;q,t]$ when $\mu = (2^a 1^b)$ has an unusual breakdown into 'atoms' as in the following formula.

Corollary 2.7

$$
H_{(2^a 1^b)}[X;q,t] = \sum_{s \in \left\{ \boxed{1}\boxed{2}, \begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix} \right\}^a} H_2^{s_1} H_2^{s_2} \cdots H_2^{s_a} H_{(1^b)}[X;t] q^{\sum_i co(s_i)}
$$

where $co(\boxed{1}\boxed{2}) = 0$ *and* $co\left(\begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix}\right) = 1$.

The interesting thing about this corollary is that the symmetric functions $H_2^{s_1} H_2^{s_2} \cdots H_2^{s_a} H_{(1^b)}[X;t]$ are each generating functions for a subset of the standard tableaux and are all Schur positive. This will be the main result of the next section and in the third section we will consider these as the atoms of the symmetric functions $H_{(2^a 1^b)}[X;q,t]$.

Because of the relation from Lemma 2.4, for $\sum_i co(s_i) = k$ we have that

$$H_2^{s_1} H_2^{s_2} \cdots H_2^{s_a} H_{(1^b)}[X;t] = t^x H_2^{\boxed{1}\boxed{2}} \cdots H_2^{\boxed{1}\boxed{2}} H_2^{\substack{\boxed{2}\\ \boxed{1}}} \cdots H_2^{\substack{\boxed{2}\\ \boxed{1}}} H_{(1^b)}[X;t] \tag{2.5}$$

for some $x \geq 0$ where the $H_2^{\boxed{1}\boxed{2}}$ occurs $a-k$ times and $H_2^{\substack{\boxed{2}\\ \boxed{1}}}$ occurs k times. In fact we may derive the following identity.

Corollary 2.8

$$H_{(2^a1^b)}[X;q,t] = \sum_{i=0}^{a} \begin{bmatrix} a \\ i \end{bmatrix}_t (H_2^{\boxed{1}\boxed{2}})^{a-i} (H_2^{\substack{\boxed{2}\\ \boxed{1}}})^i H_{(1^b)}[X;t] q^i$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_t = \frac{(t^n;t^{-1})_k}{(t^k;t^{-1})_k}$$

Proof Let $T(a) = \left\{\boxed{1}\boxed{2}, \substack{\boxed{2}\\ \boxed{1}}\right\}^a$, the set of tuples of length a with entries that are standard tableaux of size 2. For $s \in T(a)$, let $co(s) = \sum_i co(s_i)$ and let
$inv(s) = \sum_{1 \leq j < i \leq a} \chi\left(co\left(s_i\right) < co\left(s_j\right)\right)$.

The expression for $H_{(2^a1^b)}[X;q,t]$ from the previous corollary and relation in Lemma 2.4 gives that

$$\begin{aligned} H_{(2^a1^b)}[X;q,t] &= \sum_{s \in T(a)} H_2^{s_1} H_2^{s_2} \cdots H_2^{s_a} H_{(1^b)}[X;t] q^{co(s)} \\ &= \sum_{l=0}^{a} q^l \sum_{\substack{s \in T(a) \\ co(s)=l}} H_2^{s_1} H_2^{s_2} \cdots H_2^{s_a} H_{(1^b)}[X;t] \\ &= \sum_{l=0}^{a} q^l \sum_{\substack{s \in T(a) \\ co(s)=l}} t^{inv(s)} \left(H_2^{\boxed{1}\boxed{2}}\right)^{b+2a-l} \left(H_2^{\substack{\boxed{2}\\ \boxed{1}}}\right)^l H_{(1^b)}[X;t] \end{aligned}$$

Note that $\sum_{\substack{s\in T(n)\\ co(s)=k}} t^{inv(s)}$ satisfies the relations

$$\sum_{\substack{s\in T(n)\\ co(s)=k}} t^{inv(s)} = t^k \sum_{\substack{s\in T(n)\\ co(s)=k}} t^{inv(s)} + \sum_{\substack{s\in T(n)\\ co(s)=k-1}} t^{inv(s)}$$

and $\sum_{\substack{s\in T(n)\\ co(s)=n}} t^{inv(s)} = \sum_{\substack{s\in T(n)\\ co(s)=0}} t^{inv(s)} = 1$. The t binomial coefficient also satisfies the same recursion $\begin{bmatrix} n \\ k \end{bmatrix}_t = t^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_t + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_t$ and $\begin{bmatrix} n \\ n \end{bmatrix}_t = \begin{bmatrix} n \\ 0 \end{bmatrix}_t = 1$ therefore they have the same values. $\square$

In the next section we will give a combinatorial interpretation to these polynomials and show that when expanded in terms of Schur functions that the coefficients are polynomials with non-negative integer coefficients. The 'atoms' that the Macdonald polynomials break down into are related to the Butler conjectures of the $K_{\lambda\mu}(q,t)$. This relation will be made more precise in the last section with the exposition of the tableaux statistics.

The H_2^{qt} operator can be expressed in terms of the Hall-Littlewood vertex operator and the action of $H_2^{\boxed{1}\boxed{2}}$ on the Schur functions is known and given by the formula,

Proposition 2.9 *Let λ be a partition of n, for $k \geq 0$ then*

$$H_k^t s_\lambda[X] = \sum_{\rho\in\mathcal{H}_{n+k}} (-1)^{ht_n(\rho)-1} t^{|\lambda/\rho^r|} s_{\rho\rfloor_n}[X]$$

Because of this last proposition, the action of the H_2^{qt} on the Schur functions follows and can be stated as

Proposition 2.10 *Let λ be a partition of n then*

$$H_2^{qt} s_\lambda[X] = \sum_{\rho\in\mathcal{H}_{n+2}} (-1)^{ht_n(\rho)-1} t^{|\lambda/\rho^r|} s_{\rho\rfloor_n}[X] + q \sum_{\rho\in\mathcal{V}_{n+2}} (-1)^{\bar{h}t_n(\rho)-1} t^{|\rho^c|} s_{\rho\rceil^n}[X]$$

3 The Tableaux Operators

Define the class of *x-strict tableaux with n cells* (denoted by XST^n) to be the tableaux in the alphabet $\{i, i'\}_{i\geq 1}$ with the following restrictions:

- For each i, the tableau contains either no cells labeled by i or i', one cell labeled by i and none by i', two cells labeled by i and none by i', or two cells labeled by i' and none by i. No other combinations are allowed.

- The cell to the right of i can be labeled with an i or higher. The cell above i must be label larger than i.

- The cell to the right of i' must have a label strictly higher than i. The cell just above i' must have a label of i' or higher.

Define the content of $T \in XST^n$ to be the tuple $s \in \left\{\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}, \boxed{1}, \cdot\right\}^k$ such that $s_i = \begin{array}{|c|c|}\hline 1&2\\\hline\end{array}$ if T contains two i, $s_i = \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}$ if T contains two i', $s_i = \boxed{1}$ if T contains just one cell labeled by i and finally $s_i = \cdot$ if it does not contain i or i'. Denote the content of the tableau T by the symbol $\mu(T)$. The standard tableaux of size n are the set of tableaux $T \in XST^n$ such that $\mu(T) = (\boxed{1}^n)$.

Example

$$T = \begin{array}{|c|c|c|c|}\cline{1-1} 6' \\ \cline{1-2} 4 & 6' \\ \cline{1-3} 2' & 3 & 3 \\ \hline 1 & 1 & 2' & 5 \\ \hline\end{array}$$

$T \in XST^{10}$ and $\mu(T) = \left(\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}, \begin{array}{|c|c|}\hline 1&2\\\hline\end{array}, \boxed{1}, \boxed{1}, \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}\right)$

Let the operation V act on $T \in XST^n$ such that $\mu(T) = (\boxed{1}, \boxed{1}, s)$. The operator V changes T to a tableau of either type $(\boxed{1}\boxed{2}, s)$ or of type $\left(\begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}, s\right)$ depending on if the label 2 lies to the right or above the 1 respectively by changing the cells labeled by 1 and 2 to either 1s or $1'$s and decreasing the labels of each of the cells labeled with a 3 or higher by 1. V^{-1} will be the operator that acts on $T \in XST^n$ with $\mu(T)_1 = \boxed{1}\boxed{2}$ or $\begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}$ that is the reverse of the operator V.

The game of Jeu-de-Taquin may be played on these tableau with the consideration that the cells have the ordering that the cell with a label i' that lies above the other i' has a value of $i + \frac{1}{2}$. This same consideration on the ordering of the cells allows us to define row and column insertion and deletion using the usual Robinson-Schenstead correspondence.

Define a symmetric group action on the type of the tableau. For $T \in XST$, let $s = \mu(T)$. The operator $(i, i+1)$ will have the property that $\mu((i, i+1)T) = (i, i+1)s = (s_1, \ldots, s_{i+1}, s_i, \ldots, s_k)$. The operation $(i, i+1)T$ has the following definition:

- If $s_i = s_{i+1}$ then $(i, i+1)T = T$.
- If $s_i \neq s_{i+1}$ then ignore all the cells in T except those with a label in the set $\{i, i', i+1, i+1'\}$ and bring them to straight shape. The possible configurations of these cells are listed below in pairs. The action of $(i, i+1)$ is to replace the configuration by the corresponding one in the same row and then play Jeu-de-Taquin in reverse to restore the cells to their original position (see [3] to justify that this is a well defined operation).

Let $k = i + 1$ (just so that in the following diagrams, $i+1$ fits in the cell)

$$\boxed{i} \qquad \longleftrightarrow \qquad \boxed{k}$$

$$\begin{array}{|c|}\hline i'\\\hline i'\\\hline\end{array} \qquad \longleftrightarrow \qquad \begin{array}{|c|}\hline k'\\\hline k'\\\hline\end{array}$$

$$\begin{array}{|c|c|}\hline i & i \\ \hline\end{array} \quad\longleftrightarrow\quad \begin{array}{|c|c|}\hline k & k \\ \hline\end{array}$$

$$\begin{array}{|c|c}\cline{1-1} k & \\ \hline i & \multicolumn{1}{c|}{k} \\ \hline\end{array} \quad\longleftrightarrow\quad \begin{array}{|c|c}\cline{1-1} k & \\ \hline i & \multicolumn{1}{c|}{i} \\ \hline\end{array}$$

$$\begin{array}{|c|c|c|}\hline i & k & k \\ \hline\end{array} \quad\longleftrightarrow\quad \begin{array}{|c|c|c|}\hline i & i & k \\ \hline\end{array}$$

$$\begin{array}{|c|}\hline k' \\ \hline k' \\ \hline i \\ \hline\end{array} \quad\longleftrightarrow\quad \begin{array}{|c|}\hline k \\ \hline i' \\ \hline i' \\ \hline\end{array}$$

$$\begin{array}{|c|c}\cline{1-1} k' & \\ \hline i & \multicolumn{1}{c|}{k'} \\ \hline\end{array} \quad\longleftrightarrow\quad \begin{array}{|c|c}\cline{1-1} i' & \\ \hline i' & \multicolumn{1}{c|}{k} \\ \hline\end{array}$$

$$\begin{array}{|c|c}\cline{1-1} k' & \\ \cline{1-1} k' & \\ \hline i & \multicolumn{1}{c|}{i} \\ \hline\end{array} \quad\longleftrightarrow\quad \begin{array}{|c|c}\cline{1-1} k & \\ \cline{1-1} i' & \\ \hline i' & \multicolumn{1}{c|}{k} \\ \hline\end{array}$$

$$\begin{array}{|c|cc}\cline{1-1} k' & & \\ \hline i & \multicolumn{1}{|c|}{i} & \multicolumn{1}{c|}{k'} \\ \hline\end{array} \quad\longleftrightarrow\quad \begin{array}{|c|cc}\cline{1-1} i' & & \\ \hline i' & \multicolumn{1}{|c|}{k} & \multicolumn{1}{c|}{k} \\ \hline\end{array}$$

Define an operator $\mathbf{H}_2^{-1}$ on x-standard tableaux of content $\mu(T) = \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}, s\right)$ or $\mu(T) = \left(\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array}, s\right)$ and transforms it into an x-standard tableau of content (s) by the following procedure:

1. If $\mu(T) = \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}, s\right)$ then let R_1 be the first row of T and $\tilde{T}$ be T with the first row removed. Row insert the cells of R_1 that are not 1 into $\tilde{T}$ from largest to smallest and decrease each label by 1 in this new tableau. The result will be $\mathbf{H}_2^{-1}T$.

2. If $\mu(T) = \left(\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array}, s\right)$ then let C_1 be the first column of T and let $\tilde{T}$ be T with the first column removed. Column insert the cells of C_1 that are not 1 or $1'$ into $\tilde{T}$ from largest to smallest and decrease by 1 each of the labels of the cells in this new tableau. The result will be $\mathbf{H}_2^{-1}T$.

Clearly, if $\mu(T) = \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}, s\right)$ or $\left(\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array}, s\right)$ then $\mu(\mathbf{H}_2^{-1}T) = (s)$.

This operator will be used to define the *type* of a standard tableau. Let $\mu = (2^a 1^b)$. Let T be a standard tableau of size $2a + b$. The $\mu - type$ will represent the orientation of the "building blocks" of the standard tableau. It will be represented by the symbol $type_\mu(T)$ and be defined as the tuple of standard tableaux of size 1 or 2 with the following properties:

- If $a = 0$ and $\mu = (1^b)$ then $type_\mu(T) = (\boxed{1}^b)$.
- If $a = 1$ then $\mu(VT)_1 = \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}$ or $\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}$ and $type_{(21^b)}(T) = (\mu(VT)_1, \boxed{1}^b)$.
- If $a > 1$ then $\mu(VT)_1 = \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}$ or $\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}$ and
  $type_{(2^a1^b)}(T) = (\mu(VT)_1, type_{(2^{a-1}1^b)}(\mathbf{H}_2^{-1}VT))$

We wish to show that there is a relation between the $\mu - type$ of a standard tableau and a method for unstandardization of the tableau so that the content matches the $\mu - type$.

Lemma 3.1 *For a* $T \in XST^n$ $(n \geq 4)$ *and* $\mu(T) = \left(\begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}, \boxed{1}, \boxed{1}, s\right)$ *or* $\mu(T) = (\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}, \boxed{1}, \boxed{1}, s)$ *(where s is the remainder of the type-list) the tableaux operators have the following relationship*

$$V\mathbf{H}_2^{-1}T = \mathbf{H}_2^{-1}(1,2)V(2,3)(1,2)T$$

Proof The V and $(1,2)V(2,3)(1,2)$ operators only change the values of the cells that are labeled with $1, 1', 2$, or 3. The relative values of the cells of T do not change so it should be clear that if we verify this is true for the 10 tableaux below that it will be true for all x-standard tableaux that contain these as sub-tableaux.

$$\begin{array}{|c|c|c|c|}\hline 1&1&2&3\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 2&3\\\hline 1&1\\\hline\end{array} \quad \begin{array}{|c|cc}\hline 3&&\\\hline 1&\multicolumn{1}{|c|}{1}&\multicolumn{1}{c|}{2}\\\hline\end{array} \quad \begin{array}{|c|cc}\hline 1'&&\\\hline 1'&\multicolumn{1}{|c|}{2}&\multicolumn{1}{c|}{3}\\\hline\end{array} \quad \begin{array}{|c|c}\hline 3&\\\hline 1'&\\\hline 1'&\multicolumn{1}{|c|}{2}\\\hline\end{array}$$

$$\begin{array}{|c|c}\hline 3&\\\hline 2&\\\hline 1&\multicolumn{1}{|c|}{1}\\\hline\end{array} \quad \begin{array}{|c|cc}\hline 2&&\\\hline 1&\multicolumn{1}{|c|}{1}&\multicolumn{1}{c|}{3}\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 1'&3\\\hline 1'&2\\\hline\end{array} \quad \begin{array}{|c|c}\hline 2&\\\hline 1'&\\\hline 1'&\multicolumn{1}{|c|}{3}\\\hline\end{array} \quad \begin{array}{|c|}\hline 3\\\hline 2\\\hline 1'\\\hline 1'\\\hline\end{array}$$

If T is any of the first 5 tableaux then $V\mathbf{H}_2^{-1}T = \mathbf{H}_2^{-1}(1,2)V(2,3)(1,2)T = \boxed{1}\boxed{1}$ and if T is any of the second 5 tableaux then $V\mathbf{H}_2^{-1}T = \mathbf{H}_2^{-1}(1,2)V(2,3)(1,2)T = \begin{array}{|c|}\hline 1' \\ \hline 1' \\ \hline\end{array}$. $\square$

Lemma 3.2 *For a $T \in XST^n$ and for $i > 1$ the tableaux operators have the following relationship*

$$(i-1,i)\mathbf{H}_2^{-1}T = \mathbf{H}_2^{-1}(i,i+1)T$$

Proof As in the previous lemma, it is only necessary to check what these two operators do to the cells that they change. This means that there is nothing to check if $\mu(T)_i = \mu(T)_{i+1}$. A brute force proof this time however has MANY more cases to check. For each of the possible 18 configurations in (1) an application of $\mathbf{H}_2^{-1}$ will rearrange the positions of cells labeled by i and $i+1$ if $\mu(T)_1 = \boxed{1}\boxed{2}$ and there are cells labeled by i or $i+1$ in the first row of T, or $\mu(T)_1 = \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array}$ and there are cells labeled by i,i',$i+1$, or $i+1'$ in the first column of T.

Take one of the 18 configurations of the cells i,i',$i+1$, or $i+1'$ in T from (1), we we will write the possible configurations of the cells that are labeled by $i-1$,$i-1'$, i and i' after an application of $\mathbf{H}_2^{-1}$ to T. We need only verify that the images of these possible configurations under $(i-1,i)$ are the same as the possible configurations of the cells that are labeled by $i-1$,$i-1'$, i and i' after an application of $\mathbf{H}_2^{-1}$ to $(i,i+1)T$.

For notational purposes, let $k = i+1$ and $h = i-1$. Cells that were in the first row of T and change position because $\mu(T)_1 = \boxed{1}\boxed{2}$ will have an underline under the label, cells that were in the first column of T and change position because $\mu(T)_1 = \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array}$ will have a bar over the label.

$\boxed{i}$ $\qquad\qquad$ $\boxed{h}, \boxed{\underline{h}}, \boxed{\bar{h}}$

$\boxed{k}$ $\qquad\qquad$ $\boxed{i}, \boxed{\underline{i}}, \boxed{\bar{i}}$

$$\begin{array}{c} i' \\ i' \end{array} \qquad \begin{array}{c} h' \\ h' \end{array}, \begin{array}{c} \underline{h}' \\ h' \end{array}, \begin{array}{c} h' \\ \bar{h}' \end{array}, \begin{array}{c} \bar{h}' \\ \bar{h}' \end{array}$$

$$\begin{array}{c} k' \\ k' \end{array} \qquad \begin{array}{c} i' \\ i' \end{array}, \begin{array}{c} \underline{i}' \\ i' \end{array}, \begin{array}{c} i' \\ \bar{i}' \end{array}, \begin{array}{c} \bar{i}' \\ \bar{i}' \end{array}$$

$$\begin{array}{cc} i & i \end{array} \qquad \begin{array}{cc} h & h \end{array}, \begin{array}{cc} \underline{h} & h \end{array}, \begin{array}{cc} \underline{h} & \underline{h} \end{array}, \begin{array}{cc} h & \bar{h} \end{array}$$

$$\begin{array}{cc} k & k \end{array} \qquad \begin{array}{cc} i & i \end{array}, \begin{array}{cc} \underline{i} & i \end{array}, \begin{array}{cc} \underline{i} & \underline{i} \end{array}, \begin{array}{cc} i & \bar{i} \end{array}$$

$$\begin{array}{cc} k & \\ i & k \end{array} \qquad \begin{array}{cc} i & \\ h & i \end{array}, \begin{array}{cc} \underline{i} & \\ h & i \end{array}, \begin{array}{ccc} \underline{h} & \underline{i} & i \end{array}, \begin{array}{ccc} h & i & \bar{i} \end{array}, \begin{array}{cc} i & \\ \bar{h} & \bar{i} \end{array}$$

$$\begin{array}{cc} k & \\ i & i \end{array} \qquad \begin{array}{cc} i & \\ h & h \end{array}, \begin{array}{cc} i & \\ \underline{h} & h \end{array}, \begin{array}{ccc} \underline{h} & \underline{h} & i \end{array}, \begin{array}{ccc} h & h & \bar{i} \end{array}, \begin{array}{cc} \bar{i} & \\ h & \bar{h} \end{array}$$

$$\begin{array}{ccc} i & k & k \end{array} \qquad \begin{array}{ccc} h & i & i \end{array}, \begin{array}{cc} \underline{i} & \\ h & i \end{array}, \begin{array}{cc} \underline{i} & \\ h & \underline{i} \end{array}, \begin{array}{ccc} \underline{h} & \underline{i} & \underline{i} \end{array}, \begin{array}{cc} k & \\ \bar{i} & k \end{array}$$

$$\begin{array}{ccc} i & i & k \end{array} \qquad \begin{array}{ccc} h & h & i \end{array}, \begin{array}{cc} k & \\ i & \underline{i} \end{array}, \begin{array}{cc} \underline{k} & \\ \underline{i} & i \end{array}, \begin{array}{ccc} \underline{i} & \underline{i} & \underline{k} \end{array}, \begin{array}{cc} k & \\ i & \bar{i} \end{array}$$

$$\begin{array}{c} k' \\ k' \\ i \end{array} \qquad \begin{array}{c} i' \\ i' \\ h \end{array}, \begin{array}{cc} i' & \\ \underline{h} & i' \end{array}, \begin{array}{cc} i' & \\ h & \bar{i}' \end{array}, \begin{array}{cc} \bar{i}' & \\ h & \bar{i}' \end{array}, \begin{array}{c} \bar{i}' \\ \bar{i}' \\ \bar{h} \end{array}$$

$$\begin{array}{c} k \\ i' \\ i' \end{array} \qquad \begin{array}{c} i \\ h' \\ h' \end{array}, \begin{array}{cc} \underline{h}' & \\ h' & i \end{array}, \begin{array}{cc} h' & \\ h' & \bar{i} \end{array}, \begin{array}{cc} h' & \\ \bar{h}' & \bar{i} \end{array}, \begin{array}{c} \bar{i} \\ \bar{h}' \\ \bar{h}' \end{array}$$

$$\begin{array}{cc} k' & \\ i & k' \end{array} \qquad \begin{array}{cc} i' & \\ h & i' \end{array}, \begin{array}{c} \underline{i}' \\ i' \\ h \end{array}, \begin{array}{cc} \underline{i}' & \\ \underline{h} & i' \end{array}, \begin{array}{cc} i' & \\ h & \bar{i}' \end{array}, \begin{array}{c} i' \\ \bar{i}' \\ \bar{h} \end{array}$$

$$\begin{array}{cc} i' & \\ i' & k \end{array} \qquad \begin{array}{cc} h' & \\ h' & i \end{array}, \begin{array}{c} \underline{i} \\ h' \\ h' \end{array}, \begin{array}{cc} \underline{h}' & \\ h' & i \end{array}, \begin{array}{cc} h' & \\ \bar{h}' & i \end{array}, \begin{array}{c} i \\ \bar{h}' \\ \bar{h}' \end{array}$$

$$\begin{array}{ll} k' & \\ k' & \\ i & i \end{array} \qquad \begin{array}{ll} i' & \\ i' & \\ h & h \end{array}, \begin{array}{ll} i' & \\ i' & \\ \underline{h} & h \end{array}, \begin{array}{ll} i' & \\ i' & \\ \underline{h} & \underline{h} \end{array}, \begin{array}{lll} i' & & \\ h & h & \bar{i}' \end{array}, \begin{array}{lll} \bar{i}' & & \\ h & h & \bar{i}' \end{array}, \begin{array}{ll} \bar{i}' & \\ \bar{i}' & \\ h & \bar{h} \end{array}$$

$$\begin{array}{ll} k & \\ i' & \\ i' & k \end{array} \qquad \begin{array}{ll} i & \\ h' & \\ h' & i \end{array}, \begin{array}{ll} \underline{i} & \\ h' & \\ h' & i \end{array}, \begin{array}{ll} \underline{i} & \\ h' & \\ \underline{h'} & i \end{array}, \begin{array}{lll} h' & & \\ h' & i & \bar{i} \end{array}, \begin{array}{lll} h' & & \\ \bar{h}' & i & \bar{i} \end{array}, \begin{array}{ll} i & \\ \bar{h}' & \\ \bar{h}' & \bar{i} \end{array}$$

$$\begin{array}{lll} k' & & \\ i & i & k' \end{array} \qquad \begin{array}{lll} i' & & \\ h & h & i' \end{array}, \begin{array}{ll} \underline{i'} & \\ i' & \\ h & h \end{array}, \begin{array}{ll} \underline{i'} & \\ i' & \\ \underline{h} & h \end{array}, \begin{array}{lll} \underline{i'} & & \\ \underline{h} & \underline{h} & i' \end{array}, \begin{array}{lll} i' & & \\ h & h & \bar{i}' \end{array}, \begin{array}{ll} i' & \\ \bar{i}' & \\ h & \bar{h} \end{array}$$

$$\begin{array}{lll} i' & & \\ i' & k & k \end{array} \qquad \begin{array}{lll} h' & & \\ h' & i & i \end{array}, \begin{array}{ll} \underline{i} & \\ h' & \\ h' & i \end{array}, \begin{array}{ll} \underline{i} & \\ h' & \\ h' & \underline{i} \end{array}, \begin{array}{lll} \underline{h'} & & \\ h' & \underline{i} & \underline{i} \end{array}, \begin{array}{lll} h' & & \\ \bar{h}' & i & i \end{array}, \begin{array}{ll} i & \\ \bar{h}' & \\ \bar{h}' & i \end{array} \qquad \square$$

Define the tableau operator $M_1 = V$ and M_i for $i \geq 2$ by the composition of the $\sigma_i = (i, i+1)$ operators and the V operators
$M_i = \sigma_{i-1}\sigma_{i-2}\cdots\sigma_1 V \sigma_2\sigma_3\cdots\sigma_i\sigma_1\sigma_2\cdots\sigma_{i-1}$ Notice that M_i is simply defined so that it has the property

$$M_i \mathbf{H}_2^{-1} T = \mathbf{H}_2^{-1} M_{i+1} T$$

These M_i operators are "unstandardization" operators in the sense of the following proposition.

Proposition 3.3 *Let $\mu = (2^a 1^b)$ and T is a standard tableau of size $2a+b$ then*

$$\mu(M_a M_{a-1} \cdots M_1 T) = type_{(2^a 1^b)}(T)$$

Proof Note that the $type_{(2^a1^b)}(T)_i = \mu(V(\mathbf{H}_2^{-1}V)^i T)_1$. We observe that $\mu(\mathbf{H}_2^{-1}T)_j = \mu(T)_{j+1}$ and hence $type_{(2^a1^b)}(T)_i = \mu((\mathbf{H}_2^{-1})^i M_i M_{i-1}\cdots M_1 T)_1 = \mu(M_i M_{i-1}\cdots M_1 T)_i = \mu(M_a M_{a-1}\cdots M_1 T)_i$ (and the last equality follows since the M_j for $j > i$ does not change the i^{th} entry in the content).

$\square$

Define the operator N_a to be the sequence of operators $M_a M_{a-1} \cdots M_1$. When N_a acts on a standard tableau, it maps it to an x-standard tableaux with the relation $\mu(N_a T) = type_{(2^a 1^b)}(T)$ for $T \in ST^{2a+b}$. This operator is a bijection between standard tableaux and x-standard tableaux with content that is a tuple in $\left\{ \boxed{1}\boxed{2}, \begin{array}{|c|}\hline 2\\ \hline 1\\ \hline\end{array} \right\}^a \times \{\boxed{1}\}^b$.

Example 3.4

$$
\begin{array}{ccccc}
\begin{array}{lll} 8 & & \\ 5 & 7 & \\ 4 & 6 & \\ 1 & 2 & 3 \end{array} &
\begin{array}{lll} 7 & & \\ 4 & 6 & \\ 3 & 5 & \\ 1 & 1 & 2 \end{array} &
\begin{array}{lll} 6 & & \\ 3 & 5 & \\ 2 & 4 & \\ 1 & 1 & 2 \end{array} &
\begin{array}{lll} 5 & & \\ 3' & 4 & \\ 2 & 3' & \\ 1 & 1 & 2 \end{array} &
\begin{array}{lll} 4' & & \\ 3' & 4' & \\ 2 & 3' & \\ 1 & 1 & 2 \end{array} \\
T & N_1T & N_2T & N_3T & N_4T
\end{array}
$$

$$
\begin{array}{cccc}
\begin{array}{lll} 8 & & \\ 5 & 7 & \\ 4 & 6 & \\ 1 & 2 & 3 \end{array} &
\begin{array}{lll} 6 & & \\ 3 & 5 & \\ 1 & 2 & 4 \end{array} &
\begin{array}{ll} 2 & 4 \\ 1 & 3 \end{array} &
\begin{array}{l} 2 \\ 1 \end{array} \\
T & \mathbf{H}_2^{-1}VT & \mathbf{H}_2^{-1}V\mathbf{H}_2^{-1}VT & \mathbf{H}_2^{-1}V\mathbf{H}_2^{-1}V\mathbf{H}_2^{-1}VT
\end{array}
$$

Note that the operators M_i are completely reversible so that they describe a procedure for mapping the standard tableaux of size $2a+b$ bijectively to the x-standard tableaux with content in the set $\left\{ \boxed{1}\boxed{2}, \begin{array}{|c|}\hline 2\\ \hline 1\\ \hline\end{array} \right\}^a \times \{\boxed{1}\}^b$.

Let $T \in ST^{2a+b}$ and let μ be a partition with two columns with $\mu = (2^a 1^b)$. We will let the statistic $b_\mu(T)$ on standard tableaux be the number of occurrences of $\begin{array}{|c|}\hline 2\\ \hline 1\\ \hline\end{array}$ in the $type_\mu(T)$. Let the statistic $a_\mu(T)$ be defined recursively with a base case of $a = 0$ so that $a_{(1^b)}(T) = c(T)$. For $a > 0$ let $a_\mu(T) = a_{\mu^r}(\mathbf{H}_2^{-1}VT) + (\lambda(T)_1 - 2)$ if $type_\mu(T)_1 = \boxed{1}\boxed{2}$ and $a_\mu(T) = a_{\mu^r}(\mathbf{H}_2^{-1}VT) + |\lambda(T)^c|$ if $type_\mu(T)_1 = \begin{array}{|c|}\hline 2\\ \hline 1\\ \hline\end{array}$.

Proposition 3.5 *Let $n = 2a+b$. The statistic $a_\mu(T)$ where $\mu = (2^a 1^b)$ satisfies the formula*

$$a_\mu(T) = c(T) - \sum_{i=1}^{a} ((n+1) - 2i)\chi(type_\mu(T)_i = \boxed{1}\boxed{2})$$

Proof By induction on a we assume that it is true for partitions with fewer than a parts equal to 2 (with the base case of $a = 0$ true by definition), then when $type_\mu(T)_1 = \boxed{1}\boxed{2}$ we have that

$$\begin{aligned} a_\mu(T) &= a_{\mu^r}(\mathbf{H}_2^{-1}VT) + (\lambda(T)_1 - 2) \\ &= c(\mathbf{H}_2^{-1}VT) - \sum_{i=1}^{a-1}((n-1) - 2i)\chi(type_{\mu^r}(\mathbf{H}_2^{-1}VT)_i = \boxed{1}\boxed{2}) + (\lambda(T)_1 - 2) \\ &= c(\mathbf{H}_2^{-1}VT) - \sum_{i=2}^{a}((n+1) - 2i)\chi(type_\mu(T)_i = \boxed{1}\boxed{2}) + (\lambda(T)_1 - 2) \end{aligned}$$

The charge statistic is well understood and several methods for computing the charge exist. For $type_\mu(T)_1 = \boxed{1}\boxed{2}$ we have that $c(\mathbf{H}_2^{-1}VT) = c(VT) - (\lambda_1 - 2)$ because $\mathbf{H}_2^{-1}$ is the operation of cyclage of the $(\lambda_1 - 2)$ cells in the first row and then reducing the content. We also know that $c(VT) = c(T) - (n-1)$ by using the word definition of charge. This implies that $c(\mathbf{H}_2^{-1}VT) = c(T) - (n-1) - (\lambda_1 - 2)$ and hence that

$$a_\mu(T) = c(T) - (n-1) - \sum_{i=2}^{a}((n+1) - 2i)\chi(type_\mu(T)_i = \boxed{1}\boxed{2}) + (\lambda(T)_1 - 2)$$

When $type_\mu(T)_1 = \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array}$ then a similar calculation gives that

$$a_\mu(T) = c(\mathbf{H}_2^{-1}VT) - \sum_{i=2}^{a}((n+1) - 2i)\chi(type_\mu(T)_i = \boxed{1}\boxed{2}) + |\lambda(T)^c|$$

The charge of the tableau $\mathbf{H}_2^{-1}VT$ can be calculated by noting that the $\mathbf{H}_2^{-1}$ operator is one cyclage operation for every cell in $|\lambda(T)^c|$ and then reducing the content so that $c(\mathbf{H}_2^{-1}VT) = c(T) - |\lambda(T)^c|$. In both cases $a_\mu(T) = c(T) - \sum_{i=1}^{a}((n+1) - 2i)\chi(type_\mu(T)_i = \boxed{1}\boxed{2})$. $\square$

We are now ready to state the main theorem of this section.

Theorem 3.6 *Let $\mu = (2^a 1^b)$. For $s \in \left\{\boxed{1}\boxed{2}, \begin{array}{c}\boxed{2}\\\boxed{1}\end{array}\right\}^a$ the symmetric functions $H_2^{s_1} H_2^{s_2} \cdots H_2^{s_a} H_{(1^b)}[X;t]$ are generating functions for the standard tableaux of $\mu - type = (s, \boxed{1}^b)$ in the sense that*

$$H_2^{s_1} H_2^{s_2} \cdots H_2^{s_a} H_{(1^b)}[X;t] = \sum_{\substack{T \in ST^{2a+b} \\ type_\mu(T) = (s, \boxed{1}^b)}} t^{a_\mu(T)} s_{\lambda(T)}[X]$$

This gives the following logical corollary that follows from the theorem and Corollary 2.7

Corollary 3.7 *Let $\mu = (2^a 1^b)$. The $H_\mu[X;q,t]$ are generating functions for the standard tableaux in the sense that*

$$H_\mu[X;q,t] = \sum_{T \in ST^{2a+b}} t^{a_\mu(T)} q^{b_\mu(T)} s_{\lambda(T)}[X]$$

We will show that this this is true by describing two procedures, one, $\mathbf{H}_2$, that takes as input a standard tableau of shape $\lambda \vdash n$ and $\mu - type = s$, and returns a tableau for every term in the expression

$$H_2^{\boxed{1}\boxed{2}} s_\lambda[X] = \sum_{\rho/\lambda \in \mathcal{H}_{n+2}} (-1)^{ht_n(\rho)-1} t^{|\lambda/\rho^r|} s_{\rho]_n}[X] \tag{3.1}$$

The other procedure, $\bar{\mathbf{H}}_2$, will take as input a standard tableau of shape λ, and return a tableau for every term in the expression

$$H_2^{\begin{array}{c}\boxed{2}\\\boxed{1}\end{array}} s_\lambda[X] = \sum_{\rho/\lambda \in \mathcal{V}_{n+2}} (-1)^{\bar{ht}_n(\rho)-1} t^{|\rho^c|} s_{\rho]^n}[X] \tag{3.2}$$

Let $n \geq 0$ and $T \in XST^n$ and then let ρ be any partition of $2n+2$ such that $\rho/\lambda(T) \in \mathcal{H}_{n+2}$. Note that $\lambda(T)/\rho^r$ is also a horizontal strip. Create a tableau $\tilde{T}$ such that $\lambda(\tilde{T}) = \rho^r$ by performing one column evacuation for the each cell in $\lambda(T)/\rho^r$ from right to left. Because the bumping paths of the cells do not cross, the cells will be evacuated in weakly increasing order (that is, it may contain two cells of label i). Let R be the row of cells that are evacuated from T. Increase all of the labels of the cells in R and $\tilde{T}$ by one. Create a new tableau by row inserting the labels 1, 1, and all of the labels in R in increasing order into $\tilde{T}$. Call this new tableau $\mathbf{H}_2^\rho T$.

The purpose of the definition of $\mathbf{H}_2^\rho$ is to create a tableau for every term in the expression

$$H_2^{\boxed{1|2}}\left(t^{a_\mu(T)} s_{\lambda(T)}[X]\right) = \sum_{\rho/\lambda(T) \in \mathcal{H}_{n+2}} t^{a_\mu(T)+|\lambda(T)/\rho^r|}(-1)^{ht_n(\rho)-1} s_{\rho\rfloor_n}[X]$$

where we have used the convention that $s_{\rho\rfloor_n}[X] = 0$ if $\rho\rfloor_n$ is not a partition. The tableaux such that $\lambda(\mathbf{H}_2^\rho T) \neq \rho\rfloor_n$ will correspond to terms that either cancel or have weight zero when $H_2^{\boxed{1|2}}$ acts on the symmetric functions $H_2^{s_1} H_2^{s_2} \cdots H_2^{s_a} H_{(1^b)}[X;t]$ (where $s_i = \boxed{1|2}$ or $\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array}$).

Example 3.8 *$n = 6$ and $T = \begin{array}{|c|c|cc}\cline{1-2} 1' & 4 & & \\ \hline 1' & 2 & 2 & \multicolumn{1}{|c|}{3} \\ \hline\end{array} \in XST^6$. Consider the three following choices for the partition ρ corresponding to the three types of resulting tableau that will be created.*

- $\rho = (9,3,2)$
  $R = \boxed{1'}$ *and* $\tilde{T} = \begin{array}{|c|c|c}\cline{1-2} 2 & 4 & \\ \hline 1' & 2 & \multicolumn{1}{|c|}{3} \\ \hline\end{array}$.

$$\mathbf{H}_2^\rho T = \begin{array}{|c|c|c}\cline{1-2} 3 & 5 & \\ \hline 2' & 3 & \multicolumn{1}{|c|}{4} \\ \hline 1 & 1 & \multicolumn{1}{|c|}{2'} \\ \hline\end{array}$$

  $\mathbf{H}_2^{-1}\mathbf{H}_2^{(9,3,2)} T = T$ *and* $\rho\rfloor_6 = \lambda(\mathbf{H}_2^\rho T)$.

- $\rho = (9,4,1)$
  $R = \boxed{1'}$ *and* $\tilde{T} = \begin{array}{|c|ccc}\cline{1-1} 4 & & & \\ \hline 1' & \multicolumn{1}{|c|}{2} & \multicolumn{1}{|c|}{2} & \multicolumn{1}{|c|}{3} \\ \hline\end{array}$.

$$\mathbf{H}_2^\rho T = \begin{array}{|c|c|c|c}\cline{1-1} 5 & \multicolumn{3}{c}{} \\ \cline{1-3} 2' & 3 & 3 & \\ \hline 1 & 1 & 2' & \multicolumn{1}{|c|}{4} \\ \hline\end{array}$$

$\rho\rfloor_6$ *is not a partition.*

- $\rho = (10, 4)$

  $R = \begin{array}{|c|c|}\hline 1' & 4 \\ \hline \end{array}$ *and* $\tilde{T} = \begin{array}{|c|c|c|c|}\hline 1' & 2 & 2 & 3 \\ \hline \end{array}$.

$$\mathbf{H}_2^{\rho} T = \begin{array}{|c|c|c|cc}\cline{1-3} 2' & 3 & 3 & & \\ \hline 1 & 1 & 2' & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{5} \\ \hline \end{array}$$

  $\lambda(\mathbf{H}_2^{(10,4)} T) \neq \rho\rfloor_6$ *but* $\rho\rfloor_6$ *is a partition.*

Notice that $\mathbf{H}_2^{\rho} T$ falls into one of the three following categories:

1. $\lambda(\mathbf{H}_2^{\rho} T) = \rho\rfloor_n$. We make the following observations in this case:
   - $\lambda(\mathbf{H}_2^{\rho} T) = \rho\rfloor_n = (\rho_1 - n, \rho^r)$
   - $ht_n(\rho) = 1$
   - $\mu(\mathbf{H}_2^{\rho} T) = (\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}, \mu(T))$
   - For $T \in ST^n$, $type_{(2^{a+1}1^b)}(V^{-1}\mathbf{H}_2^{\rho} T) = (\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}, type_{(2^a1^b)}(T))$
   - $\mathbf{H}_2^{-1}\mathbf{H}_2^{\rho} T = T$
   - If $T \in ST^n$ then $a_{(2^{a+1}1^b)}(V^{-1}\mathbf{H}_2^{\rho} T) = a_{(2^a1^b)}(T) + (\rho_1 - n - 2) = a_{(2^a1^b)}(T) + |\lambda(T)/\rho^r|$
2. $\rho\rfloor_n$ does not exist.
3. $\rho\rfloor_n$ exists but $\lambda(\mathbf{H}_2^{\rho} T) \neq \rho\rfloor_n$. We still have $\mu(\mathbf{H}_2^{\rho} T) = (\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}, \mu(T))$, but we no longer have the relationship $\mathbf{H}_2^{-1}\mathbf{H}_2^{\rho} T = T$. About $\lambda(\mathbf{H}_2^{\rho} T)$ we can really only say that $\rho/\lambda(\mathbf{H}_2^{\rho} T)$ is a horizontal strip of size n.

Similarly, there is a procedure that adds a column block of size 2. Let $n \geq 0$ and $T \in XST^n$ and then let ρ be any partition of $2n+2$ such that $\rho/\lambda(T) \in \mathcal{V}_{n+2}$. Note that

$\lambda(T)/\rho^c$ is also a vertical strip. Create a tableau $\tilde{T}$ such that $\lambda(\tilde{T}) = \rho^c$ by performing one row evacuation for the each cell in $\lambda(T)/\rho^c$ from top to bottom. Because the bumping paths of the cells do not cross, the cells will be evacuated in strictly increasing order (C may contain two cells of label i'). Let C be the column of cells that are evacuated from T. Increase all of the labels of the cells in C and $\tilde{T}$ by one. Create a new tableau by column inserting the labels $1'$, $1'$, and all of the labels in C in order into $\tilde{T}$. Call this new tableau $\bar{\mathbf{H}}_2^\rho T$.

Again, we observe the following three categories for $\bar{\mathbf{H}}_2^\rho$:

1. $\lambda(\bar{\mathbf{H}}_2^\rho T) = \rho]^n$. We make the following observations in this case:
   - $\lambda(\bar{\mathbf{H}}_2^\rho T) = \rho]^n = (\rho_1, \rho_2, \ldots, \rho_{l(\rho)-n})$
   - $\bar{ht}_n(\rho) = 1$
   - $\mu(\bar{\mathbf{H}}_2^\rho T) = \left(\begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix}, \mu(T)\right)$
   - For $T \in ST^n$, $type_{(2^{a+1}1^b)}(V^{-1}\bar{\mathbf{H}}_2^\rho T) = \left(\begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix}, type_{(2^a1^b)}(T)\right)$
   - $\mathbf{H}_2^{-1}\bar{\mathbf{H}}_2^\rho T = T$
   - If $T \in ST^n$ then $a_{(2^{a+1}1^b)}(V^{-1}\bar{\mathbf{H}}_2^\rho T) = a_{(2^a1^b)}(T) + (2n+2-\rho_1') = a_{(2^a1^b)}(T) + |\rho^c|$
2. $\rho]^n$ does not exist.
3. $\rho]^n$ exists but $\lambda(\bar{\mathbf{H}}_2^\rho T) \neq \rho]^n$. We still have $\mu(\bar{\mathbf{H}}_2^\rho T) = \left(\begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix}, \mu(T)\right)$, but we no longer have the relationship $\mathbf{H}_2^{-1}\bar{\mathbf{H}}_2^\rho T = T$. We also have that $\rho/\lambda(\bar{\mathbf{H}}_2^\rho T)$ is a vertical strip of size n.

The operators $\mathbf{H}_2^{-1}$ and $\mathbf{H}_2^\rho/\bar{\mathbf{H}}_2^\rho$ are not true inverses of each other, but $\mathbf{H}_2^\rho$ and $\bar{\mathbf{H}}_2^\rho$ are both invertible and there is no problem reversing the steps to find T from $\mathbf{H}_2^\rho T$ or $\bar{\mathbf{H}}_2^\rho T$ as long as ρ is known. The $\bar{\mathbf{H}}_2^\rho$ and $\mathbf{H}_2^\rho$ operators do have a similar relationship with the $(i, i+1)$ and V operators.

Lemma 3.9 *For a $T \in XST^n$ $(n \geq 2)$ and $\mu(T) = (\boxed{1}, \boxed{1}, s)$ (where s is the remainder of the type-list) the tableaux operators have the following relationship*

$$(1,2)V(2,3)(1,2)\mathbf{H}_2^\rho T = \mathbf{H}_2^\rho VT$$

$$(1,2)V(2,3)(1,2)\bar{\mathbf{H}}_2^\rho T = \bar{\mathbf{H}}_2^\rho VT$$

Lemma 3.10 *For a $T \in XST^n$ and for $1 < i \leq n$ the tableaux operators have the following relationship*

$$(i,i+1)\mathbf{H}_2^\rho T = \mathbf{H}_2^\rho (i-1,i)T$$

$$(i,i+1)\bar{\mathbf{H}}_2^\rho T = \bar{\mathbf{H}}_2^\rho (i-1,i)T$$

The proofs of these lemmas are nearly the same as in the corresponding lemmas for the $\mathbf{H}_2^{-1}$ operator.

The main result that we need from these relationships can be stated as follows:

Corollary 3.11 *Let $n = 2a + b$ and $T \in ST^n$ and let ρ be a partition of $2n + 2$ such that $\rho/\lambda(T) \in \mathcal{H}_{n+2}$ then*

$$type_{(2^{a+1}1^b)}(V^{-1}\mathbf{H}_2^\rho T) = (\boxed{1}\boxed{2}, type_{(2^a1^b)}(T))$$

Similarly, if $\rho/\lambda(T) \in \mathcal{V}_{n+2}$ then

$$type_{(2^{a+1}1^b)}(V^{-1}\bar{\mathbf{H}}_2^\rho T) = \left(\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array}, type_{(2^a1^b)}(T)\right)$$

Proof From the previous two lemmas we may derive that $M_{i+1}\mathbf{H}_2^\rho = \mathbf{H}_2^\rho M_i$ and $M_{i+1}\bar{\mathbf{H}}_2^\rho = \bar{\mathbf{H}}_2^\rho M_i$ for $i > 1$. Therefore since the relation between the type of $T \in ST^n$ and the content of $N_aT \in XST^n$ is known then

$$\begin{aligned}
type_{(2^{a+1}1^b)}(V^{-1}\mathbf{H}_2^\rho T) &= \mu(N_{a+1}V^{-1}\mathbf{H}_2^\rho T) \\
&= \mu(M_{a+1}M_a \cdots M_1 V^{-1}\mathbf{H}_2^\rho T) \\
&= \mu(M_{a+1}M_a \cdots M_2 \mathbf{H}_2^\rho T) \\
&= \mu(\mathbf{H}_2^\rho M_a M_{a-1} \cdots M_1 T) \\
&= (\boxed{1|2}, \mu(N_aT)) = (\boxed{1|2}, type_{(2^a1^b)}(T))
\end{aligned}$$

The proof is exactly analogous for the statement for the $\bar{\mathbf{H}}_2^\rho$ operator. □

This also gives the following result about the a_μ statistic

Corollary 3.12 *Let $n = 2a+b$ and $T \in ST^n$ and let ρ be a partition of $2n+2$ such that $\rho/\lambda(T) \in \mathcal{H}_{n+2}$ then*

$$a_{(2^{a+1}1^b)}(V^{-1}\mathbf{H}_2^\rho T) = a_{(2^a1^b)}(T) + |\lambda(T)/\rho^r|$$

and if $\rho/\lambda(T) \in \mathcal{V}_{n+2}$ then

$$a_{(2^{a+1}1^b)}(V^{-1}\bar{\mathbf{H}}_2^\rho T) = a_{(2^a1^b)}(T) + |\rho^c|$$

Proof Using the previous lemma and Proposition 3.5 we know that

$$\begin{aligned}
a_{(2^{a+1}1^b)}(V^{-1}\mathbf{H}_2^\rho T) &= c(V^{-1}\mathbf{H}_2^\rho T) - (n+1) \\
&\quad - \sum_{i=2}^{a+1}((n+3)-2i)\chi(type_{(2^a1^b)}(T)_{i-1} = \boxed{1|2}) \\
&= c(T) + |\lambda(T)/\rho^r| + (n+1) - (n+1) \\
&\quad - \sum_{i=1}^{a}((n+1)-2i)\chi(type_{(2^a1^b)}(T)_i = \boxed{1|2}) \\
&= a_{(2^a1^b)}(T) + |\lambda(T)/\rho^r|
\end{aligned}$$

using the methods that we have to calculate the charge.

Similarly for the $\bar{\mathbf{H}}_2^\rho$ operator, we have that

$$\begin{aligned} a_{(2^{a+1}1^b)}(V^{-1}\bar{\mathbf{H}}_2^\rho T) &= c(V^{-1}\bar{\mathbf{H}}_2^\rho T) - \sum_{i=2}^{a+1}((n+3)-2i)\chi(type_{(2^a1^b)}(T)_{i-1} = \boxed{1}\boxed{2}) \\ &= c(T) + |\rho^c| - \sum_{i=1}^{a}((n+1)-2i)\chi(type_{(2^a1^b)}(T)_i = \boxed{1}\boxed{2}) \\ &= a_{(2^a1^b)}(T) + |\rho^c| \end{aligned}$$

□

The result of this is that we have a formula for the action of $H_2^{\boxed{1}\boxed{2}}$ and $H_2^{\begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix}}$ on $t^{a_\mu(T)}s_{\lambda(T)}[X]$ for a standard tableau T in terms of pictures that follows directly from equations (3.1), (3.2) and the previous corollaries.

Proposition 3.13 *Let $T \in ST^n$ and $\mu = (2^a1^b)$ where $n = 2a+b$ then*

$$H_2^{\boxed{1}\boxed{2}}\left(t^{a_\mu(T)}s_{\lambda(T)}[X]\right) = \sum_{\rho/\lambda(T)\in\mathcal{H}_{n+2}} (-1)^{ht_n(\rho)-1}t^{a_\mu(V^{-1}\mathbf{H}_2^\rho T)}s_{\rho\rfloor_n}[X]$$

and

Proposition 3.14 *Let $T \in ST^n$ and $\mu = (2^a1^b)$ where $n = 2a+b$ then*

$$H_2^{\begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix}}t^{a_\mu(T)}s_{\lambda(T)}[X] = \sum_{\rho/\lambda(T)\in\mathcal{V}_{n+2}} (-1)^{\bar{h}t_n(\rho)-1}t^{a_\mu(V^{-1}\bar{\mathbf{H}}_2^\rho T)}s_{\rho\rceil^n}[X]$$

We are now ready to prove Theorem 3.6.

Proof (of Theorem 3.6) Let $\mu = (2^a1^b)$ and $n = 2a+b$ and $s \in \left\{\boxed{1}\boxed{2}, \begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix}\right\}$. If $a = 0$ then it is a well known result that $H_{(1^b)}[X;t] = \sum_{T\in ST^b} t^{c(T)}s_{\lambda(T)}[X]$ so the base case is true.

Assume that

$$H_2^{s_1}H_2^{s_2}\cdots H_2^{s_a}H_{(1^b)}[X;t] = \sum_{\substack{T\in ST^{2a+b}\\ type_\mu(T)=(s,\boxed{1}^b)}} t^{a_\mu(T)}s_{\lambda(T)}[X]$$

then by Proposition 3.13 we have that

$$H_2^{\boxed{1}\boxed{2}} H_2^{s_1} H_2^{s_2} \cdots H_2^{s_a} H_{(1^b)}[X;t] = \tag{3.3}$$

$$\sum_{\substack{T \in ST^{2a+b} \\ type_\mu(T)=(s,\boxed{1}^b)}} \sum_{\rho/\lambda(T) \in \mathcal{H}_{n+2}} (-1)^{ht_n(\rho)-1} t^{a_{(2,\mu)}(\mathbf{H}_2^\rho T)} s_{\rho\rfloor_n}[X] \tag{3.4}$$

As was mentioned in the definition of $\mathbf{H}_2^\rho$, $\mathbf{H}_2^\rho T$ falls into three categories. If $\lambda(\mathbf{H}_2^\rho T) = \rho\rfloor_n$ then we have that $ht_n(\rho) - 1 = 0$ and so

$$(-1)^{ht_n(\rho)-1} t^{a_{(2,\mu)}(\mathbf{H}_2^\rho T)} s_{\rho\rfloor_n}[X] = t^{a_{(2,\mu)}(\mathbf{H}_2^\rho T)} s_{\lambda(\mathbf{H}_2^\rho T)}[X]$$

If $\rho\rfloor_n$ does not exist then $s_{\rho\rfloor_n}[X] = 0$. These terms may be ignored since they do not change the sum (3.4).

If $\rho\rfloor_n$ does exist but $\lambda(\mathbf{H}_2^\rho T) \neq \rho\rfloor_n$ then the corresponding term $(-1)^{ht_n(\rho)-1}$ $t^{a_{(2,\mu)}(\mathbf{H}_2^\rho T)} s_{\rho\rfloor_n}[X]$ will cancel, but it is necessary to demonstrate a sign reversing involution on this set. We require the following lemma

Lemma 3.15 *There exists an involution I_λ^n on partitions ρ such that $\rho/\lambda \in \mathcal{H}_n$, $\rho\rfloor_n$ exists and $\lambda \neq \rho\rfloor_n$ with the property that $ht_n(I_\lambda^n(\rho)) = ht_n(\rho) \pm 1$ and $\rho\rfloor_n = I_\lambda^n(\rho)\rfloor_n$.*

Proof (of Lemma) Let $h = ht_n(\rho)$. Let $\gamma = \rho\rfloor_n$. I_λ^n maps the set of ρ that satisfy the conditions of the lemma with $\lambda_h > \gamma_h$ to the set of ρ that satisfy the conditions of the lemma with $\lambda_h \leq \gamma_h$.

If $\lambda_h > \gamma_h$ then $I_\lambda^n(\rho) = \gamma$ with an $n-snake$ of height $h+1$ added. If $\lambda_h \leq \gamma_h$ then $I_\lambda^n(\rho) = \gamma$ with an $n-snake$ of height $h-1$ added (note that if $h=1$ and $\lambda_1 \leq \gamma_1$ then $\lambda = \gamma$).

If $\lambda_h > \gamma_h$ then $\lambda_{h+1} \leq \rho_{h+1} = \gamma_{h+1}$ so that $\lambda_{h+1} \leq \gamma_{h+1}$. Also if $\lambda_h \leq \gamma_h$ then $\lambda_{h-1} \geq \rho_h > \rho_h - 1 = \gamma_{h-1}$ so that $\lambda_{h-1} > \gamma_{h-1}$. These two statements together show that I_λ^n is an involution. $\square$

With this lemma we have a sign reversing involution on the tableaux $\mathbf{H}_2^\rho T$ such that $\lambda(\mathbf{H}_2^\rho T) \neq \rho\rfloor_n$. For if there is a tableau $\mathbf{H}_2^\rho T$ with the property that $\lambda(\mathbf{H}_2^\rho T) \neq \rho$ and $\rho\rfloor_n$ exists then if we let $\tilde{\rho} = I_\lambda^n(\rho)$, the involution says that $S = (\mathbf{H}_2^{\tilde{\rho}})^{-1}\mathbf{H}_2^\rho T$ also has this property and $(-1)^{ht_n(\rho)-1} t^{a_{(2,\mu)}(\mathbf{H}_2^\rho T)} s_{\rho\rfloor_n}[X] = -(-1)^{ht_n(\tilde{\rho})-1} t^{a_{(2,\mu)}(\mathbf{H}_2^{\tilde{\rho}} S)} s_{\tilde{\rho}\rfloor_n}[X]$.

Therefore equation (3.4) has only positive terms and the negative ones cancel. The terms that survive all have the property that
$\mathbf{H}_2^{-1}\mathbf{H}_2^\rho T = T$. Every tableau T such that $type_\mu(T) = (\boxed{1}\boxed{2}, s)$ will correspond to exactly one of the terms in this sum since $\mathbf{H}_2^{(n+\lambda(T)_1, \lambda(T)^r)}\mathbf{H}_2^{-1} T = T$. Therefore equation (3.4) becomes

$$H_2^{\boxed{1}\boxed{2}} H_2^{s_1} H_2^{s_2} \cdots H_2^{s_a} H_{(1^b)}[X;t] = \sum_{\substack{T \in ST^{2a+b+2} \\ type_{(2,\mu)}(T) = \left(\boxed{1}\boxed{2}, s, \boxed{1}^b\right)}} t^{a_{(2,\mu)}(T)} s_{\lambda(T)}[X]$$

The proof that

$$H_2^{\begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix}} H_2^{s_1} H_2^{s_2} \cdots H_2^{s_a} H_{(1^b)}[X;t] = \sum_{\substack{T \in ST^{2a+b+2} \\ type_{(2,\mu)}(T) = \left(\begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix}, s, \boxed{1}^b\right)}} t^{a_{(2,\mu)}(T)} s_{\lambda(T)}[X]$$

is exactly analogous. $\square$

4 More About Statistics on Tableaux

Some very interesting properties about the standard tableaux follow from the definitions in the previous section. The 'atoms' of the Macdonald polynomials and the $\mu - type$ of the standard tableaux suggest that the tableaux naturally fall into standard tableaux classes. For a sequence $s \in \left\{\boxed{1}\boxed{2}, \begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix}\right\}^a \times \{\boxed{1}\}^b$ set $STC^s = \{T \in ST^{2a+b} | type_{(2^a1^b)}(T) = s\}$.

It will never be clear how beautiful this breakdown of the standard tableaux into classes is until the picture of where the 'atoms' lie in the standard tableaux when they are ranked by the charge is clear. The figures at the end of this paper are the posets of the standard tableaux of size $4, 5$ and 6 when they are ranked by the charge. The standard tableau classes are grouped together in this poset and shaded so that each class is separated. The horizontal position of each tableau is slightly related to cyclage, but not as much as it was in the case of the column strict tableaux. Many of the properties of the Macdonald polynomials can be observed in these diagrams (especially my favorite: $\omega H_{(2^a 1^b)}[X; q, t] = q^a t^{n(2^a 1^b)} H_{(2^a 1^b)}[X; 1/q, 1/t]$) and expansions for $H_{(2^a 1^b)}[X; q, t]$ in terms of Schur functions can be immediately written down.

Example 4.1 *The tableaux class* $STC^{\left(\boxed{1}\boxed{2},\begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix},\begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix}\right)}$ *is given by*

$$\begin{array}{|c|c}\hline 6 & \\\hline 5 & \\\hline 4 & \\\hline 3 & \multicolumn{1}{|c|}{} \\\hline 1 & \multicolumn{1}{c|}{2} \\\hline\end{array}$$

$$\begin{array}{|c|cc}\hline 5 & & \\\hline 4 & & \\\hline 3 & & \\\hline 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{6} \\\hline\end{array} \qquad \begin{array}{|c|c|}\hline 5 & \multicolumn{1}{c}{} \\\hline 4 & \multicolumn{1}{c}{} \\\hline 3 & 6 \\\hline 1 & 2 \\\hline\end{array}$$

$$\begin{array}{|c|c|}\hline 4 & 6 \\\hline 3 & 5 \\\hline 1 & 2 \\\hline\end{array} \qquad \begin{array}{|c|cc}\hline 6 & & \\\hline 4 & & \\\hline 3 & & \\\hline 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{5} \\\hline\end{array} \qquad \begin{array}{|c|cc}\hline 4 & & \\\hline 3 & \multicolumn{1}{c|}{6} & \\\hline 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{5} \\\hline\end{array}$$

$$\begin{array}{|c|cc}\hline 6 & & \\\hline 3 & \multicolumn{1}{c|}{5} & \\\hline 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{4} \\\hline\end{array} \qquad \begin{array}{|c|ccc}\hline 4 & & & \\\hline 3 & & & \\\hline 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{6} \\\hline\end{array}$$

$$\begin{array}{|c|c|cc}\hline 3 & 5 & & \\\hline 1 & 2 & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{6} \\\hline\end{array}$$

The symmetric function that this corresponds to is

$$H_2^{\boxed{1}\boxed{2}} H_2^{\begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix}} H_2^{\begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix}} 1 = s_{(21111)}[X] + ts_{(3111)}[X] + ts_{(2211)}[X] + t^2 s_{(222)}[X] + t^2 s_{(3111)}[X]$$

$$+t^2 s_{(321)}[X] + t^3 s_{(321)}[X] + t^3 s_{(411)}[X] + t^4 s_{(42)}[X]$$

These classes have the property that $STC^{\left(s,\begin{array}{|c|c|}\hline 1&2\\\hline\end{array},\boxed{1}^{b}\right)} \cup STC^{\left(s,\begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array},\boxed{1}^{b}\right)} = STC^{(s,\boxed{1}^{b+2})}$ simply by definition of the type. There is also a relation between the a_μ and b_μ statistics over this set of tableaux.

Proposition 4.2 *If* $type_{(2^{a+1}1^b)}(T)_{a+1} = \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}$ *then* $a_{(2^{a+1}1^b)}(T) = a_{(2^a1^{b+2})}(T)$ *and* $b_{(2^{a+1}1^b)}(T) = b_{(2^{a+1}1^b)}(T) + 1$.

If $type_{(2^{a+1}1^b)}(T)_{a+1} = \begin{array}{|c|c|}\hline 1&2\\\hline\end{array}$ *then* $a_{(2^{a+1}1^b)}(T) = a_{(2^a1^{b+2})}(T) + (b+1)$ *and* $b_{(2^{a+1}1^b)}(T) = b_{(2^{a+1}1^b)}(T)$.

Example 4.3

$$T = \begin{array}{|c|c|}\hline 4&6\\\hline 2&5\\\hline 1&3\\\hline\end{array}$$

$type_{(2^3)}(T) = \left(\begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}, \begin{array}{|c|c|}\hline 1&2\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}\right)$

$a_{(2^3)}(T) = 3,\ a_{(2^21^2)}(T) = 3,\ a_{(21^4)}(T) = 6,\ a_{(1^6)}(T) = 6$

$b_{(2^3)}(T) = 2,\ b_{(2^21^2)}(T) = 1,\ b_{(21^4)}(T) = 1,\ b_{(1^6)}(T) = 0$

This relationship is consistent with observations made by Lynne Butler [1] about adjacent rows of the q,t-Kostka matrix. Comparing $K_{\lambda(2^a1^{b+2})}(q,t)$ to $K_{\lambda(2^{a+1}2^b)}(q,t)$, one notices that every term either changes by a factor of q or a factor of t^{b+1}.

In fact we derive the following Corollary from the proposition.

Corollary 4.4 *For* $s \in \left\{\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}\right\}^a$ *we have the following relationship*

$$H_2^{s_1}H_2^{s_2}\cdots H_2^{s_a}H_2^{\begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}}H_{(1^b)}[X;t] + t^{b+1}H_2^{s_1}H_2^{s_2}\cdots H_2^{s_a}H_2^{\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}}H_{(1^b)}[X;t]$$
$$= H_2^{s_1}H_2^{s_2}\cdots H_2^{s_a}H_{(1^{b+2})}[X;t]$$

Notice that equation (2.5) suggests that the standard tableaux classes are isomorphic if the type has the same number of occurrences of $\begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}$. The isomorphism between these classes is given by the composition of the operators that were introduced in the last section. The N_a (invertible) operator changes a standard tableau in the class STC^s to an x-standard tableau such that the content is s. The $(i, i+1)$ operators change the tableau to define a symmetric group action on the content. To make this more precise we define the bijection in the following proposition:

Proposition 4.5 *For* $1 \leq i < a$ *and* $s \in \left\{\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}\right\}^a \times \{\boxed{1}^b\}$ *the operator* $N_a^{-1}(i, i+1)N_a$ *is a bijection between* STC^s *and* $STC^{(i,i+1)s}$*. Furthermore this operator has the property that if* $s_i = \begin{array}{|c|c|}\hline 1&2\\\hline\end{array}$ *and* $s_{i+1} = \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}$ *then* $a_{(2^a1^b)}(N_a^{-1}(i, i+1)N_aT) = a_{(2^a1^b)}(T) + 1$ *and* $b_{(2^a1^b)}(N_a^{-1}(i, i+1)N_aT) = b_{(2^a1^b)}(T)$*.*

Example 4.6

$$\begin{array}{ccc}
\begin{array}{|c|c|}\hline 4&6\\\hline 2&5\\\hline 1&3\\\hline\end{array} & \begin{array}{|c|c|}\hline 4&6\\\hline 3&5\\\hline 1&2\\\hline\end{array} & \begin{array}{|c|c|}\hline 5&6\\\hline 2&4\\\hline 1&3\\\hline\end{array} \\[1em]
T & N_3^{-1}(1,2)N_3T & N_3^{-1}(2,3)N_3T
\end{array}$$

T *has* $type_{(2^3)}(T) = \left(\begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}, \begin{array}{|c|c|}\hline 1&2\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}\right)$*, so* $type_{(2^3)}(N_3^{-1}(1,2)N_3T) = \left(\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}\right)$ *and* $type_{(2^3)}(N_3^{-1}(2,3)N_3T) = \left(\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}\right)$.

$a_{(2^3)}(T) = 3,\ a_{(2^3)}(N_3^{-1}(1,2)N_3T) = 2,\ a_{(2^3)}(N_3^{-1}(2,3)N_3T) = 4.$

Taking the transpose (flipping the shape and entries of the diagram about the $x = y$ line) of a standard tableau T will be represented by the operator ωT. It has the property that if the $type_\mu(T) = s$ then the $type_\mu(\omega T) = (\omega s_1, \omega s_2, \ldots, \omega s_k)$. This follows directly from the definition of the $\mu - type$ since $\mathbf{H}_2^{-1}V\omega T = \omega\mathbf{H}_2^{-1}VT$. This gives a simple method for computing the a_μ and b_μ statistics of ωT from the a_μ and b_μ statistics of T.

Proposition 4.7 *$a_{(2^a1^b)}(\omega T) = \binom{a+b}{2} + \binom{a}{2} - a_{(2^a1^b)}(T)$ and $b_{(2^a1^b)}(\omega T) = a - b_{(2^a1^b)}(T)$.*

This result follows from the fact that $c(\omega T) = \binom{n}{2} - c(T)$ and Proposition 3.5 and the definition of b_μ. This is consistent with the symmetric function identity $\omega H_\mu[X;q,t] = q^{n(\mu')}t^{n(\mu)}H_\mu[X;1/q,1/t]$ since $n((2^a1^b)) = \binom{a+b}{2} + \binom{a}{2}$ and $n((2^a1^b)') = a$.

Before the end of this paper, we would like to point out a less obvious observation about the standard tableaux classes. After a conversation with Will Brockman on about the Hall-Littlewood polynomials, he showed me several conjectures about the number of standard tableaux that fall in a catabolism type when ranked by charge. Since the standard tableaux classes that we have defined here are generalizations for the catabolism type, it seems likely that the same conjectures will hold true for these classes. Again we let $s \in \left\{\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}\right\}^a \times \left\{\begin{array}{|c|}\hline 1\\\hline\end{array}\right\}^b$ then define the symbol $A_s^i = \#\{T | T \in STC^s, a_{(2^a1^b)}(T) = i\}$.

Conjecture 4.8 *The sequence $A_s^* = (A_s^0, A_s^1, A_s^2, \ldots)$ is a unimodal sequence (that is, it increase and then decreases).*

Example 4.9

$$\mu = (2^3) \qquad s = \left(\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}, \begin{array}{|c|c|}\hline 1&2\\\hline\end{array}, \begin{array}{|c|c|}\hline 1&2\\\hline\end{array}\right) \qquad A_s^* = (1,1,2,3,2,1,1)$$

$$s = \left(\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}, \begin{array}{|c|c|}\hline 1&2\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}\right) \qquad A_s^* = (1,2,3,2,1)$$

$$s = \left(\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}\right) \qquad A_s^* = (1,2,3,2,1)$$

$$s = \left(\begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}\right) \qquad A_s^* = (1,1,2,3,2,1,1)$$

$$\mu = (2^21^2) \qquad s = \left(\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}, \begin{array}{|c|c|}\hline 1&2\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline\end{array}^2\right) \qquad A_s^* = (1,3,4,4,4,2,1,1)$$

$$s = \left(\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline\end{array}^2\right) \qquad A_s^* = (1,2,4,4,4,2,1)$$

$$s = \left(\begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline\end{array}^2\right) \qquad A_s^* = (1,1,2,4,4,4,3,1)$$

$$\mu = (2^1 1^4) \qquad s = \left(\boxed{1}\boxed{2}, \boxed{1}^4\right) \qquad A_s^* = (1,2,4,5,7,6,5,4,2,1,1)$$

$$s = \left(\begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix}, \boxed{1}^4\right) \qquad A_s^* = (1,1,2,4,5,6,7,5,4,2,1)$$

$$\mu = (1^6) \qquad s = \left(\boxed{1}^6\right) \qquad A_s^* = (1,1,2,4,5,7,9,9,9,9,7,5,4,2,1,1)$$

We list here the A_s^* sequences for only the classes $\left(\boxed{1}\boxed{2}^l, \begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix}^{a-l}, \boxed{1}^b\right)$ since the other classes are isomorphic to these. By the observations from Proposition 4.2 we know that for $s \in \left\{\boxed{1}\boxed{2}, \begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix}\right\}^a$ we have that the sequence $A^*_{\left(s,\boxed{1}^{b+2}\right)}$ can be calculated from the sequences $A^*_{\left(s,\boxed{1}\boxed{2},\boxed{1}^b\right)}$ and $A^*_{\left(s,\begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix},\boxed{1}^b\right)}$ since $A^i_{\left(s,\boxed{1}^{b+2}\right)} = A^{i-b-1}_{\left(s,\boxed{1}\boxed{2},\boxed{1}^b\right)} + A^i_{\left(s,\begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix},\boxed{1}^b\right)}$.

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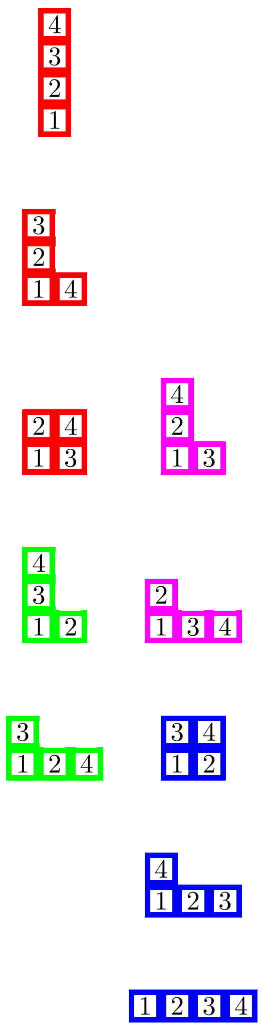

Figure 1: Charge Poset for $n = 4$

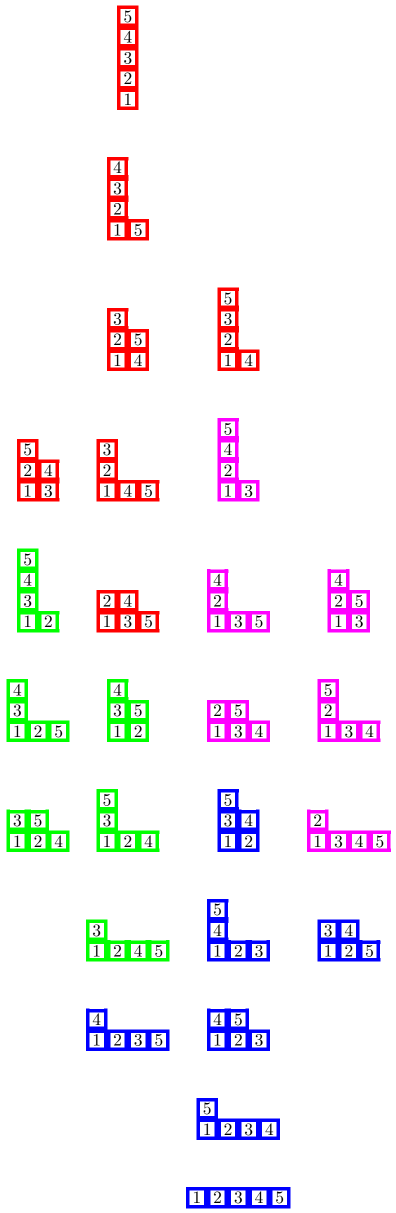

Figure 2: Charge Poset for $n = 5$

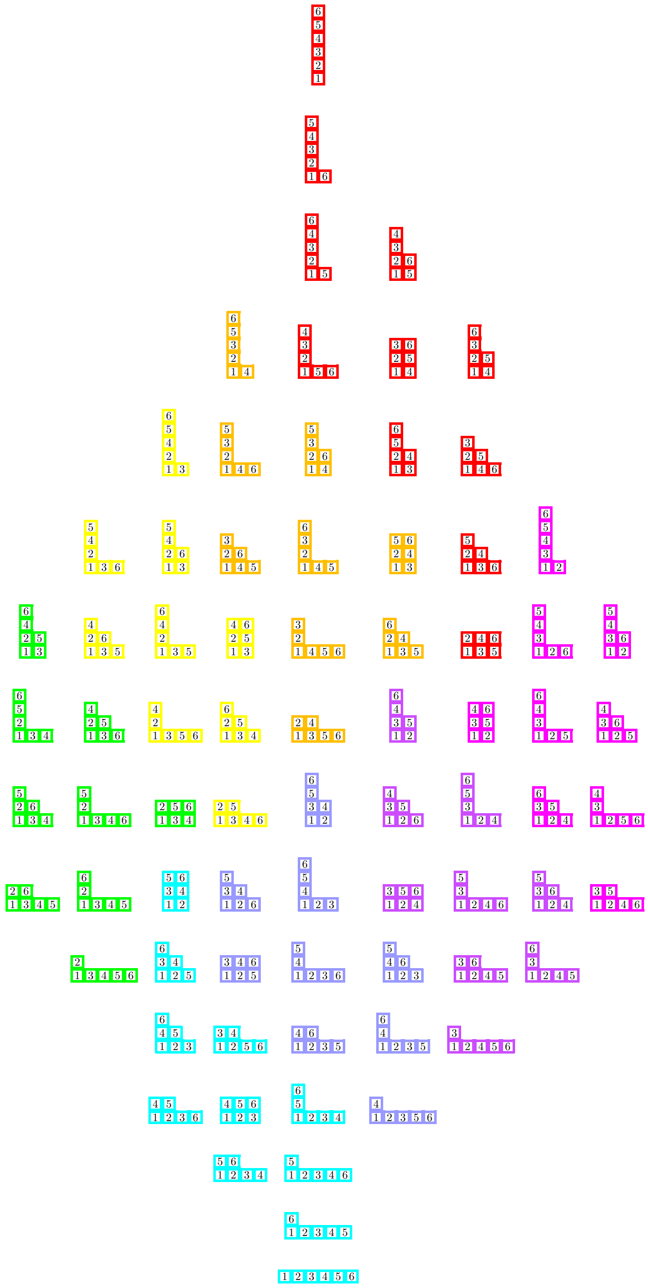

Figure 3: Charge Poset for $n = 6$