On the Stanley-Wilf conjecture for the number of permutations avoiding a given pattern

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ABSTRACT. Consider, for a permutation $\sigma \in \mathcal{S}_k$, the number $F(n,\sigma)$ of permutations in $\mathcal{S}_n$ which avoid σ as a subpattern. The conjecture of Stanley and Wilf is that for every σ there is a constant $c(\sigma) < \infty$ such that for all n, $F(n,\sigma) \le c(\sigma)^n$. All the recent work on this problem also mentions the "stronger conjecture" that for every σ, the limit of $F(n,\sigma)^{1/n}$ exists and is finite. In this short note we prove that the two versions of the conjecture are equivalent, with a simple argument involving subadditivity.

We also discuss n-permutations, containing all $\sigma \in \mathcal{S}_k$ as subpatterns. We prove that this can be achieved with $n = k^2$, we conjecture that asymptotically $n \sim (k/e)^2$ is the best achievable, and we present Noga Alon's conjecture that $n \sim (k/2)^2$ is the threshold for random permutations.

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1. Introduction

Consider, for a permutation $\sigma \in \mathcal{S}_k$, the set $\mathcal{A}(n,\sigma)$ of permutations $\tau \in \mathcal{S}_n$ which avoid σ as a subpattern, and its cardinality, $F(n,\sigma) := |\mathcal{A}(n,\sigma)|$. Recall that "$\tau$ contains σ" as a subpattern means that there exist $1 \le x_1 < x_2 < \cdots < x_k \le n$ such that for $1 \le i, j \le k$,

$$\tau(x_i) < \tau(x_j) \quad \text{if and only if} \quad \sigma(i) < \sigma(j). \tag{1}$$

An outstanding conjecture is that for every σ there is a finite constant $c(\sigma)$ such that for all n, $F(n,\sigma) \le c(\sigma)^n$. In the 1997 Ph.D. thesis of Bóna [2], supervised by

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Stanley, this conjecture is attributed to "Wilf and Stanley [oral communication] from 1990." All the recent work on this problem also mentions the "stronger conjecture" that for every σ, the limit of $F(n,\sigma)^{1/n}$ exists and is finite. According to Wilf (private communication, 1999) the original conjecture was of this latter form.

In this short note we give, as Theorem 1, a simple argument, involving subadditivity, which shows that the two versions of the conjecture are equivalent.

Here is some background information on the current status of the Stanley-Wilf conjecture. Represent $\sigma \in \mathcal{S}_k$ by the word $\sigma(1)\,\sigma(2)\,\cdots\sigma(k)$. For the case of the increasing pattern, i.e the identity permutation, $\sigma = 12\cdots k$, the upper bound $F(n,\sigma) \le ((k-1)^2)^n$ is well known, and follows from the Robinson-Schensted-Knuth correspondence; also Regev [7] gives the asymptotics

$$F(n, 12\cdots k) \sim \lambda_k \frac{(k-1)^{2n}}{n^{k(k-2)/2}},$$

with an explicit constant λ_k. Simion and Schmidt [8] give a bijective proof that for each $\sigma \in \mathcal{S}_3$, $F(n,\sigma) = \frac{1}{n+1}\binom{2n}{n}$; see also Knuth [6], section 2.2.1, exercises.

For $\sigma = 1342$, Bóna [2] finds the explicit generating function for $F(n,\sigma)$, showing that for all n, $F(n,1342) < 8^n$, and $\lim F(n,1342)^{1/n} = 8$. Note in contrast that $\lim F(n,1234)^{1/n} = 9$. Bóna observes that indeed, in all cases for which $\lim F(n,\sigma)^{1/n}$ is known explicitly, it is an integer! For the special class of "layered patterns," such as $\sigma = 67\,345\,12$, Bóna [3] has shown that $\sup_n F(n,\sigma)^{1/n}$ is finite. Alon and Friedgut [1] prove an upper bound for the general case which is tantalizingly close to the goal; they relate the problem to a result on generalized Davenport-Schinzel sequences from Klazar [5], and show that for every $\sigma \in \mathcal{S}_k$ there exists $c(\sigma) < \infty$ such that for all n, $F(n,\sigma) \le c(\sigma)^{n\gamma^*(n)}$, where $\gamma^*(n)$ is an extremely slowly growing function, given explicitly in terms of the inverse of the Ackermann function.

Theorem 1. *For every $k \ge 2$ and $\sigma \in \mathcal{S}_k$, for every $m, n \ge 1$,*

$$F(m+n,\sigma) \ge F(m,\sigma)\ F(n,\sigma) \tag{2}$$

and $F(n,\sigma) \ge 1$; hence by Fekete's lemma on subadditive sequences,

$$c(\sigma) := \lim_{n\to\infty} F(n,\sigma)^{1/n} \ \in [1,\infty] \text{ exists,} \tag{3}$$

and $\forall n \ge 1, \quad F(n,\sigma) \le c(\sigma)^n$.

Proof. First we will show (2) by constructing, from an m-permutation and an n-permutation which avoid τ, an $(m+n)$-permutation which avoids τ, injectively.

Without loss of generality, we may assume that k precedes 1 in σ (since, with $(\cdot)^r$ to denote the left-right reverse of a permutation, τ avoids σ iff τ^r avoids σ^r, and hence for all n, $F(n,\sigma) = F(n,\sigma^r)$.)

Let $\tau' \in \mathcal{S}_m$ and $\tau'' \in \mathcal{S}_n$, where each of τ' and τ'' avoids σ. Let τ''' be the result of adding m to each symbol in the word for τ'', so that τ''' is a word in which each of the symbols $m+1, \ldots, m+n$ appears exactly once.

Consider the concatenation τ of τ' with τ''' as a permutation, $\tau \in \mathcal{S}_{m+n}$. Clearly, τ avoids σ, establishing (2).

[In detail, suppose to the contrary that τ contains σ, say at the k-tuple of positions given by $1 \le x_1 < x_2 < \cdots < x_k \le m+n$. Recall that k precedes 1 in σ; say that $\sigma(a) = 1$ and $\sigma(b) = k$ with $1 \le b < a \le k$, so that by (1), for $1 \le i \le k$, $\tau(x_a) \le \tau(x_i) \le \tau(x_b)$. If $x_k \le m$ then τ' contains σ, and if $x_1 > m$ then τ'' contains σ. If neither of these, then the $x_1 \le m$ so that $\tau(x_1) \le m$, hence $\tau(x_a) \le \tau(x_1) \le m$ and therefore $x_a \le m$; similarly $x_k > m$ so that $\tau(x_k) > m$, hence $\tau(x_b) \ge \tau(x_k) > m$ and therefore $x_b > m$, contradicting $b < a$.]

Recalling that k precedes 1 in σ, the identity permutation in $\mathcal{S}_n$ avoids σ and demonstrates that $F(n, \sigma) \ge 1$ for every $n \ge 1$. Fekete's lemma [4], see also [9], is that if $a_1, a_2, \ldots \in \mathbb{R}$ satisfy for all $m, n \ge 1$, $a_m + a_n \le a_{m+n}$, then $\lim_{n\to\infty} a_n/n = \inf_{n\ge1} a_n/n \in [-\infty, \infty)$. Applying this with $a_n := -\log F(n, \sigma)$ completes our proof. $\square$

There exist [10] examples with $\sigma, \sigma' \in \mathcal{S}_k$, with σ' the identity permutation, and $F(n, \sigma) > F(n, \sigma')$, and Bóna [2], Theorem 4 shows that for all $n \ge 7$, $F(n, 1324) > F(n, 1234)$. Nevertheless, it is possible that for every k, the largest exponential growth rate is the $(k-1)^2$ achieved by the identity permutation.

Conjecture 1. *(\$100.00) For all $\sigma \in \mathcal{S}_k$ and $n \ge 1$, $F(n, \sigma) \le (k-1)^{2n}$.*

The problem of the shortest common superpattern.

Define $G(n, k)$ to be the number of permutations $\tau \in \mathcal{S}_n$ which avoid *at least one* permutation in $\mathcal{S}_k$, i.e.

$$G(n,k) := |\cup_{\sigma \in \mathcal{S}_k} \mathcal{A}(n,\sigma)|, \text{ where } F(n,\sigma) := |\mathcal{A}(n,\sigma)|.$$

Simion and Schmidt [8], p. 398, give a formula for $n! - G(n, 3)$, the number of n-permutations which contain all six patterns of length 3. In considering $G(n, k)$, it is natural to consider the length $m(k)$ of the shortest permutation which contains every $\sigma \in \mathcal{S}_k$ as a subpattern, i.e. to consider

$$m(k) := \min\{n \colon G(n,k) < n!\ \} = \min\{n \colon \cup_{\sigma \in \mathcal{S}_k} \mathcal{A}(n,\sigma) \ \ne \mathcal{S}_n\ \}.$$

For a trivial lower bound on $m(k)$, since $\tau \in \mathcal{S}_n$ contains at most $\binom{n}{k}$ subpatterns, to contain every subpattern requires $\binom{n}{k} \ge k!$, hence $\liminf_k m(k)/k^2 \ge 1/e^2$.

Theorem 2. *There exists an n-permutation, with $n = k^2$, containing every k-permutation as a subpattern; i.e. $m(k) \le k^2$.*

Proof. Consider the lexicographic order on $[k]^2$ as a one-to-one map specifying the ranks of the ordered pairs, i.e. let $r : [k]^2 \to [k^2]$, with $(i,j) \mapsto (i-1)k + j$. Also consider the transposed lexicographic order $t : [k]^2 \to [k^2]$ given by $t(i,j) := r(j,i)$. Consider the permutation $\tau \in \mathcal{S}_{k^2}$ given by $\tau = r \circ t^{-1}$; for example, with $k = 3$, this is $\tau = 147258369$. Then, clearly, τ contains every $\sigma \in \mathcal{S}_k$ as a subpattern. In detail, with the positions $x_1 := t(\sigma(1), 1), \ldots, x_k := t(\sigma(k), k)$ we have $x_1 < \cdots < x_k$ and for $m = 1$ to k, $\tau(x_m) = (r \circ t^{-1})(t(\sigma(m), m)) = r(\sigma(m), m)$ so that $\tau(x_a) < \tau(x_b)$ iff $\sigma(a) < \sigma(b)$. $\square$

Conjecture 2. *As $k \to \infty$, $m(k) \sim (k/e)^2$.*

In contrast, from the known behavior of the length L_n of the longest increasing subsequence, $L_n \sim 2\sqrt{n}$ with high probability, one cannot hope to use *random* permutations to show that $\liminf m(k)/k^2 \leq (1/e)^2$. The probability that a random n-permutation does *not* contain every $\sigma \in \mathcal{S}_k$ as a subpattern is $G(n,k)/n!$. Define the threshold $t(k)$ by $t(k) = \min\{n \colon G(n,k)/n! \leq 1/2\}$, so that trivially $m(k) \leq t(k)$, and hence $\liminf\ t(k)/k^2 \geq 1/4$.

Conjecture 3. *(Noga Alon) The threshold length $t(k)$, for a random permutation to contain all k-permutations with substantial probability, has $t(k) \sim (k/2)^2$.*

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