An exploration of the permanent-determinant method

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Abstract

The permanent-determinant method and its generalization, the Hafnian-Pfaffian method, are methods to enumerate perfect matchings of plane graphs that were discovered by P. W. Kasteleyn. We present several new techniques and arguments related to the permanent-determinant with consequences in enumerative combinatorics. Here are some of the results that follow from these techniques:

1. If a bipartite graph on the sphere with $4n$ vertices is invariant under the antipodal map, the number of matchings is the square of the number of matchings of the quotient graph.

2. The number of matchings of the edge graph of a graph with vertices of degree at most 3 is a power of 2.

3. The three Carlitz matrices whose determinants count $a \times b \times c$ plane partitions all have the same cokernel.

4. Two symmetry classes of plane partitions can be enumerated with almost no calculation.

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The permanent-determinant method and its generalization, the Hafnian-Pfaffian method, is a method to enumerate perfect matchings of plane graphs that was discovered by P. W. Kasteleyn [18]. Given a bipartite plane graph Z, the method produces a matrix whose determinant is the number of perfect matchings of Z. Given a non-bipartite plane graph Z, it produces a Pfaffian with the same property. The method can be used to enumerate symmetry classes of plane partitions [21, 22] and domino tilings of an Aztec diamond [45] and is related to some recent factorizations of the number of matchings of plane graphs with symmetry [5, 15]. It is related to the Gessel-Viennot lattice path method [12], which has also been used to enumerate plane partitions [2,38]. The method could lead to a fully unified enumeration of all ten symmetry classes of plane partitions. It may also lead to a proof of the conjectured q-enumeration of totally symmetric plane partitions.

In this paper, we will discuss some basic properties of the permanent-determinant method and some simple arguments that use it. Here are some original results that follow from the analysis:

1. If a bipartite graph on the sphere with $4n$ vertices is invariant under the antipodal map, the number of matchings is the square of the number of matchings of the quotient graph.

2. The number of matchings of the edge graph of a graph with vertices of degree at most 3 is a power of 2.

3. The three Carlitz matrices whose determinants count $a \times b \times c$ plane partitions all have the same cokernel.

4. Two symmetry classes of plane partitions can be enumerated with almost no calculation. (This result was independently found by Ciucu [5]).

The paper is largely written in the style of an expository, emphasizing techniques for using the permanent-determinant method rather than specific theorems that can be proved with the techniques. Here is a summary for the reader interested in comparing with previously known results: Sections I, II, and III are a review of well-known linear algebra and results of Kasteleyn, except for III A and III B, which are new. Sections IV, V, and VI are mostly new. Parts of Section IV were discovered independently by Regge and Rasetti, Jockusch, Ciucu, and Tesler. Obviously the Gessel-Viennot method, the Ising model, and tensor calculus themselves are due to others. Section VII consists entirely of new and independently discovered results about plane partitions. Finally Section VIII is strictly a historical survey.

A Acknowledgements

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I Graphs and determinants

A *sign-ordering* of a finite set is a linear ordering chosen up to an even permutation. Given two disjoint sets A and B, a bijection $f : A \to B$ induces a sign-ordering of $A \cup B$ as follows. Order the elements of A arbitrarily, and then list

$$a_1, f(a_1), a_2, f(a_2), \ldots .$$

More generally, an oriented matching of a finite set A, meaning a partition of A into ordered pairs, induces a sign-ordering of A by the same construction. A sign-ordering of $A \cup B$ is also equivalent to a linear ordering of A and a linear ordering of B, chosen up to simultaneous odd or even permutations, by choosing f to be order-preserving.

Let Z be a weighted bipartite graph with black and white vertices, where the weights of the edges lie in some field $\mathbb{F}$. (Usually $\mathbb{F}$ will will be $\mathbb{R}$ or $\mathbb{C}$.) The graph Z has a *weighted, bipartite adjacency matrix*, $M(Z)$, whose rows are indexed by the black vertices of Z and whose columns are indexed by the white vertices. The matrix entry $M(Z)_{v,w}$ is the total weight of all edges from v to w. If the vertices of Z are sign-ordered, then $\det(M(Z))$ is well-defined (and taken to be 0 unless $M(Z)$ is square). By abuse of notation, we define

$$\det(Z) = \det(M(Z)).$$

The sign of $\det(Z)$ is determined by choosing linear orderings of the rows and columns compatible with the sign-ordering of Z. If the vertices are not sign-ordered, the absolute determinant $|\det(Z)|$ is still well-defined.

Just as matrices are a notation for linear transformations, a weighted bipartite graph Z can also denote a linear transformation

$$L(Z) : \mathbb{F}[B] \to \mathbb{F}[W].$$

Here B is the set of black vertices, W is the set of white vertices, and $\mathbb{F}[X]$ denotes the set of formal linear combinations of elements of X with coefficients in $\mathbb{F}$. The map $L(Z)$ is the one whose matrix is $M(Z)$. Note that Z is not uniquely determined by $L(Z)$: if Z has multiple edges, the linear transformation only depends on the sum of the weights of these edges. If Z has an edge with weight 0, the edge is synonymous with an absent edge. Row and column operations on $M(Z)$ can be viewed as operations on Z itself modulo these ambiguities.

These observations also hold for weighted, oriented non-bipartite graphs. Given such a graph Z, the *antisymmetric adjacency matrix* $A(Z)$ has a row and column for every vertex of Z. The matrix entry $A(Z)_{v,w}$ is the total weight of all edges from v to w minus the total weight of edges from w to v. This matrix has a Pfaffian $\mathrm{Pf}(A(Z))$ whose sign is well-defined if the vertices of Z are sign-ordered. We also define

$$\mathrm{Pf}(Z) = \mathrm{Pf}(A(Z)).$$

Recall that the Pfaffian $\mathrm{Pf}(M)$ of an antisymmetric matrix M is a sum over matchings in the set of rows of M. The sign of the Pfaffian depends on a sign-ordering of the rows of M. In these respects, the Pfaffian generalizes the determinant. The Pfaffian also satisfies the relation

$$\det(M) = \mathrm{Pf}(M)^2. \tag{1}$$

This relation has a bijective proof: If M is antisymmetric, the terms in the determinant indexed by permutations with odd-length cycles vanish or cancel in pairs. The remaining terms are bijective with pairs of matchings of the rows of M, and the signs agree. This argument, and the permanent-determinant method generally, blur the distinction between bijective and algebraic proofs in enumerative combinatorics.

In particular,

$$\det(Z) = \mathrm{Pf}(Z)$$

when Z is bipartite if all edges are oriented from black to white. (If this seems inconsistent with equation (1), recall that the implicit matrix on the right, $A(Z)$, has two copies of the one on the left, $M(Z)$.) If Z has indeterminate weights, the polynomial $\det(Z)$ or $\mathrm{Pf}(Z)$ has one term for each perfect matching m of Z. The term may be written

$$t(m) = (-1)^m \omega(m),$$

where $(-1)^m$ is the sign of m relative to the sign-ordering of vertices of Z, and $\omega(m)$ is the product of the weights of edges of m. Thus $\det(Z)$ or $\mathrm{Pf}(Z)$ for an arbitrary graph Z is a basic object in enumerative combinatorics.

II The permanent-determinant method

Let Z be a connected, bipartite planegraph. By planarity we mean that Z is embedded in an (oriented) sphere. The faces of Z are disks; together with the edges and vertices they form a cell structure, or CW complex, on the sphere. Since the sphere is oriented, each face is oriented. The edges of Z have a preferred orientation, namely the one in which all edges point from black to white. The Kasteleyn curvature (curvature for short) of Z at a face F is defined as

$$c(F) = (-1)^{|F|/2+1} \frac{\prod_{e \in F_+} \omega(e)}{\prod_{e \in F_-} \omega(e)},$$

where $|F|$ is the number of edges in F, F_+ is the set of edges whose orientation agrees with the orientation of F, and F_- is the set of edges whose orientation disagrees with that of F. (Each face inherits its orientation from that of the sphere.) A face F is *flat* if $c(F)$ is 1. See Figure 1.

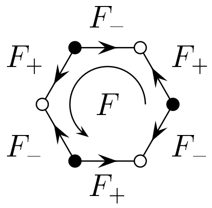

Figure 1: Computing Kasteleyn curvature.

Theorem 1 (Kasteleyn). *If Z is unweighted, a flat weighting exists.*

The theorem depends on the following lemma.

Lemma 2. *If Z has an even number of vertices, and in particular if black and white vertices are equinumerous, then there are an even number of faces with $4k$ sides.*

Proof. Let n_V, n_E, and n_F be the number of vertices, edges, and faces of Z, respectively. The Euler characteristic equation of the sphere is

$$\chi = n_F - n_E + n_V = 2.$$

The term n_V is even. Divide the contribution to n_E from each edge, namely -1, evenly between the two incident faces. Then the contribution to $n_F - n_E$ of a face with $4k$ sides is an odd integer, while the contribution of a face with $4k+2$ sides is an even integer. Therefore there are an even number of the former. $\square$

Proof of theorem. Consider the cohomological chain complex of the cell structure given by Z with coefficients in the multiplicative group $\mathbb{F}^*$. (Since it may be confusing to consider homological algebra with multiplicative coefficients, we will sometimes denote a "sum" of $\mathbb{F}^*$-cochains as $a \dot{+} b$.) Consider the same orientations of the edges and faces of Z as above. With these orientations, we can view a function from n-cells to $\mathbb{F}^*$ as an n-cochain. In particular, a weighting ω of Z is equivalent to a 1-cochain. Let ω_k, the Kasteleyn cochain, be a 2-cochain which assigns $(-1)^{|F|/2+1}$ to each face f. The coboundary $\delta\omega$ of ω is related to the curvature by

$$c(F) = \omega_k \dot{+} \delta\omega.$$

Thus, a flat weighting exists if and only if ω_k is a coboundary. By the lemma, ω_k has an even number of faces with weight -1 and the rest have weight 1. Thus, ω_k represents the trivial second cohomology class of the sphere. Therefore it is a coboundary. $\square$

Following the terminology of the proof, the curvature of any weighting is a coboundary, because it is the sum (in the sense of "$\dot{+}$") of two coboundaries, ω_k and the coboundary of the weighting. Thus the product of all curvatures of all faces is 1.

Theorem 3 (Kasteleyn). *If Z is flat, the number of perfect matchings is $\pm\det(Z)$, because $t(m)$ has the same sign for all m.*

A complete proof is given in Reference 21, but the result also follows from a more general result.

By a *loop* we mean a collection of edges of Z whose union is a simple closed curve. If the loop ℓ is the difference between two matchings m_1 and m_2, then all edges of ℓ point in the same direction if we reverse the edges of $\ell \cap m_2$. Of the two regions of the sphere separated by ℓ, the positive one is the one whose orientation agrees with ℓ.

Theorem 4. *If m_1 and m_2 are two matchings of Z that differ by one loop ℓ, the ratio of their terms $t(m_1)/t(m_2)$ in the expansion of $\det(Z)$ equals the product of the curvatures of the faces on the positive side of ℓ.*

Proof. The loop ℓ has an even number of sides and also must enclose an even number vertices on the positive side S_+. If we remove the vertices and edges on the negative side S_-, we obtain a new graph Z' such that the loop ℓ bounds a face F that replaces S_-. Since the total curvature of all faces of Z' is 1, the curvature of F is the reciprocal of the total curvature of all other faces. Finally,

$$\begin{aligned} c(F) &= (-1)^{|F|/2+1} \frac{\prod_{e \in F_+} \omega(e)}{\prod_{e \in F_-} \omega(e)} \\ &= (-1)^{m_1}(-1)^{m_2} \frac{\prod_{e \in \ell \cap m_2} \omega(e)}{\prod_{e \in \ell \cap m_1} \omega(e)} \\ &= \frac{t(m_2)}{t(m_1)}. \end{aligned}$$

The signs agree because m_1 and m_2 differ by an even cycle, which is an odd permutation if and only if F has $4k$ sides. □

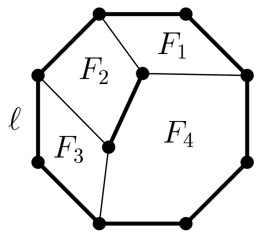

Figure 2: A loop enclosing four faces.

Figure 2 illustrates the proof of Theorem 4. The loop encloses four faces. Edges in bold appear in at least one of two terms m_1 and m_2 that differ by ℓ. The theorem in this case says that

$$\frac{t(m_2)}{t(m_1)} = c(F_1)c(F_2)c(F_3)c(F_4).$$

In light of Theorem 4, if Z is an unweighted graph, a curvature function and a reference matching m are enough to define $\det(Z)$, because we can choose a weighting and a sign-ordering with the desired curvature and such that $t(m) = 1$. The matrix $M(Z)$ will then have the following ambiguity. In general, if ω_1 and ω_2 are two weightings with the same curvature, then $\omega_1 \dotplus \omega_2^{-1}$ is a 1-cocycle. Since the first homology of the sphere is trivial, the ratio is a 1-coboundary, *i.e.*, ω_1 and ω_2 differ by a 0-cochain. The corresponding matrices $M(Z_{\omega_1})$ and $M(Z_{\omega_2})$ then differ by multiplication by diagonal matrices on the left and the right.

III The Hafnian-Pfaffian method

Kasteleyn's method for non-bipartite plane graphs expresses the number of perfect matchings as a Pfaffian. For simplicity, we consider unweighted, oriented graphs. The analysis has a natural generalization to weighted graphs in which the orientation is completely separate from the weighting. The *curvature* of an orientation at a face is 1 if an odd number of edges point clockwise around the face and -1 otherwise. The orientation is flat if the curvature is 1 everywhere. A routine generalization of Theorem 3 shows that if Z has a flat orientation, $\mathrm{Pf}(Z)$ is the number of matchings [18].

A graph Z, whether planar or not, is *Pfaffian* if it admits an orientation such that all terms in $\mathrm{Pf}(Z)$ have the same sign [24].

If Z is bipartite, orienting Z is equivalent to giving each edge a 1 if it points to black to white and -1 otherwise. The weighting is flat if and only if the orientation is flat.

Theorem 5 (Kasteleyn). *If Z is an unoriented plane graph, a flat orientation exists.*

In particular, planar graphs are Pfaffian.

Proof. The proof follows that of Theorem 1. Fix an orientation o, and again consider the mod 2 Euler characteristic of the sphere. Ignoring the vertices, we transfer the Euler characteristic of each edge to the incident face whose orientation agrees with that of the edge. The net Euler characteristic of a face is then 0 if it is flat and 1 if it is not, therefore there must be an even number of non-flat faces. Let k be the curvature of o.

The Euler characteristic calculation shows that k is a coboundary of a 1-cochain c with coefficients in the multiplicative group $\{\pm 1\}$. Let $o' = c \dotplus o$ be the "sum" of c and o, defined by the rule that o' and o agree on those edges where c is 1 and disagree where c is -1. Then o' is flat. □

In the same vein, suppose that Z and Z' are the same graph with two different flat orientations. By homology considerations, the "difference" of the two orientations (1 where they agree, -1 where they disagree) is a 1-coboundary. Thus they differ by the coboundary of a 0-cochain c, which is a function from the vertices to $\{\pm 1\}$. Let D be the diagonal matrix whose entries are the values of c. Then $A(Z)$, $A(Z')$, and D satisfy the relation

$$A(Z) = DA(Z')D^T.$$

Note that c and D are unique up to sign.

III A Spin structures

We conclude with some comments about flat orientations of a graph Z on a surface of genus g. Kasteleyn [18] proved that the number of matchings of such a graph is given by a sum of 4^g Pfaffians defined using inequivalent flat orientations of Z. (See also Tesler [40].) There is an interesting relationship between these flat orientations and spin structures. A spin structure on a surface is determined by a vector field with even-index singularities. We can make such a vector field using an orientation and a matching m. At each vertex, make the vectors point to the vertex. Then replace each edge by a continuous family of edges such that in the middle of the edge, the vector field is 90 degrees clockwise relative to the orientation of the edge Figure 3 shows this operation applied to the four edges of a square.

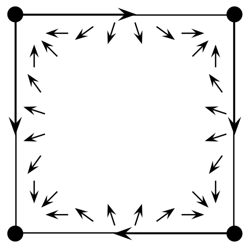

Figure 3: Vectors describing a spin structure.

Because the orientation is flat, the vector field extends to the faces with even-index singularities, but the singularities at the vertices are odd. Contract the odd-index singularities in pairs along edges of the matching; the resulting vector field induces a spin structure. For a fixed orientation, inequivalent matchings yield distinct spin structures. Here two matchings are equivalent if they are homologous. For a fixed matching, two inequivalent orientations yield distinct spin structures.

III B Projective-plane graphs

An expression for the number of matchings of a non-planar graph may in general require many Pfaffians. But there is an interesting near-planar case when a single Pfaffian suffices.

A graph is a *projective-plane graph* if it is embedded in the projective plane. A graph embedded in a surface is *locally bipartite* if all faces are disks and have an even number of sides. It is *globally bipartite* if it is bipartite. If Z is locally but not globally bipartite, then it has a non-contractible loop, but all non-contractible loops have odd length while all contractible loops have even length.

Theorem 6. *If Z is a connected, projective-plane graph which is locally but not globally bipartite, then it is Pfaffian.*

Proof. Assume that Z has an even number of vertices.

The curvature of an orientation of Z is well-defined even though the projective plane is non-orientable: Since each face has an even number of sides, the curvature is 1 if an odd number of edges point in both directions and -1 otherwise. If the curvature of an arbitrary orientation o is a coboundary, meaning that an even number of faces have curvature -1, then there is a flat orientation by the homology argument of Theorem 5.

To prove that the curvature of o must be a coboundary, we cut along a non-contractible loop ℓ, which must have odd length, to obtain an oriented plane graph Z'. Every face of Z becomes a face of Z', and in addition Z' has an outside face with $2|\ell|$ sides. A face in Z' has the same curvature as in Z assuming that it is a face of both graphs. The graph Z' has an odd number of vertices, because it has $|\ell|$ more vertices than Z does. By the argument of Theorem 5, Z' has an odd number of faces with curvature -1. Moreover, the outside face is one of them, because each of the edges of ℓ appears twice, both times pointing either clockwise or counterclockwise. Therefore Z must have an even number of faces with curvature -1.

Finally, we show that a flat orientation of Z is in fact Pfaffian. Let m_1 and m_2 be two matchings that differ by a single loop. Since the loop has even length, it is contractible. By the usual argument, the ratio $t(m_1)/t(m_2)$ of the corresponding terms in the expansion of $\mathrm{Pf}(A(Z))$ equals the product of the curvatures of the faces that the loop bounds. Since Z is flat, this product is 1. $\square$

IV Symmetry

IV A Generalities

Let V be a vector space over $\mathbb{C}$, the complex numbers. If a linear transformation $L : V \to V$ commutes with the action of a reductive group G, then $\det(L)$ factors according to the direct sum decomposition of V into irreducible representations of G. At the abstract level, for each distinct irreducible representation R, we can make a vector space V_R such that $V_R \otimes R$ is an isotypic summand of V, and there exist isotypic blocks

$$L_R : V_R \to V_R$$

such that

$$L = \bigoplus_R (L_R \otimes I_R),$$

where I_R is the identity on R. Then

$$\det(L) = \prod_R \det(L_R)^{\dim R}. \tag{2}$$

More concretely, if M is a matrix and a group G has a matrix representation ρ such that

$$\rho(g)M = M\rho(g),$$

then after a change of basis, M decomposes into blocks, with $\dim R$ identical blocks of some size for each irreducible representation R of G, so its determinant factors.

Suppose that L is an endomorphism of some integral lattice X in V (concretely, if M is an integer matrix) and R is some rational representation. After choosing a rational basis $\{r_i\}$ for R, we can realize copies V_{r_i} of V_R as rational subspaces of V. The lattice L preserves each V_{r_i} and acts acts on it as L_R. Then $X \cap V_{r_i}$ is a lattice in V_{r_i}, and L is an endomorphism of this lattice as well. The conclusion is that each $\det(L_R)$ must be an integer because L_R is an endomorphism of a lattice. Indeed, this argument works for any number field (such as the Gaussian rationals) and its ring of integers (such as the Gaussian integers) if R is not a rational representation, which tells us that equation 2 is in general a factorization into algebraic integers if L is integral. The determinant $\det(L_R)$ is, a priori, in the same field as the representation R. A refinement of the argument shows that it is in the same field as the character of R, which may lie in a smaller field than R itself.

The general principle of factorization of determinants applies to enumeration of matchings in graphs with symmetry via the Hafnian-Pfaffian method. As discussed in Sections I and III, an oriented graph Z yields an antisymmetric map

$$A(Z) : \mathbb{C}[Z] \to \mathbb{C}[Z].$$

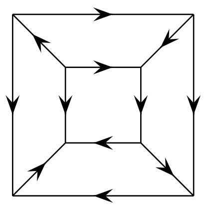

Figure 4: A Pfaffian orientation with broken symmetry.

Any symmetry of the oriented graph intertwines this map, and the factorization principle applies. However there are three complications. First, the orientation may have less symmetry than the graph itself (Figure 4). Second, the principle of factorization gives information about the determinant and not the Pfaffian. (If a summand R of an orthogonal representation V is orthogonal and L is an antisymmetric endomorphism of V, then the factor $\det(L_R)$ is the square of $\mathrm{Pf}(L_R)$. But if R is symplectic or complex, then $\det(L_R)$ need not be a square. In these cases the factorization principle is less informative.) Third, the number of matchings might only factor into algebraic integers, which is less informative than a factorization into ordinary integers.

Let Z be a connected plane graph, and suppose that a group G acts on the sphere and preserves Z and the orientation of the sphere. Then G acts by permutation matrices on $V = \mathbb{C}[Z]$, the vector space generated by vertices of Z. Although G commutes with the adjacency matrix of Z, it does not in general commute with the antisymmetric adjacency matrix $A(Z)$ if Z is oriented, because G might not preserve the orientation of Z. However, if Z has a flat orientation o and $g \in G$, then go differs from o by a coboundary in the sense of Sections II and III. This means that there is a signed permutation matrix $\widetilde{g}$ which does commute with $A(Z)$. These signed permutation matrices together form a linear representation of some group $\widetilde{G}$ which extends G. At first glance it may appear as if the fiber of this extension is as big as $\{\pm 1\}^{|Z|}$. But because Z is connected, among diagonal signed matrices only the identity and its negative commute with $A(Z)$; only constant 0-cochains leave alone the orientation of every edge of Z. Therefore $\widetilde{G}$ is a central extension of G by the two-element group $G_0 = \{\pm 1\}$. The subgroup G_0 either acts trivially on V or negates it. Thus, in decomposing V into irreducible representations, we need only consider those where G_0 acts trivially (by definition the *even* representations) or only those where G_0 acts by negation (by definition the *odd* representations), depending on whether the action on V is even or odd.

If G has odd order, the central extension must split. In this case, by averaging, Z has a flat orientation invariant under G [41]. If G a cyclic group of rotations of order $2n$, then the central extension might not split, but it is not very interesting as an extension; if Z is bipartite, one can find a flat weighting using $4n$th roots of unity which is invariant under G [15]. But in the most complicated case, when Z has icosahedral symmetry, the central extension $\widetilde{G}$ (when it is non-trivial) is the binary icosahedral group $\widetilde{A_5}$, which is quite interesting. This central extension seems related to the connection between flat orientations and spin structures mentioned in Section III, because the symmetry group of a spun sphere is an analogous central extension of $SO(3)$. Irrespective of G, the representation theory of $\widetilde{G}$ reveals a factorization of the number of matchings of Z.

If Z is bipartite, then there are two important changes to the story. First, after including signs, one can make orientation-reversing symmetries commute with $A(Z)$ as well, because in the bipartite case they take flat orientations to flat orientations. If Z is not bipartite, the best signed versions of orientation-reversing symmetries instead anticommute with $A(Z)$. Anticommutation is less informative than commutation, but they still sometimes provide information together with the following fact from linear algebra: If A and B anticommute and B is invertible, then

$$\mathrm{tr}(A^n) = \mathrm{tr}(-A^n) = 0$$

for n odd, because

$$-A^n = B^{-1}A^nB.$$

Second, if Z is bipartite, then color-reversing symmetries yield no direct information via representation theory. In this case, it is better to apply representation theory to the bipartite adjacency matrix $M(Z)$. This matrix represents a linear map

$$L : \mathbb{C}[B] \to \mathbb{C}[W],$$

where B is the set of black vertices and W is the set of white vertices, rather than a linear endomorphism of a single space. The color-preserving symmetries act on both $V = \mathbb{C}[B]$ and $U = \mathbb{C}[W]$ and $M(Z)$ intertwines these actions. Hence for each irreducible R there is an isotypic block

$$L_R : V_R \to U_R.$$

Hence for each R we must choose volume elements on V_R and U_R so that $\det L_R$ is well-defined. Nevertheless, equation 2 still holds if it is properly interpreted.

IV B Cyclic symmetry

Suppose that G is generated by a single rotation g of order n, and let ω be an nth root of unity when n is odd or G fixes a vertex (the split case), or an odd $2n$th root of unity when n is even and G does not fix a vertex (the non-split case). Then the vertex space $\mathbb{C}[Z]$ has an isotypic summand $\mathbb{C}[Z]_\omega$. A suitable set of vectors of the form

$$v + \omega g v + \ldots + \omega^n g^n v,$$

where v is a vertex of Z, form a basis of $\mathbb{C}[Z]_\omega$ (assuming certain sign conventions in the non-split case). In this case the isotypic blocks of $A(Z)$ are all represented by weighted plane graphs whose Kasteleyn curvature can be easily derived from that of Z [15].

If Z is bipartite, then a reflection symmetry produces a similar factorization, and again the resulting matrices are represented by plane graphs [5].

The other possibility for a color-preserving cyclic symmetry is a glide-reflection. In particular, Z may be invariant under the antipodal map on the sphere. In this case, the blocks of $M(Z)$ cannot be represented by plane graphs. Instead, they produce projective-plane graphs. So the number of matchings factors, but the permanent-determinant method does not identify either factor as an unweighted enumeration.

IV C Color-reversing symmetry

If Z is bipartite, then a color-reversing symmetry does not a priori lead to an interesting factorization of the number of matchings of Z. For example, if the symmetry is an involution, then if we use it to establish a bijection between black and white vertices, we learn only that $M(Z)$ is symmetric, which says little about its determinant. However, color-reversing symmetries do have two interesting and related consequences.

First, if Z has color-reversing and color-preserving symmetries, the color-reversing symmetries sometimes imply that the numerical factors of $\det(Z)$ from the color-preserving symmetries lie in smaller-than-expected number-field rings. For example, if Z has a color-reversing 90-degree rotational symmetry g, then the symmetry g^2 yields

$$\det(Z) = \det(Z_i)\det(Z_{-i}),$$

where $Z_{\pm i}$ is the quotient graph Z/g^2 with curvature $\pm i$ at the faces fixed by g^2. (Recall that Z is on the sphere, so there are two fixed faces.) Then the remaining symmetry tells us that Z_i and Z_{-i} have a curvature-preserving isomorphism, which means that their determinants are equal up to a unit in the Gaussian integers. At the same time, their determinants are complex conjugates. Thus, $\det(Z_i)$ is, up to a unit, in the form a or $(1+i)a$ for some rational integer a. The conclusion is that $\det(Z)$ is either a square or twice a square [15]. Similarly, if Z has a color-reversing 60-degree symmetry,

$$\det(Z) = ab^2$$

for some integers a and b coming from enumerations in quotient graphs.

Second, if Z has a color-reversing involution g which does not fix any edges, then the antisymmetric adjacency matrix $A(Z/g)$ of the quotient graph Z/g can be interpreted as the bipartite adjacency matrix of Z with some weighting. Since the determinant is the square of the Pfaffian,

$$\det(Z) = \mathrm{Pf}(Z/g).$$

If Z/g is flat (which implies that Z/g has an even number of vertices), then Z with its induced weighting may or may not be flat, depending on g. Assuming Z is connected, g may be the antipodal involution in the sphere or it may be rotation by 180 degrees. In the first case, Z is flat, while Z/g is projective-plane graph which is locally but not globally bipartite. In the second case, Z is not flat, but has curvature -1 at the two faces fixed by rotation. The first case is a new theorem:

Theorem 7. *If Z is a bipartite graph on the sphere with $4n$ vertices which is invariant under the antipodal involution g, and if g exchanges colors of vertices of Z, then the number of matchings of Z is the square of the number of matchings of Z/g.*

For example, the surface of a Rubik's cube satisfies these conditions (Figure 5).

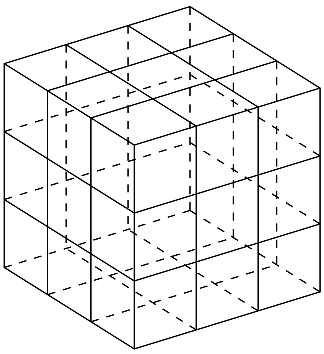

Figure 5: The Rubik's cube graph.

Exercise. Prove Theorem 7 with an explicit bijection.

This exercise is a special case of the bijective argument that

$$\det(M) = \mathrm{Pf}(M)^2$$

for any antisymmetric matrix M.

IV D Icosahedral symmetry

The results of this section were discovered independently by Rasetti and Regge [32].

The easiest realization of the binary icosahedral group $\widetilde{A_5}$ is as the subgroup of the unit quaternions $a + b\vec{\imath} + c\vec{\jmath} + d\vec{k}$ for which (a, b, c, d) is one of the points

$$\frac{1}{2}(\tau, 1, \frac{1}{\tau}, 0) \qquad (1, 0, 0, 0) \qquad \frac{1}{2}(1, 1, 1, 1)$$

or the points obtained from these by changing signs or even permutations of coordinates. Here τ is the golden ratio. Note that two elements of $\widetilde{A_5}$ are conjugate if and only if they have the same real part a.

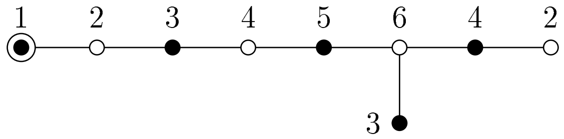

Figure 6: Extended E_8, a graph of representations of $\widetilde{A_5}$.

This realization also describes a two-dimensional representation π of $\widetilde{A_5}$. The character of π is twice the real parts of the elements of $\widetilde{A_5}$. By the McKay correspondence, the irreducible representations of $\widetilde{A_5}$ together form an E_8 graph, where the trivial representation is the extending vertex, and two representations R and R' are joined by an edge if $R \otimes \pi$ contains R' as a summand. This diagram is given in Figure 6 together with the dimensions of the representations. The trivial representation is circled. A black vertex is an even representation in the sense of Section IV A, while a white vertex is an odd representation. This graph can be used to compute the character table of $\widetilde{A_5}$, which is given in Table 1. In this table,

$$\overline{\tau} = -\frac{1}{\tau}$$

is the Galois conjugate of τ. The table indicates various properties of the representations. The conjugacy class c_0 contains only 1, so its row is the trace of the identity or the dimension of each representation. The conjugacy class c_8 contains only -1, so its row indicates which representations are even and which are odd. The representation R_1 equals the defining representation π. Apparently, five of the characters are rational, while the other four lie in the golden field $\mathbb{Q}(\tau)$. Less superficially, the character table can be used to find the direct sum decomposition of an arbitrary representation from its character, or to decompose an equivariant map into its isotypic blocks.

Suppose that a graph Z has icosahedral symmetry. If a rotation by 180 degrees fixes a vertex of Z, then the action of $\widetilde{A_5}$ on $\mathbb{C}[Z]$ is even, but if such a rotation fixes an edge or a face, then the action is odd (exercise). In the second case, the factorization principle says that $\det(A(Z))$ is in the form $a^2\overline{a}^2b^4c^6$, where b and c are integers and a and $\overline{a}$ are conjugate elements in $\mathbb{Z}(\tau)$, because the available representations have

χ	R_0	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8
c_0	1	2	3	4	5	6	4	2	3
c_1	1	τ	τ	1	0	-1	-1	$\overline{\tau}$	$\overline{\tau}$
c_2	1	1	0	-1	-1	0	1	1	0
c_3	1	$-\overline{\tau}$	$\overline{\tau}$	-1	0	1	-1	$-\tau$	τ
c_4	1	0	-1	0	1	0	0	0	-1
c_5	1	$\overline{\tau}$	$\overline{\tau}$	1	0	-1	-1	τ	τ
c_6	1	-1	0	1	-1	0	1	-1	0
c_7	1	$-\tau$	τ	-1	0	1	-1	$-\overline{\tau}$	$\overline{\tau}$
c_8	1	-2	3	-4	5	-6	4	-2	3

Table 1: Character table of $\widetilde{A_5}$

dimensions 2,2,4, and 6. Thus the number of matchings factors as $a^1\overline{a}^1b^2c^3$. In the first case, $\mathbb{C}[Z]$ decomposes entirely into orthogonal representations, which implies that

$$\mathrm{Pf}(Z) = a^1b^3\overline{b}^3c^4d^5$$

using the dimensions of the representations and the number fields in which they lie. Here are four interesting examples. In the first three, the author explicitly computed the factorization from symmetry by computing the trace of $gM(Z)^n$ for different g and n.

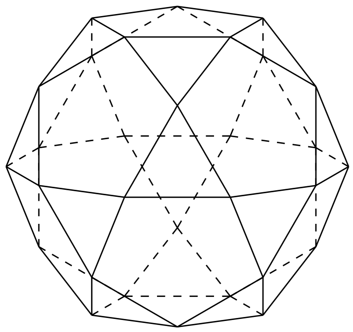

Figure 7: The edge graph of a dodecahedron.

1. If Z is an icosahedron, then $\mathbb{C}[Z]$ consists of two copies of the 6-dimensional representation R_6, which means that $A(Z)$ is, after a change of basis, six copies of a 2×2 matrix M. Moreover, $A(Z)$ anticommutes with antipodal inversion,

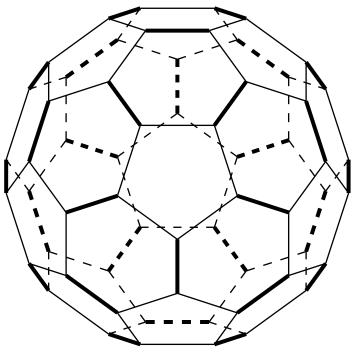

Figure 8: The C_{60} graph with two kinds of edges.

so M must have vanishing trace. The matrix M^2, and therefore $A(Z)$ must be a multiple of the identity. In fact,

$$A(Z)^2 = 5I.$$

Thus there are $5^3 = 125$ matchings.

2. If Z is a dodecahedron, then

$$\mathrm{Pf}(Z) = 36 = 1^3 6^2.$$

The space $\mathbb{C}[Z]$ has two 6-dimensional summands, which contribute the factors of 1, and two 4-dimensional summands, which contribute the factors of 6.

3. If Z is the edge graph of a dodecahedron or icosahedron (Figure 7), then

$$\mathrm{Pf}(Z) = (4 + 2\sqrt{5})^3 (4 - 2\sqrt{5})^3 1^4 2^5 = -2^{11}.$$

The space $\mathbb{C}[Z]$ has two of each of the non-trivial even representations.

4. Let Z be the bond graph of the fullerene C_{60} (Figure 8). Suppose that its hexagonal edges (the thicker ones in the figure) have weight 1 and its pentagonal edges (the thinner ones) have weight p. According to Tesler [42], the total weight of all matchings is

$$\begin{aligned}\mathrm{Pf}(Z) = {} & (1 - 2p + \frac{5+\sqrt{5}}{2}p^2)(1 - 2p + \frac{5-\sqrt{5}}{2}p^2) \\ & (1 + 2p + 2p^3 + 5p^4)^2 (1 + p^2 + 2p^3 + p^4)^3.\end{aligned}$$

In this case the space $\mathbb{C}[Z]$ is 60-dimensional and decomposes as two copies of each 2-dimensional representation, four copies of the odd 4-dimensional representation, and six copies of the 6-dimensional representation. The factors from the 2-dimensional representation are at most quadratic and the factors of the 4-dimensional representation are at most quartic. It follows that the factorization above coincidences with the factorization given by the symmetry principle.

V Gessel-Viennot

Some of the results in the section were independently discovered by Horst Sachs et al [1].

The Gessel-Viennot method is another method that counts combinatorial objects using determinants [12]. Let Z be a directed, plane graph with n sources (univalent vertices with outdegree 1) and n sinks (univalent vertices with indegree 1). Suppose further that the edges at each vertex are segregated, meaning that no four edges alternate in, out, in, out. Then the Gessel-Viennot method produces an $n \times n$ matrix whose determinant is the number of collections of n disjoint, directed paths in Z from the sources to the sinks. Each entry of the matrix is the number of paths from a source to a sink. The method has some ad hoc generalizations that produce Pfaffians [27,39]. It has been used to enumerate several classes of plane partitions [2, 38].

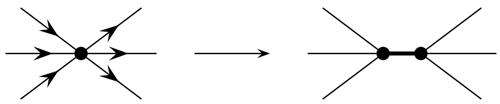

Figure 9: Transforming Gessel-Viennot to Kasteleyn.

Given a graph Z suitably decorated for the Gessel-Viennot method, there is a related graph Z' to which the permanent-determinant method applies. Namely, split each vertex of Z into an edge e, with all inward arrows at one end of e and all outward arrows at the other end (Figure 9). This operation induces a bijection betwen disjoint path collections in Z and matchings in Z'.

There is a corresponding relation between the Gessel-Viennot matrix $GV(Z)$ and a Kasteleyn matrix $M(Z')$. Define a *pivot operation* to be the act of replacing an $n \times n$ matrix of the form

$$\left(\begin{array}{c|c} M & v \\ \hline w & 1 \end{array}\right)$$

by $M - (v \otimes w)$, where v and w are vectors and M is an $(n-1) \times (n-1)$ matrix. The determinant does not change under pivot operations. Starting with $M(Z')$, if we pivot at all entries corresponding to edges in Z' which are contracted in Z, the result is the Gessel-Viennot matrix $GV(Z)$ with some rows and columns negated. This proves that $\det GV(Z)$ is the number of matchings of Z'. It also suggests that any enumeration derived using the Gessel-Viennot method can also be understood using the permanent-determinant method.

It was pointed out to the author by Mihai Ciucu that there is a version of the Gessel-Viennot method for arbitrary graphs, whether planar or not. In this case the transformation of Figure 9 still produces the relation

$$\det GV(Z) = \det Z',$$

where the right side is a weighted enumeration of matchings of Z' in our sense. However, this more general setting cannot be interpreted via Theorem 3.

V A Cokernels

An integer $n \times n$ matrix M can be interpreted as a homomorphism from $\mathbb{Z}^n$ to itself. The cokernel is defined as the target divided by the image:

$$\operatorname{coker} M = \mathbb{Z}^n / (\operatorname{im} M).$$

Alternatively, the cokernel is the abelian group on n generators whose relations are given by M. The cokernel is invariant under pivot operations, and the number of elements in the cokernel is $|\det(M)|$. Moreover, the cokernel of a Kasteleyn matrix $M(Z)$ depends only on the unweighted graph Z and not the particular choice of a flat weighting, and any corresponding Gessel-Viennot matrix also has the same cokernel.

Question 1. *Is there a natural bijection, or an algebraic generalization of a bijection, between the cokernel of $M(Z)$ and the set of matching of Z?*

For any integer matrix M, the cokernel of M^T is naturally the Pontryagin dual of the cokernel of M. In other words, there is a natural Fourier transform map

$$\Psi : \mathbb{C}[\operatorname{coker} M] \to \mathbb{C}[\operatorname{coker} M^T].$$

This map can be interpreted as a generalized bijection.

VI Other tricks

VI A Forcing planarity

Let Z be an arbitrary bipartite graph with some weighting. Then $\det(Z)$ is interesting as a weighted enumeration of the matchings of Z and as an algebraic quantity. At the same time, there is a simple way to convert Z to a plane graph without changing its determinant. Then ideas related to the permanent-determinant method apply. (Note that the number of matchings of Z in general will change. Together with the

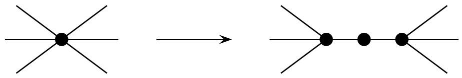

Figure 10: Vertex splitting.

assumption that $NP \neq P$, this would otherwise contradict the fact that the number of matchings for a general non-planar graph is a $\#P$-hard quantity [43].)

In such a graph Z, we can *triple* an edge, *i.e.*, replace it by three edges in series. If the weight of the original edge is w and the weights of three edges replacing it are a, b, and c, then $\det(Z)$ will not change if $w = ac$ and $b = -1$. Edge tripling is a special case of *vertex splitting*, a more general operation in which a vertex is replaced by two edges in series (Figure 10). Vertex splitting changes $\det(Z)$ by a factor of $\pm a$ if one of the new edges has weight a and the other has weight $-a$.

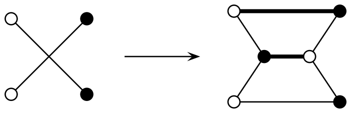

Figure 11: A crossing replaced by a butterfly.

Pick a projection of Z in the plane, *i.e.*, a drawing where edges are straight but may cross. After tripling the edges sufficiently, every edge crosses at most one other edge, and the convex hull of two crossing edges contain no vertices other than the endpoints of those edges. Then two edges that cross can be replaced by seven that do not cross according to Figure 11. Call the new subgraph a *butterfly*. If the top two horizontal edges of the butterfly (which are in bold in the figure) are given weight -1 and all other edges of the butterfly and the edges that cross have weight 1, then the operation of replacing the crossing by the butterfly can be reproduced by row and column operations on $M(Z)$. It does not change the determinant if the edges are given suitable weights. Thus, at the expense of more vertices and edges, Z becomes planar.

VI B The Ising model

The partition function of the unmagnetized Ising model on a graph is the following weighted enumeration. Given two numbers a and b, called Boltzmann weights, and given a graph Z, compute the total weight of all functions s (states) from the vertices of Z to the set of spins $\{\uparrow, \downarrow\}$, where the weight of a state s is the product of the weights of the edges, and the weight of an edge is a if the spins of its vertices agree and b if they disagree. In a generalization of the model, a and b can be different for different edges. If $a = b = 1$ for a given edge in this generalization, the edge can be ignored.

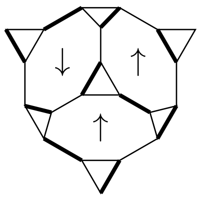

Figure 12: An Ising state and its matching.

If Z is a plane graph, we can make Z triangular, meaning that all faces are triangles, at the expense of adding ignorable edges. Let Z' be the dual graph with a vertex added in the middle of each edge. Let Z'' be the edge graph of Z'. Then (exercise) there is a 2-to-1 map from the Ising states of Z to the perfect matchings of Z''. The weights of Ising states can be matched up to a global factor by assigning weights to edges of Z''. Thus the Hafnian-Pfaffian method can be used to find the total weight of all Ising states of Z.

More generally, if Z is any plane graph whose vertices have valence 1, 2, or 3, then the number of matchings of the edge graph Z' of Z is a power of two. Indeed, the set of matchings can always be interpreted as an affine vector space over $\mathbb{Z}/2$. A matching of Z' is equivalent to an orientation of Z such that at each vertex, an odd number of edges are oriented outward (exercise). If the orientations of each individual edge are arbitrarily labelled 0 and 1, the constraints at the vertices are all linear. For example, the number of matchings of the edge graph of a dodecahedron (Section IV D) is a power of two. The bachelorhood vertex [21] for symmetric plane partitions is a similar construction.

VI C Tensor calculus

This section relates determinants of graphs to the formal setting of quantum link invariants [33]. We present no complete mathematical results, only a brief summary of how to discuss determinants of graphs in more algebraic terms. Another class of enumeration problems, planar ice and alternating-sign matrices, are related to the Jones polynomial and other quantum invariants based on the quantum group $U_q(\mathrm{sl}(2))$.

The Ising model is an example of a *state model*, a general scheme where one has a set of atoms, a set of states for each atom, local weights which depend on the states of particular clusters of atoms, and the global weight of a state which is defined as the product of all local weights. (Most weighted enumerations of objects such as matchings and plane partitions can be described as state models.) Many natural state models can be interpreted as tensor expressions. In index notation, one can write an expressions of tensors over a common vector space V such as

$$A^{ab}_c B^c_d C^d_b \ldots$$

Each index may only appear twice, once covariantly (as a subscript) and once contravariantly (as a superscript). Repeated indices are summed. For simplicity, suppose all indices are repeated so that the expression is scalar-valued. Although this description of index notation depends on choosing a basis for V, such a tensor expression is basis-independent. In state model terms, each index is called an atom and the matrix of each tensor is a set of local weights. Note also that there an oriented graph associated to a tensor expression, where each vertex corresponds to a tensor and each edge connecting to vertices corresponds to indices connecting the tensors. A tensor expression is a more versatile form for a state model because arbitrary an arbitrary linear transformation of V extends to transformations of tensors. In particular, V might be a group representation and the tensors might be invariant under the group action.

If Z is a bipartite, trivially weighted graph, then $\det(Z)$ has such an interpretation, except that the tensors are invariant under a supergroup action instead of a group action. Recall that a supervector space is a $\mathbb{Z}/2$-graded vector space. An algebra structure on such a vector space is supercommutative if it is graded-commutative (odd vectors anticommute with each other, even-graded vectors commute with everything). A supergroup is a group object in the category of supercommutative algebras. It is like a group except that instead of a commutative algebra of real- or complex-valued functions on the group (with multiplication meaning pointwise multiplication), there is a supercommutative algebra A. The group law (in the finite-dimensional case) is expressed by a multiplication operation

$$m : A^* \otimes A^* \to A^*$$

on the dual vector space. The particular supergroup of interest to us has the property that both A and A^* are isomorphic to $\Lambda(\mathbb{R})$, an exterior algebra with one generator x. We consider tensors and tensor expressions over A viewed as a representation of itself.

The space A has an invariant multilinear function

$$y_n : A^{\otimes n} \to \mathbb{R}$$

given by

$$a_1 \otimes a_2 \otimes \ldots \otimes a_n \mapsto \mu(a_1 a_2 \ldots a_n),$$

where the dual vector

$$\mu : A \to \mathbb{R}$$

is defined by

$$\mu(1) = 0 \qquad \mu(x) = 1.$$

The multilinear function y_n can be viewed as an element of $(A^*)^{\otimes n}$. Similarly there is a dual tensor

$$x_n \in A^{\otimes n}$$

using the algebra structure on A^*. If each black vertex of Z is replaced by a copy of x_n and each white vertex by a copy of y_n in such a way that there is an index for each edge, then the tensors together form a scalar-valued expression, because each index appears twice and every index is contracted. This scalar-valued expression is exactly $\det(Z)$.

VII Plane partitions

Plane partitions are one of the most interesting applications of the permanent-determinant method [21]. A *plane partition in a box* is a collection of unit cubes in an $a \times b \times c$ box (rectangular prism) such that below, behind, and to the left of each cube is either another cube or a wall. A plane partition in a box is equivalent to a tiling of a hexagon with sides of length a, b, c, a, b, and c by unit 60° triangles (Figure 13).

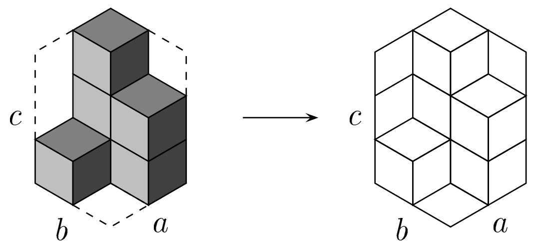

Figure 13: A plane partition and a tiling.

Such a lozenge tiling is equivalent to a matching in a hexagonal mesh (technically known as chicken wire) graph $Z(a, b, c)$ (Figure 14).

The total number of plane partitions in an $a \times b \times c$ box is given by MacMahon's formula:

$$N(a, b, c) = \frac{H(a)H(b)H(c)H(a+b+c)}{H(a+b)H(a+c)H(b+c)}, \tag{3}$$

where

$$H(n) = (n-1)!(n-2)!\ldots 3!2!1!$$

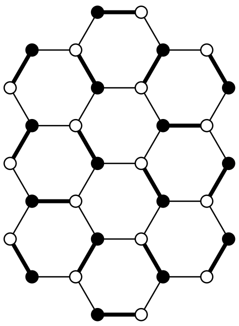

Figure 14: The graph $Z(2, 2, 3)$ and a matching.

is the hyperfactorial function. One can enumerate plane partitions with symmetry, each corresponding to matchings which are invariant under some group action G on $Z(a, b, c)$. If G acts freely, these are just the matchings of the quotient graph $Z(a, b, c)/G$. When G does not act freely, there is always a way to modify the quotient graph so that its matchings are equinumerous with the invariant matchings of $Z(a, b, c)$. The number of plane partitions in each symmetry class has a formula similar to equation (3).

Number	Kind	Acronym
$N_3(a, a, a)$	Cyclically symmetric	CSPP
$N_5(a, b, c)$	Self-complementary	SCPP
$N_9(2a, 2a, 2a)$	Cyclically symmetric, self-complementary	CSSCPP
$N_6(2a, b, b)$	Transpose-complement	TCPP
$N_8(2a, 2a, 2a)$	Cyclically symmetric, transpose-complement	CSTCPP

Table 2: Plane partitions and their numbers

A plane partition is *cyclically symmetric* if the corresponding matching is invariant under rotation by 120°. It is *self-complementary* if the matching is invariant under rotation by 180°. It is *transpose-complement* if the matching is invariant under a color-preserving reflection. The particular symmetry classes of plane partitions based on these symmetries that we will consider are given in Table 2. In the table the parameters of an enumeration $N_i(a, b, c)$ refer to the three dimensions of the box. For example $N_6(2a, b, b)$ is the number of TCPPs in a $2a \times b \times b$ box.

The invariant matchings corresponding to the first three symmetry classes are equivalent to matchings of $Z(a, b, c)/G$ for suitable G. The partitions in the last two symmetry classes correspond to matchings of $Z(a, b, c)//G$, a graph which is obtained from the quotient graph by deleting those vertices that have a non-trivial stabilizer

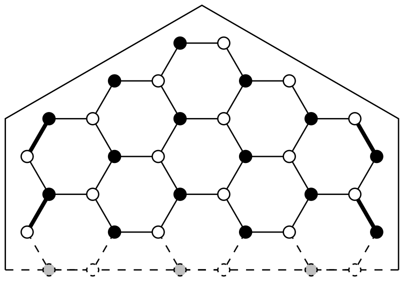

Figure 15: A deleted quotient graph for TCPPs.

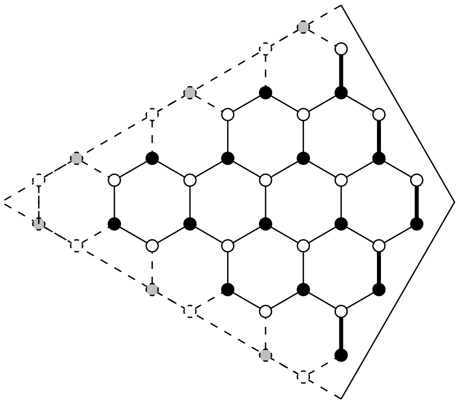

Figure 16: A deleted quotient graph for CSTCPPs.

in G (Figures 15 and 16). In each case let $Z_i(a,b,c)$ be the graph corresponding to $N_i(a,b,c)$. In the last two cases, some of the vertices are *vestigial*, meaning that they can only be matched in one way. (These vertices are matched with bold edges in the figures.) Let $Z_i'(a,b,c)$ be the graph $Z(a,b,c)$ with vestigial vertices removed.

The q-enumeration $N(a,b,c)_q$ of plane partitions is given by the natural q-analogue of equation (3), namely the one where each factorial is replaced by a q-factorial. Here the q-weight of a plane partition is q^k if there are k cubes. The q-enumeration is also the determinant of $Z(a,b,c)$ if it is weighted with curvature q in each hexagonal face. Three of the other symmetry classes can also be q-enumerated, but we will consider only the q-enumeration of cyclically symmetric plane partitions, where as with unrestricted plane partitions, the weight of each cube is q.

The six symmetry classes that do involve complementation have no obvious q-enumerations, but they do have natural -1 enumerations. In these cases the symmetry group G acts on individual cubes just as it does without complementation. But whereas in the four classes without complementation each orbit of cubes is either filled or empty, in the six classes with complementation each orbit is always half-filled.

Nonetheless, there is a natural move between symmetric plane partitions which replaces half of an orbit of cubes by the opposite half. Any two plane partitions in a given symmetry class differ by either an odd or an even number of moves. This defines a relative sign between them. The corresponding signed enumeration is a natural generalization of q-enumeration with $q = -1$. We conjecture that there is a product formula for the -1-enumeration of every symmetry class of plane partitions.

Stembridge [37] has found interesting patterns in signed enumerations of cyclically symmetric plane partitions called *strange* enumerations. In a strange enumeration, the weight of each plane partition is the product of the weights of cubes. Each cube has weight 1 or -1 depending on its position in the box. Consider plane partitions in the box $[0, a]^3$ in Cartesian coordinates, and denote each unit cube by the coordinates of the corner farthest away from the origin. We will consider strange enumerations $N_3(a, a, a)_{s,t,u}$, where s is the sign (or weight) of cubes (i, i, i), t is the sign of cubes (i, j, j) with $i \neq j$, and u is the sign of cubes (i, j, k) with i, j, and k all different. Similarly, let $N(2a, 2b, 2c)_{s,t}$ be a signed (or weighted) enumeration of unrestricted plane partitions in a $2a \times 2b \times 2c$ box, where the cubes $(a + i, b + i, c + i)$ (where i can be positive or negative) have weight s and the other cubes have weight t.

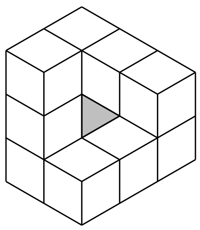

Figure 17: A Penrose-style lozenge tiling.

Finally, let a, b, and c have the same parity. We will consider lozenge tilings of an a, $b+1$, c, $a+1$, b, $c+1$ hexagon with the middle unit triangle removed. Such tilings resemble a Penrose impossible triangle (Figure 17). Let $N_P(a, b, c)$ be the number of these and let $Z_P(a, b, c)$ be the corresponding graph. There is a product formula for $N_P(a, b, c)$ [31] which has been proved by Okada and Krattenthaler [28]. A very short argument has been found by Ciucu for the case $b = c$ [6] and generalized [4].

VII A Various relations

The properties of the permanent-determinant method yield various relations between the enumerations of plane partitions defined above. Many enumerations of plane partitions are *round*. An integer is round if it is a product of relatively small numbers. A polynomial is round if it is a product of cyclotomic polynomials of low degree. A round enumeration in combinatorics almost always has an explicit product formula.

Rotational symmetry of $Z(a,a,a)$ implies that

$$N(a,a,a)_{q^3} = N_3(a,a,a)_q N_3(a,a,a)_{\omega q} N_3(a,a,a)_{\omega^2 q},$$

where ω is a cube root of unity. This relation together with MacMahon's formula for $N(a,a,a)_q$ implies that $N_3(a,a,a)_q$ is round, but it does not imply the explicit formula for $N_3(a,a,a)_q$ conjectured by MacDonald and proved by Mills, Robbins, and Rumsey [25].

Rotational symmetry also tells us that

$$\begin{aligned} N(a,a,a)_{1,-1} = {} & N_3(a,a,a)_{1,-1,-1} \\ & N_3(a,a,a)_{\omega,-1,-1} \\ & N_3(a,a,a)_{\omega^2,-1,-1}. \end{aligned}$$

The factor $N_3(a,a,a)_{1,-1,-1}$ is a strange enumeration. This case is entirely analogous to $N_3(a,a,a)_q$ because we know that $N(a,a,a)_{1,-1}$ is round (see below). Therefore $N_3(a,a,a)_{1,-1,-1}$ is round also, but the explicit formula conjectured by Stembridge is still open.

Reflection symmetry in $Z(2a,b,b)$ tells us that

$$N(2a,b,b) = N_6(2a,b,b) N_6(2a,b,b)_2,$$

and reflection symmetry in $Z(2a+1,b,b)$ tells us that

$$N(2a+1,b,b) = 2N_6(2a,b,b) N_6(2a,b,b)_2.$$

Here the second factor is a certain 2-enumeration of TCPP's (a weighting where certain faces have curvature 2). These two relations imply the known formula for $N_6(2a,b,b)$ [36].

The method of Section IV C establishes the identities

$$\begin{aligned} |N_1(2a,2b,2c)_{-1,1}| &= N_5(2a,2b,2c)^2 \\ |N_3(2a,2a,2a)_{-1,1,1}| &= N_9(2a,2a,2a)^2. \end{aligned}$$

The second identity is one of Stembridge's strange enumerations. Two other strange enumerations,$|N_3(2a,2a,2a)_{-1,-1,1}|$ and $|N_3(2a,2a,2a)_{-1,1,-1}|$, can be recognized as perfect squares by the same method. However, these enumerations are not round numbers; in general they have large prime factors.

The construction in Section VI A applies to plane partitions as follows: Let Z be the disjoint union of two copies of the plane partition graph $Z(a,b,c)$ with flat weighting. Arrange Z so that most of the horizontal edges of one copy are concentric with hexagons of the other copy, is in Figure 18. Now replace all of the crossings

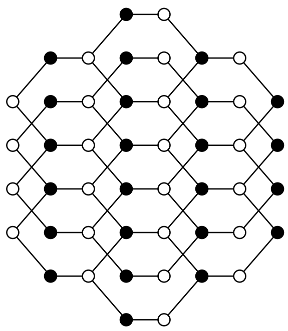

Figure 18: Two copies of $Z(2,2,2)$ overlaid.

by butterflies. Most of the long horizontal edges of the butterflies cancel in pairs (they are multiple edges with opposite weight). The new graph, after a little vertex splitting, is exactly the graph $Z(2a,2b,2c)$, but the weighting is no longer flat. Rather it has curvature -1 in every face. Since

$$\det(Z) = N(a,b,c)^2,$$

the conclusion is that

$$N(2a,2b,2c)_{-1} = N(a,b,c)^2.$$

This relation is independently a corollary of MacMahon's q-enumeration of plane partitions. Via Stembridge's $q=-1$ phenomenon, it is also related to the enumeration of self-complementary plane partitions. Since the original graph Z could have been given a weighting with indeterminate curvature instead of a flat one, it generalizes to a new relation between generating functions for plane partitions in a $2a \times 2b \times 2c$ box and plane partitions in an $a \times b \times c$ box.

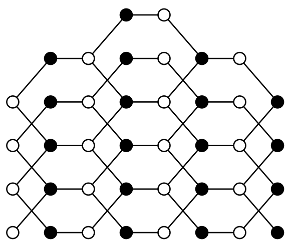

Figure 19: Overlaying $Z_6(4,2,2)$ and $Z'_6(4,3,3)$.

Forcing planarity also establishes the identities

$$
\begin{aligned}
N(2a+1,2b,2c)_{-1} &= N(a,b,c)N(a+1,b,c)\\
N(2a+1,2b+1,2c)_{-1} &= N(a,b+1,c)N(a+1,b,c)\\
N_6(4a,2b,2b)_{-1} &= N_6(2a,b,b)^2\\
N_6(4a+2,2b+1,2b+1)_{-1} &= N_6(2a,b+1,b+1)\\
&\quad N_6(2a+2,b,b)\\
N_6(4a,2b+1,2b+1)_{-1} &= N_6(2a,b,b)\\
&\quad N_6(2a,b+1,b+1).
\end{aligned}
$$

The last three identities require some manipulations with vestigial vertices. For example, the last identity is derived by overlaying $Z_6(2a,b,b)$ and $Z_6'(2a,b+1,b+1)$ and then forcing planarity (Figure 19).

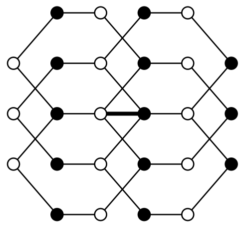

Figure 20: Two joined copies of $N_P(1,1,1)$.

A more complicated use of Section VI A establishes the identities

$$|N(2a,2b,2c)_{1,-1}| = N(a,b,c)N(a,b,c)^2_{-1} \tag{4}$$

and

$$|N(2a+1,2b+1,2c+1)_{1,-1}| = 2N_P(a,b,c)^2. \tag{5}$$

To derive equation (4), take two copies of $Z(a,b,c)$, one flat and one with curvature -1 at the center face, and overlay them as in Figure 18. The first copy has determinant $N(a,b,c)$, while by Section IV C, the second copy has determinant $N(a,b,c)^2_{-1}$. For equation (5), take two copies of $Z_P(a,b,c)$ overlaid, but instead of deleting the middle vertices, connect them by an edge of weight 2, as in Figure 20. (In the figure, the connecting edge of weight 2 is in bold.) If each copy of $Z_P(a,b,c)$ is weighted so that it would be flat if its middle vertex were removed, then the determinant of the combined graph is $2N_P(a,b,c)^2$, since every matching has to contain the middle edge. But forcing planarity converts this graph to $Z(2a+1,2b+1,2c+1)$ with a certain weighting. The determinant of this graph is the left side of equation (5), as desired.

Symmetry of $Z(a,a,a)$ yields the factorization:

$$\begin{aligned} N(a,a,a)_{1,-1} = {} & N_3(a,a,a)_{1,-1,-1} \\ & N_3(a,a,a)_{\omega,-1,-1} \\ & N_3(a,a,a)_{\omega^2,-1,-1}. \end{aligned}$$

This implies that $N_3(a,a,a)_{1,-1,-1}$ is a round number, but does not establish its explicit formula.

Finally, plane partitions in an $a \times b \times c$ box are a good example of the connection between the permanent-determinant method and the Gessel-Viennot method. Here there are three Gessel-Viennot matrices, of order $a \times a$, $b \times b$, and $c \times c$. They are also known as Carlitz matrices.

Question 2. *What is the cokernel of a Kasteleyn matrix for plane partitions in an $a \times b \times c$ box?*

By Gessel-Viennot, if

$$\min(a,b,c) = 1,$$

the cokernel is cyclic. But if

$$a = b = c = 2,$$

the cokernel is the non-cyclic abelian group $\mathbb{Z}/2 \times \mathbb{Z}/10$.

VIII Historical note

Kasteleyn [18] gives an excellent exposition of both the Hafnian-Pfaffian method and its application to problems in statistical mechanics. His bibliography constitutes a nearly complete chronicle of the series of papers that led to its discovery. In this section, we give a summary of that chronicle.

Onsager's solution of the Ising model [29] was the first convincing analysis of any interesting statistical mechanical model of a crystal. However, this paper contained difficult mathematics which left open the possibility of a simpler approach. With this motivation, Kac and Ward [17] announced a new solution eight years later, in which the partition function of the Ising model was expressed as the determinant of a relatively small matrix. The permutation expansion of the determinant was interpreted as a weighted enumeration. This paper attracted the attention of many strong physicists and mathematicians [26], but it contained a serious combinatorial error. Feynman recast the calculations in a correct but incomplete form in unpublished work. He replaced the error by a conjecture which was popularized by Harary, who was organizing a book on applications of graph theory to physics. Sherman proved the conjecture [35] and generalized both it and the Kac-Ward determinant to the

Ising model for arbitrary plane graphs. Harary eventually substituted his reference to Sherman's method by Kasteleyn's survey [18].

Independently of Sherman, Hurst and Green [14] found a cleaner approach to the square-lattice Ising model in which the partition function is the Pfaffian of the anti-symmetric incidence matrix of a nearly planar, weighted graph. The paper of Hurst and Green inspired Fisher and Temperley, and independently Kasteleyn, to enumerate perfect matchings in a square lattice with a special case of the Hafnian-Pfaffian method. Fisher saw that the method also works for finite hexagonal meshes (which already includes some enumerations of current interest). Independently Kasteleyn generalized the method to all plane graphs.

Later Percus [30] saw that since the square lattice is bipartite, the Hafnian-Pfaffian method reduces to the permanent-determinant method. With hindsight it seems that the only reason that Pfaffians and not determinants were used for the square lattice dimer model was that they arise naturally in the planar Ising model. (Mathematical physicists call weighted enumerations of perfect matchings *dimer models*.) Much later still, Little showed that $K_{3,3}$-free graphs are Pfaffian [24].

These results seemed to complete the story of the permanent-determinant method, until interest was renewed more recently in the context of enumerative combinatorics [9, 10, 15, 21, 22, 31, 40, 45]. In the author's opinion, one of the nicest new observations about the method is the fact that some of the minors of the inverse of a large Kasteleyn matrix are the probabilities of local configurations of edges [19]. For this reason and others, the method could well have some bearing on arctic circle phenomena in domino and lozenge tilings [8, 16]. Another recent development is a satisfactory classification of bipartite Pfaffian graphs [34].

The Gessel-Viennot method was discovered by Gessel and Viennot [11, 12] in the context of enumerative combinatorics, and completely independently of the permanent-determinant method. The method was independently anticipated by Lindström [23] in the context of matroid theory. It has been widely influential in enumeration of plane partitions, even more so than the permanent-determinant method [2–4, 7, 20, 27, 28, 38]. Yet another determinant variation which is likely related to the permanent-determinant method was recently found by Helfgott and Gessel [13].

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